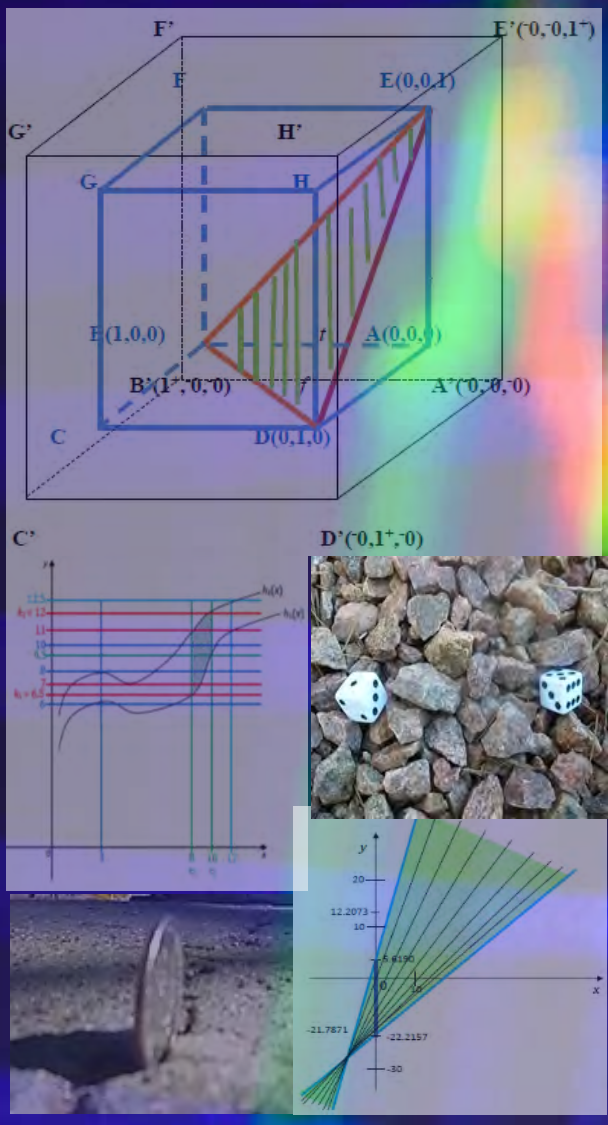
Volume 75,2025

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering



⟨A⟩ ⟨neutA⟩ ⟨antiA⟩

Editors-in-Chief
Florentin Smarandache
Mohamed Abdel-Baset
Maikel Leyva-Vázquez

ISSN 2331-6055 (Print)
ISSN 2331-608X (Online)

Neutrosophic Science
International Association (NSIA)

ISSN 2331-6055 (print) *ISSN 2331-608X (online)*

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering

University of New Mexico

ISSN 2331-6055 (print) ISSN 2331-608X (online)

University of New Mexico

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering

Copyright Notice

Copyright @ Neutrosophics Sets and Systems

All rights reserved. The authors of the articles do hereby grant Neutrosophic Sets and Systems non-exclusive, worldwide, royalty-free license to publish and distribute the articles in accordance with the Budapest Open Initiative: this means that electronic copying, distribution and printing of both full-size version of the journal and the individual papers published therein for non-commercial, academic or individual use can be made by any user without permission or charge. The authors of the articles published in Neutrosophic Sets and Systems retain their rights to use this journal as a whole or any part of it in any other publications and in any way they see fit. Any part of Neutrosophic Sets and Systems howsoever used in other publications must include an appropriate citation of this journal.

Information for Authors and Subscribers

*"*Neutrosophic Sets and Systems*"* has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (*T*), a degree of indeterminacy (*I*), and a degree of falsity (*F*), where *T, I, F* are standard or non-standard subsets of $]^{-}0, 1^{+}[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

All submissions should be designed in MS Word format using our template file:

http://fs.unm.edu/NSS/NSS-paper-template.doc.

A variety of scientific books in many languages can be downloaded freely from the Digital Library of Science:

http://fs.unm.edu/ScienceLibrary.htm.

To submit a paper, mail the file to the Editor-in-Chief. To order printed issues, contact the Editor-in-Chief. This journal is non-commercial, academic edition. It is printed from private donations.

Information about the neutrosophics you get from the UNM website:

http://fs.unm.edu/neutrosophy.htm. The

home page of the journal is accessed on

http://fs.unm.edu/NSS.

ISSN 2331-6055 (print) ISSN 2331-608X (online)

University of New Mexico

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering

**** NSS has been accepted by SCOPUS. Starting with Vol. 19, 2018, the NSS articles are indexed in Scopus.**

NSS ABSTRACTED/INDEXED IN
SCOPUS,
Google Scholar,
Google Plus,
Google Books,
EBSCO,
Cengage Thompson Gale (USA),
Cengage Learning (USA),
ProQuest (USA),
Amazon Kindle (USA),
University Grants Commission (UGC) - India,
DOAJ (Sweden),
International Society for Research Activity (ISRA),
Scientific Index Services (SIS),
Academic Research Index (ResearchBib),
Index Copernicus (European Union),
CNKI (Tongfang Knowledge Network Technology Co.,
Beijing, China),
Baidu Scholar (China),
Redalyc - Universidad Autonoma del Estado de Mexico (IberoAmerica),
Publons,
Scimago, etc.

Google Dictionaries have translated the neologisms "**neutrosophy**" (1) and"**neutrosophic**" (2), coined in 1995 for the first time, into about 100 languages.
FOLDOC Dictionary of Computing (**1, 2**), Webster
Dictionary (**1, 2**), Wordnik (**1**),Dictionary.com, The Free
Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

Copyright © Neutrosophic Sets and Systems, 2025

ISSN 2331-6055 (print) NSS **Editorial Board** *ISSN 2331-608X (online)*

University of New Mexico

Editors-in-Chief

Prof. Emeritus Florentin Smarandache, PhD, Postdoc, Mathematics, Physical and Natural Sciences Division, University of New Mexico, Gallup Campus, NM 87301, USA, Email: smarand@unm.edu.

Dr. Mohamed Abdel-Baset, Head of Department of Computer Science, Faculty of Computers and Informatics, Zagazig University, Egypt, Email: mohamedbasset@ieee.org.

Prof. Dr. Maikel Leyva-Vázquez, Instituto Superior Tecnológico Bolivariano de Tecnología, Ecuador: maikel.leyvav@ug.edu.ec.

Associate Editors

Dr. Said Broumi, Laboratory of Information Processing, Faculty of Science Ben M'Sik, University of Hassan II, Casablanca, Morocco, Email: s.broumi@flbenmsik.ma.

Assoc. Prof. Alok Dhital, Mathematics, Physical and Natural Sciences Division, University of New Mexico, Gallup Campus, NM 87301, USA, Email: adhital@unm.edu.

Dr. S. A. Edalatpanah, Department of Applied Mathematics, Ayandegan Institute of Higher Education, Tonekabon, Iran, Email: saedalatpanah@gmail.com.

Charles Ashbacker, Charles Ashbacher Technologies, Box 294, 118 Chaffee Drive, Hiawatha, IA 52233, United States, Email: cashbacher@prodigy.net.

Prof. Dr. Xiaohong Zhang, Department of Mathematics, Shaanxi University of Science &Technology, Xian 710021, China, Email: zhangxh@shmtu.edu.cn.

Prof. Dr. W. B. Vasantha Kandasamy, School of Computer Science and Engineering, VIT, Vellore 632014, India, Email: vasantha.wb@vit.ac.in.

Editors

Yanhui Guo, University of Illinois at Springfield, One University Plaza, Springfield, IL 62703, United States,Email: yguo56@uis.edu.

Giorgio Nordo, MIFT - Department of Mathematical and Computer Science, Physical Sciences and Earth Sciences, Messina University, Italy,Email: giorgio.nordo@unime.it.

Mohamed Elhoseny, American University in the Emirates,Dubai,UAE,
Email: mohamed.elhoseny@aue.ae.

Le Hoang Son, VNU Univ. of Science, Vietnam National Univ. Hanoi, Vietnam, Email: sonlh@vnu.edu.vn.

Huda E. Khalid, Head of Scientific Affairs and Cultural Relations Department, Nineveh Province, Telafer University, Iraq, Email: dr.huda-ismael@uotelafer.edu.iq.

A. A. Salama, Dean of the Higher Institute of Business and Computer Sciences, Arish, Egypt, Email: ahmed_salama_2000@sci.psu.edu.eg.

Young Bae Jun, Gyeongsang National University, South Korea, Email: skywine@gmail.com.

Yo-Ping Huang, Department of Computer Science and Information, Engineering National Taipei University, New Taipei City, Taiwan, Email: yphuang@ntut.edu.tw.

Tarek Zayed, Department of Building and Real Estate, The Hong Kong Polytechnic University, Hung Hom, 8 Kowloon, Hong Kong, China, Email: tarek.zayed@polyu.edu.hk.

Vakkas Ulucay, Kilis 7 Aralık University, Turkey, Email: vulucay27@gmail.com.

Peide Liu, Shandong University of Finance and Economics, China, Email: peide.liu@gmail.com.

Jun Ye, Ningbo University, School of Civil and Environmental Engineering, 818 Fenghua Road, Jiangbei District, Ningbo City, Zhejiang Province, People's Republic of China, Email: yejun1@nbu.edu.cn.

Memet Şahin, Department of Mathematics, Gaziantep University, Gaziantep 27310, Turkey, Email: mesahin@gantep.edu.tr.

Muhammad Aslam & Mohammed Alshumrani, King Abdulaziz Univ., Jeddah, Saudi Arabia, Emails magmuhammad@kau.edu.sa, maalshmrani @kau.edu.sa.

Mutaz Mohammad, Department of Mathematics, Zayed University, Abu Dhabi 144534, United Arab Emirates. Email: Mutaz.Mohammad@zu.ac.ae.

Abdullahi Mohamud Sharif, Department of Computer Science, University of Somalia, Makka Al-mukarrama Road, Mogadishu, Somalia, Email: abdullahi.shariif@uniso.edu.so.

.

Copyright © Neutrosophic Sets and Systems, 2025

Katy D. Ahmad, Islamic University of Gaza, Palestine,Email: katyon765@gmail.com.
NoohBany Muhammad, American University of Kuwait,Kuwait,Email: noohmuhammad12@gmail .com.
Soheyb Milles, Laboratory of Pure and Applied Mathematics, University of Msila, Algeria, Email: soheyb.milles@univ-msila.dz.
Pattathal Vijayakumar Arun, College of Science and Technology,Phuentsholing,Bhutan,
Email: arunpv2601@gmail.com.
Endalkachew Teshome Ayele, Department of Mathematics, Arbaminch University, Arbaminch, Ethiopia,Email: endalkachewteshome83@yahoo. com.
A. Al-Kababji, College of Engineering, Qatar University,Doha,Qatar,
Email: ayman.alkababji@ieee.org.
Xindong Peng, School of Information Science and Engineering, Shaoguan University, Shaoguan 512005, China, Email: 952518336@qq.com.
Xiao-Zhi Gao, School of Computing, University of Eastern Finland, FI-70211 Kuopio, Finland, xiao-zhi.gao@uef.fi.
Madad Khan, Comsats Institute of Information Technology,Abbottabad,Pakistan,
Email: madadmath@yahoo.com.
G. Srinivasa Rao, Department of Statistics, The University of Dodoma, Dodoma, PO. Box: 259, Tanzania,Email: gaddesrao@gmail.com.
Ibrahim El-henawy, Faculty of Computers and Informatics,ZagazigUniversity,Egypt,
Email: henawy2000@yahoo.com.
Muhammad Saeed, Department of Mathematics, University of Management and Technology, Lahore, Pakistan,
Email: muhammad.saeed@umt.edu.pk.
A. A. A. Agboola, Federal University of Agriculture, Abeokuta, Nigeria,
Email: agboolaaaa@funaab.edu.ng.
Abduallah Gamal, Faculty of Computers and Informatics, Zagazig University, Egypt,
Email: abduallahgamal@zu.edu.eg.
Ebenezer Bonyah, Department of Mathematics Education, Akenten Appiah-Menka University of Skills Training and Entrepreneurial Development, Kumasi 00233, Ghana,
Email: ebbonya@gmail.com.
Roan Thi Ngan, Hanoi University of Natural Resources and Environment, Hanoi, Vietnam,
Email: rtngan@hunre.edu.vn.
Sol David Lopezdomínguez Rivas, Universidad Nacional de Cuyo, Argentina.
Email: sol.lopezdominguez@fce.uncu.edu.ar.
Maikel Yelandi Leyva Vázquez, Universidad Regional Autónoma de los Andes (UNIANDES), Avenida Jorge Villegas, Babahoyo, Los Ríos, Ecuador,
Email: ub.c.investigacion@uniandes.edu.ec.
Arlen Martín Rabelo, Exxis, Avda. Aviadores del Chaco N° 1669 c/ San Martin, Edif. Aymac I, 4to. piso, Asunción, Paraguay,
Email: arlen.martin@exxis-group.com.
Carlos Granados, Estudiante de Doctorado en Matematicas, Universidad del Antioquia, Medellın, Colombia,
Email: carlosgranadosortiz@outlook.es.
Tula Carola Sanchez Garcia, Facultad de Educacion de la Universidad Nacional Mayor de San Marcos, Lima, Peru,
Email: tula.sanchez1@unmsm.edu.pe.
Carlos Javier Lizcano Chapeta, Profesor - Investigador de pregrado y postgrado de la Universidad de Los Andes, Mérida 5101, Venezuela, Email: lizcha_4@hotmail.com.
Noel Moreno Lemus, Procter & Gamble International Operations S.A., Panamá,
Email: nmlemus@gmail.com.
Asnioby Hernandez Lopez, Mercado Libre, Montevideo, Uruguay.
Email: asnioby.hernandez@mercadolibre.com.
Muhammad Akram, University of the Punjab, New Campus, Lahore, Pakistan,
Email: m.akram@pucit.edu.pk.
Tatiana Andrea Castillo Jaimes, Universidad de Chile, Departamento de Industria, Doctorado en Sistemas de Ingeniería, Santiago de Chile, Chile,
Email: tatiana.a.castillo@gmail.com.
Irfan Deli, Muallim Rifat Faculty of Education, Kilis 7 Aralik University, Turkey,
Email: irfandeli@kilis.edu.tr.
Ridvan Sahin, Department of Mathematics, Faculty of Science, Ataturk University, Erzurum 25240, Turkey, Email: mat.ridone@gmail.com.
Ibrahim M. Hezam, Department of computer, Faculty of Education, Ibb University, Ibb City,

Yemen, Email: ibrahizam.math@gmail.com.
Moddassir khan Nayeem, Department of Industrial and Production Engineering, American International University-Bangladesh, Bangladesh; nayeem@aiub.edu.
Aiyared Iampan, Department of Mathematics, School of Science, University of Phayao, Phayao 56000, Thailand, Email: aiyared.ia@up.ac.th.
Ameirys Betancourt-Vázquez, 1 Instituto Superior Politécnico de Tecnologias e Ciências (ISPTEC), Luanda, Angola, Email: ameirysbv@gmail.com.
H. E. Ramaroson, University of Antananarivo, Madagascar, Email: erichansise@gmail.com.
G. Srinivasa Rao, Department of Mathematics and Statistics, The University of Dodoma, Dodoma PO. Box: 259, Tanzania.
Onesfole Kuramaa, Department of Mathematics, College of Natural Sciences, Makerere University, P.O Box 7062, Kampala, Uganda, Email: onesfole.kurama@mak.ac.ug.
Karina Pérez-Teruel, Universidad Abierta para Adultos (UAPA), Santiago de los Caballeros, República Dominicana, Email: karinaperez@uapa.edu.do.
Neilys González Benítez, Centro Meteorológico Pinar del Río, Cuba, Email: neilys71@nauta.cu.
Jesus Estupinan Ricardo, Centro de Estudios para la Calidad Educativa y la Investigation Cinetifica, Toluca,Mexico,Email: jestupinan2728@gmail.com
Victor Christianto, Malang Institute of Agriculture (IPM), Malang, Indonesia, Email: victorchristianto@gmail.com.
Wadei Al-Omeri, Department of Mathematics, Al-Balqa Applied University, Salt 19117, Jordan, Email: wadeialomeri@bau.edu.jo.
Ganeshsree Selvachandran, UCSI University, Jalan Menara Gading, Kuala Lumpur, Malaysia, Email: Ganeshsree@ucsiuniversity.edu.my.
Ilanthenral Kandasamy, School of Computer Science and Engineering (SCOPE), Vellore Institute of Technology (VIT), Vellore 632014, India, Email: ilanthenral.k@vit.ac.in
Kul Hur, Wonkwang University, Iksan, Jeollabukdo, South Korea, Email: kulhur@wonkwang.ac.kr.
Kemale Veliyeva & Sadi Bayramov, Department of Algebra and Geometry, Baku State University, 23 Z. Khalilov Str., AZ1148, Baku, Azerbaijan, Email: kemale2607@mail.ru,
Email: baysadi@gmail.com.
Irma Makharadze & Tariel Khvedelidze, Ivane Javakhishvili Tbilisi State University, Faculty of Exact and Natural Sciences, Tbilisi, Georgia.
Inayatur Rehman, College of Arts and Applied Sciences, Dhofar University Salalah, Oman, Email: irehman@du.edu.om.
Mansour Lotayif, College of Administrative Sciences, Applied Science University, P.O. Box 5055, East Al-Ekir, Kingdom of Bahrain.
Riad K. Al-Hamido, Math Department, College of Science, Al-Baath University, Homs, Syria, Email: riad-hamido1983@hotmail.com.
Saeed Gul, Faculty of Economics, Kardan University, Parwan-e- Du Square, Kabil, Afghanistan, Email: s.gul@kardan.edu.af.
Faruk Karaaslan, Çankırı Karatekin University, Çankırı, Turkey, Email: fkaraaslan@karatekin.edu.tr.
Morrisson Kaunda Mutuku, School of Business, Kenyatta University, Kenya
Surapati Pramanik, Department of Mathematics, Nandalal Ghosh B T College, India, Email: drspramanik@isns.org.in.
Suriana Alias, Universiti Teknologi MARA (UiTM) Kelantan, Campus Machang, 18500 Machang, Kelantan, Malaysia, Email: suria588@kelantan.uitm.edu.my.
Arsham Borumand Saeid, Dept. of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran, Email: arsham@uk.ac.ir.
Ahmed Abdel-Monem, Department of Decision support, Zagazig University, Egypt, Email: aabdelmounem@zu.edu.eg.
Çağlar Karamasa, Anadolu University, Faculty of Business,
Turkey, Email: ckaramasa@anadolu.edu.tr.
Mohamed Talea, Laboratory of Information Processing, Faculty of Science Ben M'Sik, Morocco, Email: taleamohamed@yahoo.fr.
Assia Bakali, Ecole Royale Navale, Casablanca, Morocco, Email: assiabakali@yahoo.fr.
V.V. Starovoytov, The State Scientific Institution «The United Institute of Informatics Problems of the National Academy of Sciences of Belarus»,

Copyright © Neutrosophic Sets and Systems, 2025

Minsk, Belarus, Email: ValeryS@newman.bas-net.by.
E.E. Eldarova, L.N. Gumilyov Eurasian National University, Nur-Sultan, Republic of Kazakhstan, Email: Doctorphd_eldarova@mail.ru. Mukhamed iyeva Dilnoz Tulkunovna & Egamberdiev Nodir Abdunazarovich, Science and innovation center for information and communication technologies, Tashkent University of Information Technologies (named after Muhammad Al-Khwarizmi), Uzbekistan.
Mohammad Hamidi, Department of Mathematics, Payame Noor University (PNU), Tehran, Iran. Email: m.hamidi@pnu.ac.ir.
Lemnaouar Zedam, Department of Mathematics, Faculty of Mathematics and Informatics, University Mohamed Boudiaf, M'sila, Algeria, Email: l.zedam@gmail.com.
M. Al Tahan, Department of Mathematics, Lebanese International University, Bekaa, Lebanon, Email: madeline.tahan@liu.edu.lb.
Mohammad Abobala, Tishreen University, Faculty of Science, Department of Mathematics, Lattakia, Syria,Email" mohammad.abobala@tishreen.edu.sy
Rafif Alhabib, AL-Baath University, College of Science, Mathematical Statistics Department, Homs, Syria, Email: ralhabib@albaath-univ.edu.sy.
R. A. Borzooei, Department of Mathematics, Shahid Beheshti University, Tehran, Iran, borzooei@hatef.ac.ir.
Selcuk Topal, Mathematics Department, Bitlis Eren University, Turkey, Email: s.topal@beu.edu.tr.
Qin Xin, Faculty of Science and Technology, University of the Faroe Islands, Tórshavn, 100, Faroe Islands.
Sudan Jha, Pokhara University,Kathmandu, Nepal, Email: jhasudan@hotmail.com.
Mimosette Makem and Alain Tiedeu, Signal, Image and Systems Laboratory, Dept. of Medical and Biomedical Engineering, Higher Technical Teachers' Training College of EBOLOWA, PO Box 886, University of Yaoundé, Cameroon, E-mail: alain_tiedeu@yahoo.fr.
Mujahid Abbas, Department of Mathematics and Applied Mathematics, University of Pretoria Hatfield 002, Pretoria, South Africa, Email: mujahid.abbas@up.ac.za.
Željko Stević, Faculty of Transport and Traffic Engineering Doboj, University of East Sarajevo, Lukavica, East Sarajevo, Bosnia and Herzegovina, Email: zeljko.stevic@sf.ues.rs.ba.
Michael Gr. Voskoglou, Mathematical Sciences School of Technological Applications, Graduate Technological Educational Institute of Western Greece, Patras, Greece, Email: voskoglou@teiwest.gr.
Saeid Jafari, College of Vestsjaelland South, Slagelse, Denmark, Email: sj@vucklar.dk.
Angelo de Oliveira, Ciencia da Computacao, Universidade Federal de Rondonia, Porto Velho - Rondonia, Brazil, Email: angelo@unir.br.
Valeri Kroumov, Okayama University of Science, Okayama, Japan, Email: val@ee.ous.ac.jp.
Rafael Rojas, Universidad Industrial de Santander, Bucaramanga, Colombia, Email: rafael2188797@correo.uis.edu.co.
Walid Abdelfattah, Faculty of Law, Economics and Management, Jendouba, Tunisia, Email: abdelfattah.walid@yahoo.com.
Akbar Rezaei, Department of Mathematics, Payame Noor University, P.O.Box 19395-3697, Tehran, Iran, Email: rezaei@pnu.ac.ir.
John Frederick D. Tapia, Chemical Engineering Department, De La Salle University - Manila, 2401 Taft Avenue, Malate, Manila, Philippines, Email: john.frederick.tapia@dlsu.edu.ph.
Darren Chong, independent researcher, Singapore, Email: darrenchong2001@yahoo.com.sg.
Galina Ilieva, Paisii Hilendarski, University of Plovdiv, 4000 Plovdiv, Bulgaria, Email: galili@uni-plovdiv.bg.
Paweł Pławiak, Institute of Teleinformatics, Cracow University of Technology, Warszawska 24 st., F-5, 31-155 Krakow, Poland, Email: plawiak@pk.edu.pl.
E. K. Zavadskas, Vilnius Gediminas Technical University, Vilnius, Lithuania, Email: edmundas.zavadskas@vgtu.lt.
Darjan Karabasevic, University Business Academy, Novi Sad, Serbia, Email: darjan.karabasevic@mef.edu.rs.
Dragisa Stanujkic, Technical Faculty in Bor, University of Belgrade, Bor, Serbia, Email: dstanujkic@tfbor.bg.ac.rs.
Katarina Rogulj, Faculty of Civil Engineering,

Architecture and Geodesy, University of Split, Matice Hrvatske 15, 21000 Split, Croatia; Email: katarina.rogulj@gradst.hr.
Luige Vladareanu, Romanian Academy, Bucharest, Romania, Email: luigiv@arexim.ro.
Hashem Bordbar, Center for Information Technologies and Applied Mathematics, University of Nova Gorica, Slovenia, Email: Hashem.Bordbar@ung.si.
N. Smidova, Technical University of Kosice, SK 88902, Slovakia, Email: nsmidova@yahoo.com.
Quang-Thinh Bui, Faculty of Electrical Engineering and Computer Science, VŠB-Technical University of Ostrava, Ostrava-Poruba, Czech Republic, Email: qthinhbui@gmail.com.
Mihaela Colhon & Stefan Vladutescu, University of Craiova, Computer Science Department, Craiova, Romania,Emails: colhon.mihaela@ucv.ro, vladutescu.stefan@ucv.ro.
Philippe Schweizer, Independent Researcher, Av. de Lonay 11, 1110 Morges, Switzerland, Email: flippe2@gmail.com.
Madjid Tavanab, Business Information Systems Department, Faculty of Business Administration and Economics University of Paderborn, D-33098 Paderborn, Germany, Email: tavana@lasalle.edu.
Rasmus Rempling, Chalmers University of Technology, Civil and Environmental Engineering, Structural Engineering, Gothenburg, Sweden.
Fernando A. F. Ferreira, ISCTE Business School, BRU-IUL, University Institute of Lisbon, Avenida das Forças Armadas, 1649-026 Lisbon, Portugal, Email: fernando.alberto.ferreira@iscte-iul.pt.
Julio J. Valdés, National Research Council Canada, M-50, 1200 Montreal Road, Ottawa, Ontario K1A 0R6, Canada, Email: julio.valdes@nrc-cnrc.gc.ca.
Tieta Putri, College of Engineering Department of Computer Science and Software Engineering, University of Canterbury, Christchurch, New Zeeland.
Phillip Smith, School of Earth and Environmental Sciences, University of Queensland, Brisbane, Australia, phillip.smith@uq.edu.au.
Sergey Gorbachev, National Research Tomsk State University, 634050 Tomsk, Russia, Email: gsv@mail.tsu.ru.
Sabin Tabirca, School of Computer Science, University College Cork, Cork, Ireland, Email: tabirca@neptune.ucc.ie.
Umit Cali, Norwegian University of Science and Technology, NO-7491 Trondheim, Norway, Email: umit.cali@ntnu.no.
Willem K. M. Brauers, Faculty of Applied Economics, University of Antwerp, Antwerp, Belgium, Email: willem.brauers@uantwerpen.be.
M. Ganster, Graz University of Technology, Graz, Austria, Email: ganster@weyl.math.tu-graz.ac.at.
Ignacio J. Navarro, Department of Construction Engineering, Universitat Politècnica de València, 46022 València, Spain, Email: ignamar1@cam.upv.es.
Francisco Chiclana, School of Computer Science and Informatics, De Montfort University, The Gateway, Leicester, LE1 9BH, United Kingdom, Email: chiclana@dmu.ac.uk.
Jean Dezert, ONERA, Chemin de la Huniere, 91120 Palaiseau, France, Email: jean.dezert@onera.fr.

.

Copyright © Neutrosophic Sets and Systems, 2025

ISSN 2331-6055 (print) NSS ISSN 2331-608X (online)
University of New Mexico

Contents

Yaser Ahmad Alhasan, Abuobida Alfahal and Raja Abdullah Abdulfatah, The 2- refined neutrosophic hyperbolic functions with its differential and integrals……..1-14
Tanmoy Chatterjee, Surapati Pramanik, Triangular fuzzy quadripartitioned neutrosophic set and its properties……15-28
Gugu Narzary, Bhimraj Basumatary, A Study on n-th Derivative of Neutrosophic Function and Neutrosophic Differential Equation……..……..29-49
Yaser Ahmad Alhasan, Issam Mohammed Abdallah Abdalaziz, Raja Abdullah Abdulfatah and Qusay Alhassan, The linear 2-refined neutrosophic differential equations……..……50-59
Prasen Boro, and Bhimraj Basumatary, A Study on some Properties of Fourier Integrals Based on Neutrosophic Function……60-82
Maha Mohammed Saeed, Samer Al Ghour, Arif Mehmood, Wahid Rehman, Fabio Souto de Azevedo, R. Seethalakshi, Aamir Abbas, M. Kaviyarasu, Pairwise Approach to Neutrosophic Soft Bi-Topological Spaces: Exploring Neutrosophic Soft Open Sets and Their Real-Life Applications..83-138
Mohammad Shafiq bin Mohammad Kamari, Zahari Bin Md. Rodzi, R.H. Al-Obaidi, Faisal Al-Sharqi, Ashraf Al-Quran, Rawan A. shlaka, Deciphering the Geometric Bonferroni Mean Operator in Pythagorean Neutrosophic Sets Framework……139-161
Priya Mathews, Lovelymol Sebastian and Baiju Thankachan, Similarity -Based Pattern Recognition for Disease Symptom Extraction and Characterization……..162-180
Ahmad A. Abubaker, Mayada Abualhomos, Khaled Matarneh, Abdallah Al-Husban, A Numerical Approach For The Algebra of Two-Fold 2×2 Fuzzy Real Matrices and Anti Fuzzy Matrices…181-195
Raed Hatamleh, on The Continuous and Differentiable Two-Fold Neutrosophic and Fuzzy Real Functions……....196-209
Hanzhong Wang, Jun Ye, Rui Yong, Single Valued Neutrosophic Number Ensemble Learning Model for Stability Classification of Open Pit Mine Slopes……210-223
S. Narasimman, M. Shanmugapriya, R. Sundareswaran, Laxmi Rathour, Lakshmi Narayan Mishra, Vinita Dewangan, Vishnu Narayan Mishra, Identification of influential factors affecting student performance in semester examinations in the educational institution using score topological indices in Single Valued Neutrosophic Graphs……..224-240
Bing Sun, Dongyan Gao, Shujuan Yin, Enhancing Multiple-Attribute Decision-Making Framework for Talent Cultivation Quality Evaluation Through Using Weighted Dual Hamy Mean and Prioritized Aggregation Operators……..241-277
Huayang Sun, Zhongyu Liu, Enhanced TFNN-GRA Technique for Multiple-Attribute Decision-Making with Triangular Fuzzy Neutrosophic Information and Applications to Performance Evaluation of Internally Cured High-Strength Concrete ……..278-301
Antonios Paraskevas and Florentin Smarandache, Introduction to algorithmic ethics under a neutrosophic framework: Re-mapping the debate……302-308
Martina Jency J, Babisha Julit R L and Pious Missier S, $N\,g^{\#}-$ Irresolvable and Strongly $N\,g^{\#}-$ Irresolvable Spaces in Neutrosophic Topological Spaces……309-318
Huihui Shi, Ting Li, Fan Yang, Lei Qiao, Modeling Uncertainties Associated with Single-Valued Neutrosophic Multi-Attribute Decision-Making for Performance Evaluation of Risk Investment in Small and Medium-Sized High-Technology Venture Enterprises……319-340
Augus Kurian, Sumathi I R, and Omaima Al-Shanqiti, The Theoretical Framework on Approximation of Neutrosophic Numbers and Their Application……...341-370
Ziyan Wang and Xifeng He, Enhanced T2NN-LogTODIM-TOPSIS Framework for Assessing Vocational College Students' Employment Quality Using Type-2 Neutrosophic Numbers……371-390
Yourong Guo, Enhancing DVNN-WCSM Technique for Double-Valued Neutrosophic Multiple-Attribute Decision-Making in Digital Economy: A Case Study on Enhancing the Quality of Development of Henan's Cultural and Tourism Industry……..391-407
Yimo Zhao, Neutrosophic-Based Enhanced Framework for Multi-Attribute Decision-Making Using Single-Valued Neutrosophic Sets in Evaluating Quality of University General Education Courses...408-426

Copyright © Neutrosophic Sets and Systems, 2025

NSS
Neutrosophic Sets and Systems, Vol. 75, 2025
University of New Mexico

The 2- refined neutrosophic hyperbolic functions with its differential and integrals

Yaser Ahmad Alhasan[1,*], Abuobida Alfahal[2] and Raja Abdullah Abdulfatah[3]

[1]Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, KSA; y.alhasan@psau.edu.sa
[2]Deanship the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj Saudi Arabia; a.alfahal@psau.edu.sa
[3]Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia; ra.raj.1960@hotmail.coma

***Corresponding author:** y.alhasan@psau.edu.sa

Abstract: This article's goal is to study the 2-refined neutrosophic hyperbolic functions by defining the 2-refined neutrosophic hyperbolic functions, discussing the 2-refined neutrosophic hyperbolic identities, introducing the rules for derivatives and integrals of the 2-refined neutrosophic hyperbolic functions, and presenting the logarithmic forms of the inverse 2-refined neutrosophic hyperbolic functions.

Keywords: 2-refined neutrosophic; hyperbolic fraction; 2-refined neutrosophic derivatives and 2-refined neutrosophic integrals.

1. Introduction and Preliminaries

As an alternative to the existing logics, Smarandache proposed the neutrosophic Logic to characterize a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, and contradiction. Refined neutrosophic numbers were made available by Smarandache in the following format: $(a, b_1I_1, b_2I_2, \dots, b_nI_n)$ where $a, b_1, b_2, \dots, b_n \in R \text{ or } C$ [1].

The notion of refined neutrosophic algebraic structures was first proposed by Agboola[2]. Furthermore, paper [3] examined the refined neutrosophic rings I, assuming that they divide into two indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. It then follows logically that: [3]

$$I_1I_1 = {I_1}^2 = I_1 \quad (1)$$
$$I_2I_2 = {I_2}^2 = I_2 \quad (2)$$
$$I_1I_2 = I_2I_1 = I_1 \quad (3)$$

Furthermore, a large number of papers [4-5-6-7-8-12-13-14-15] present research on revised neutrosophic numbers. A study on the refined ah-isometry and its applications in refined neutrosophic surfaces was given by Mehmet Celik and Ahmed Hatip. Smarandache talked about

neutrosophic indefinite integral [11]. Additionally, Alhasan gave multiple calculus presentations in which he covered neutrosophic definite and indefinite integrals. Also, he demonstrated the most significant uses of definite integrals in neutrosophic logic [9–10].

This study addressed a variety of topics: the introduction and preliminary information were offered in the first part, while the 2-refined neutrosophic numbers that contain two parts of indeterminacy (I_1, I_2) were discussed in the major discussion part. Conclusion is provided in the final section.

2. Main Discussion

2.1. Neutrosophic hyperbolic functions

Let $f(x, I_1, I_2) = e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2}$, then we can express of the 2- refined neutrosophic function as the following:

$$f(x, I_1, I_2) = e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2}$$
$$= \frac{e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2} + e^{-((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}}{2}$$
$$+ \frac{e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2} - e^{-((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}}{2}$$

The odd function is called the 2- refined neutrosophic hyperbolic sine of x and the even function is called the 2- refined neutrosophic hyperbolic cosine of x.

Hence:

$$sinh\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)$$
$$= \frac{e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2} - e^{-((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}}{2}$$

$$cosh\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)$$
$$= \frac{e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2} + e^{-((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}}{2}$$

For all $x \in (-\infty, +\infty)$, Where $a_1, b_1, c_1, a_2, b_2, c_2$ are real numbers, while $I_1, I_2 =$ indeterminacy.

After that, we can write the definitions that follow:

Definition 1

The 2- refined neutrosophic function

$$cosh\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)$$
$$= \frac{e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2} - e^{-((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}}{2}$$

is called 2- refined neutrosophic hyperbolic cosine.

Definition 2

The 2- refined neutrosophic function

$$cosh\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)$$

$$= \frac{e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2} + e^{-((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}}{2}$$

is called 2- refined neutrosophic hyperbolic sine.

Definition 3
The 2- refined neutrosophic function
$tanh\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)$

$$= \frac{e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2} - e^{-((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}}{e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2} + e^{-((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}}$$

is called 2- refined neutrosophic hyperbolic tangent.

Definition 4
The 2- refined neutrosophic function
$coth\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)$

$$= \frac{e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2} - e^{-((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}}{e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2} + e^{-((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}}$$

is called 2- refined neutrosophic hyperbolic cotangent.

Definition 5
The 2- refined neutrosophic function

$$sech\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big) = \frac{1}{cosh\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)}$$

$$= \frac{2}{e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2} + e^{-((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}}$$

is called 2- refined neutrosophic hyperbolic secant.

Definition 6
The 2- refined neutrosophic function

$$csch\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big) = \frac{1}{sinh\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)}$$

$$= \frac{2}{e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2} - e^{-((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}}$$

is called 2- refined neutrosophic hyperbolic cosecant.

2.2 2- Refined neutrosophic hyperbolic identities

1) $cosh\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big) + sinh\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)$

$= e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2}$

2) $cosh\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big) - sinh\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)$

$= e^{-((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}$

3) $cosh^2\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big) - sinh^2\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big) = 1$

4) $1-tanh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)=sech^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$

5) $coth^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)-1=csch^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$

6) $sinh\Big(2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\Big)$

$$=2\,sinh\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\,cosh\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$$

7) $cosh\Big(2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\Big)$

$$=cosh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)+sinh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$$

$$or\quad =2\,cosh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)-1$$

$$or\quad =2\,sinh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)+1$$

8) $cosh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$

$$=\frac{1}{2}\left(cosh\Big(2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\Big)+1\right)$$

9) $sinh^2\Big(2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\Big)$

$$=\frac{1}{2}\left(cosh\Big(2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\Big)-1\right)$$

Note:

We can easily prove of these identities, for example:

Prove (3)

$$cosh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)-sinh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$$

$$=\left(\frac{e^{((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}+e^{-((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}}{2}\right)^2$$
$$-\left(\frac{e^{((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}-e^{-((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}}{2}\right)^2$$

$$=\frac{e^{2((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}+2+e^{-2((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}}{4}$$
$$-\left(\frac{e^{2((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}-2+e^{-2((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}}{4}\right)=\frac{4}{4}=1$$

Prove (4)

$$cosh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)-sinh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)=1$$

divide both sides by: $cosh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$, we get

$$1-\frac{sinh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)}{cosh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)}=\frac{1}{cosh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)}$$

Hence:

$$1-tanh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)=sech^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$$

2.3 Derivatives of the 2- refined neutrosophic hyperbolic functions

1) $\frac{d}{dx}\big[sinh\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\big]$
$=(a_1+b_1I_1+c_1I_2)\ cosh\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$

2) $\frac{d}{dx}\big[cosh\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\big]$
$=(a_1+b_1I_1+c_1I_2)\ sinh\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$

3) $\frac{d}{dx}\big[tanh\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\big]$
$=(a_1+b_1I_1+c_1I_2)\ sech^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$

4) $\frac{d}{dx}\big[coth\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\big]$
$=-(a_1+b_1I_1+c_1I_2)\ csch^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$

5) $\frac{d}{dx}\big[sech\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\big]$
$=-(a_1+b_1I_1$
$+c_1I_2)\ sech\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\ tanh\big((a_1+b_1I_1+c_1I_2)x+a_2$
$+b_2I_1+c_2I_2\big)$

6) $\frac{d}{dx}\big[csch\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\big]$
$=-(a_1+b_1I_1$
$+c_1I_2)\ csch\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\ coth\big((a_1+b_1I_1+c_1I_2)x+a_2$
$+b_2I_1+c_2I_2\big)$

Example 1

1) $\frac{d}{dx}\big[sinh\big((5+3I_1-I_2)x+2I_1+7I_2\big)\big]=(5+3I_1-I_2)\ cosh\big((5+3I_1-I_2)x+2I_1+7I_2\big)$

2) $\frac{d}{dx}\big[tanh\big((-7+I_1+4I_2)x+4-I_1\big)\big]=(-7+I_1+4I_2)\ sech^2\big((-7+I_1+4I_2)x+4-I_1\big)$

3) $\frac{d}{dx}\big[csch\big((4+I_1+I_2)x+2+I_1-7I_2\big)\big]$

$=(-4-I_1-I_2)\ csch\big((4+I_1+I_2)x+2+I_1-7I_2\big)\ coth\big((4+I_1+I_2)x+2+I_1-7I_2\big)$

4) $\frac{d}{dx}[cosh(1+9I_1-11I_2)x^4]=(4+36I_1-44I_2)x^3\ sinh(1+9I_1-11I_2)x^4$

5) $\frac{d}{dx}\Big[sech\sqrt{(1+41I_2)x-3+2I_1-4I_2}\Big]$

$=\frac{1+41I_2}{2\sqrt{(1+41I_2)x-3+2I_1-4I_2}}\ sech\sqrt{(1+41I_2)x-3+2I_1-4I_2}\ tanh\sqrt{(1+41I_2)x-3+2I_1-4I_2}$

6) $\frac{d}{dx}\left[ln\left(tanh\left((1+I_1+I_2)x+3+I_1-2I_2\right)\right)\right] = \frac{(1+I_1+I_2)\,sech^2\left((1+I_1+I_2)x+3+I_1-2I_2\right)}{tanh\left((1+I_1+I_2)x+3+I_1-2I_2\right)}$

2.4 Derivatives of the inverse 2- refined neutrosophic hyperbolic functions

1) $\frac{d}{dx}\left[sinh^{-1}\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)\right]$

$$= \frac{a_1+b_1I_1+c_1I_2}{\sqrt{1+\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)^2}}$$

2) $\frac{d}{dx}\left[cosh^{-1}\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)\right]$

$$= \frac{a_1+b_1I_1+c_1I_2}{\sqrt{\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)^2-1}} \; ; \; x > \frac{1-a_2-b_2I_1-c_2I_2}{a_1+b_1I_1+c_1I_2}$$

3) $\frac{d}{dx}\left[tanh^{-1}\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)\right]$

$$= \frac{a_1+b_1I_1+c_1I_2}{1-\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)^2} \; ; \; |(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2| < 1$$

4) $\frac{d}{dx}\left[coth^{-1}\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)\right]$

$$= \frac{a_1+b_1I_1+c_1I_2}{1-\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)^2} \quad ; \; |(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2| > 1$$

5) $\frac{d}{dx}\left[sech^{-1}\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)\right]$

$$= \frac{-a_1-b_1I_1-c_1I_2}{\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)\sqrt{1-\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)^2}}$$

$$; \frac{-a_2-b_2I_1-c_2I_2}{a_1+b_1I_1+c_1I_2} < x < \frac{1-a_2-b_2I_1-c_2I_2}{a_1+b_1I_1+c_1I_2}$$

6) $\frac{d}{dx}\left[csch^{-1}\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)\right]$

$$= \frac{-a_1-b_1I_1-c_1I_2}{\left|\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)\right|\sqrt{1+\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)^2}}$$

$$; x \neq \frac{-a_2-b_2I_1-c_2I_2}{a_1+b_1I_1+c_1I_2}$$

Example 2

1) $\frac{d}{dx}\left[sinh^{-1}\left((4+I_1+I_2)x+2+I_1-7I_2\right)\right]=\frac{4+I_1+I_2}{\sqrt{1+\left((4+I_1+I_2)x+2+I_1-7I_2\right)^2}}$

2) $\frac{d}{dx}\left[cosh^{-1}\left((2+2I_1+2I_2)x+3+4I_1+5I_2\right)\right]=\frac{2+2I_1+2I_2}{\sqrt{\left((2+2I_1+2I_2)x+3+4I_1+5I_2\right)^2-1}}$

3) $\frac{d}{dx}\left[tanh^{-1}\left(sin\left((7+8I_1+2I_2)x+1+I_1+I_2\right)\right)\right]$

$$=\frac{(7+8I_1+2I_2)\,cos\left((7+8I_1+2I_2)x+1+I_1+I_2\right)}{1-sin^2\left((7+8I_1+2I_2)x+1+I_1+I_2\right)}$$

$$=\frac{(7+8I_1+2I_2)\,cos\left((7+8I_1+2I_2)x+1+I_1+I_2\right)}{cos^2\left((7+8I_1+2I_2)x+1+I_1+I_2\right)}$$

$$=\frac{7+8I_1+2I_2}{cos\left((7+8I_1+2I_2)x+1+I_1+I_2\right)}$$

$$=(7+8I_1+2I_2)sec\left((7+8I_1+2I_2)x+1+I_1+I_2\right)$$

5) $\frac{d}{dx}\left[sech^{-1}\left((-1+5I_1+3I_2)x+4+I_1+3I_2\right)\right]$

$$=\frac{-1+5I_1+3I_2}{\left((-1+5I_1+3I_2)x+4+I_1+3I_2\right)\sqrt{1-\left((-1+5I_1+3I_2)x+4+I_1+3I_2\right)^2}}$$

6) $\frac{d}{dx}\left[csch^{-1}\left((10+12I_1+7I_2)x+1-3I_1+14I_2\right)\right]$

$$=\frac{10+12I_1+7I_2}{|(10+12I_1+7I_2)x+1-3I_1+14I_2|\sqrt{1+\left((10+12I_1+7I_2)x+1-3I_1+14I_2\right)^2}}$$

3. Integrals of the 2- refined neutrosophic hyperbolic functions

Let $a_1\neq 0,\ a_1\neq -c_1, a_1\neq -b_1-c_1\ and\ a_2,b_2,c_2$ are real numbers, while $I_1, I_2=$ indeterminacy, then:

1) $\int cosh\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)dx$

$$=\left(\frac{1}{a_1}+\left[\frac{-b_1}{(a_1+c_1)(a_1+b_1+c_1)}\right]I_1-\left[\frac{c_1}{a_1(a_1+c_1)}\right]I_2\right)sinh\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)+C$$

2) $\int sinh\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)dx$

$$= \left(\frac{1}{a_1} + \left[\frac{-b_1}{(a_1+c_1)(a_1+b_1+c_1)}\right] I_1 - \left[\frac{c_1}{a_1(a_1+c_1)}\right] I_2\right) cosh\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big) + C$$

3) $\int sech^2\big((a+bI)x + c + dI\big)\, dx$

$$= \left(\frac{1}{a_1} + \left[\frac{-b_1}{(a_1+c_1)(a_1+b_1+c_1)}\right] I_1 - \left[\frac{c_1}{a_1(a_1+c_1)}\right] I_2\right) tanh\big((a+bI)x + c + dI\big) + C$$

4) $\int csch^2\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)\, dx$

$$= -\left(\frac{1}{a_1} + \left[\frac{-b_1}{(a_1+c_1)(a_1+b_1+c_1)}\right] I_1 - \left[\frac{c_1}{a_1(a_1+c_1)}\right] I_2\right) coth\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big) + C$$

5) $\int sech\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)\, tanh\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)\, dx$

$$= -\left(\frac{1}{a_1} + \left[\frac{-b_1}{(a_1+c_1)(a_1+b_1+c_1)}\right] I_1 - \left[\frac{c_1}{a_1(a_1+c_1)}\right] I_2\right) sech\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big) + C$$

6) $\int csch\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)\, coth\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)\, dx$

$$= -\left(\frac{1}{a_1} + \left[\frac{-b_1}{(a_1+c_1)(a_1+b_1+c_1)}\right] I_1 - \left[\frac{c_1}{a_1(a_1+c_1)}\right] I_2\right) csch\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big) + C$$

Note 1

Whereas $C = a_0 + b_0I_1 + c_0I_2$ *and* $a_0,\ b_0,\ c_0$ are real numbers.

Example 3

1) $\int cosh^7\big((2 + I_1 + 2I_2)x + 2 - 1I_1 + 15I_2\big)\, sinh\big((2 + I_1 + 2I_2)x + 2 - 1I_1 + 15I_2\big)\, dx$

Solution:

$u = cosh\big((2 + I_1 + 2I_2)x + 2 - 1I_1 + 15I_2\big)$

$\Longrightarrow du = (2 + I_1 + 2I_2)\, sinh\big((2 + I_1 + 2I_2)x + 2 - 1I_1 + 15I_2\big)dx$

$$\frac{du}{2 + I_1 + 2I_2} = sinh\big((2 + I_1 + 2I_2)x + 2 - 1I_1 + 15I_2\big)\, dx$$

Then:

$$\int cosh^7\big((2 + I_1 + 2I_2)x + 2 - 1I_1 + 15I_2\big)\, sinh\big((2 + I_1 + 2I_2)x + 2 - 1I_1 + 15I_2\big)\, dx$$

$$= \int u^7 \frac{du}{2 + I_1 + 2I_2} = \frac{1}{2 + I_1 + 2I_2} \frac{u^8}{8} + C$$

$$= \left(\frac{1}{2} - \frac{1}{20} I_1 - \frac{1}{4} I_2\right) \frac{u^8}{8} + C = \left(\frac{1}{2} - \frac{1}{20} I_1 - \frac{1}{4} I_2\right) cosh^8\big((2 + I_1 + 2I_2)x + 2 - 1I_1 + 15I_2\big) + C$$

$$2) \int coth\big((5 + 2I_1 + I_2)x - 7 + 11I_1 + 9I_2\big)\, dx = \int \frac{cosh\big((5 + 2I_1 + I_2)x - 7 + 11I_1 + 9I_2\big)}{sinh\big((5 + 2I_1 + I_2)x - 7 + 11I_1 + 9I_2\big)} dx$$

$$u = sinh\big((5 + 2I_1 + I_2)x - 7 + 11I_1 + 9I_2\big)$$

$$\Longrightarrow \quad du = (5 + 2I_1 + I_2) cosh\big((5 + 2I_1 + I_2)x - 7 + 11I_1 + 9I_2\big)\, dx$$

$$\frac{du}{5 + 2I_1 + I_2} = cosh\big((5 + 2I_1 + I_2)x - 7 + 11I_1 + 9I_2\big)\, dx$$

Then:

$$\int \frac{cosh\big((5 + 2I_1 + I_2)x - 7 + 11I_1 + 9I_2\big)}{sinh\big((5 + 2I_1 + I_2)x - 7 + 11I_1 + 9I_2\big)} dx = \int \frac{1}{u} \frac{du}{5 + 2I_1 + I_2} = \frac{1}{5 + 2I_1 + I_2} \int \frac{1}{u}\, du$$

$$= \left(\frac{1}{5} - \frac{1}{24} I_1 - \frac{1}{30} I_2\right) ln|u| + C$$

$$= \left(\frac{1}{5} - \frac{1}{24} I_1 - \frac{1}{30} I_2\right) ln\big|sinh\big((5 + 2I_1 + I_2)x - 7 + 11I_1 + 9I_2\big)\big| + C$$

$$3) \int sech\big((4 + 5I_1 + 6I_2)x + 2 + I_1 + 8I_2\big)\, tanh\big((4 + 5I_1 + 6I_2)x + 2 + I_1 + 8I_2\big)\, dx$$

$$= \left(\frac{-1}{4} + \frac{1}{30} I_1 + \frac{3}{20} I_2\right) sech\big((4 + 5I_1 + 6I_2)x + 2 + I_1 + 8I_2\big) + C$$

3.1 Logarithmic forms of inverse 2- refined neutrosophic hyperbolic functions

$$1)\ sinh^{-1}\big((a_1 + b_1 I_1 + c_1 I_2)x + a_2 + b_2 I_1 + c_2 I_2\big)$$

$$= ln\left((a_1 + b_1 I_1 + c_1 I_2)x + a_2 + b_2 I_1 + c_2 I_2 + \sqrt{\big((a_1 + b_1 I_1 + c_1 I_2)x + a_2 + b_2 I_1 + c_2 I_2\big)^2 + 1}\right)$$

$$2)\ cosh^{-1}\big((a_1 + b_1 I_1 + c_1 I_2)x + a_2 + b_2 I_1 + c_2 I_2\big)$$

$$= ln\left((a_1 + b_1 I_1 + c_1 I_2)x + a_2 + b_2 I_1 + c_2 I_2 + \sqrt{\big((a_1 + b_1 I_1 + c_1 I_2)x + a_2 + b_2 I_1 + c_2 I_2\big)^2 - 1}\right)$$

$$3)\ tanh^{-1}\big((a_1 + b_1 I_1 + c_1 I_2)x + a_2 + b_2 I_1 + c_2 I_2\big)$$

$$= \frac{1}{2}\ ln\left(\frac{1 + (a_1 + b_1 I_1 + c_1 I_2)x + a_2 + b_2 I_1 + c_2 I_2}{1 - (a_1 + b_1 I_1 + c_1 I_2)x + a_2 + b_2 I_1 + c_2 I_2}\right)$$

$$4)\ coth^{-1}\big((a_1 + b_1 I_1 + c_1 I_2)x + a_2 + b_2 I_1 + c_2 I_2\big)$$

$$= \frac{1}{2}\ ln\left(\frac{(a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2 + 1}{(a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2 - 1}\right)$$

5) $sech^{-1}\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)$

$$= ln\left(\frac{1 + \sqrt{1 - \big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)^2}}{(a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2}\right)$$

6) $csch^{-1}\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)$

$$= ln\left(\frac{1}{(a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2} + \frac{\sqrt{1 + \big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)^2}}{|(a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2|}\right)$$

Proof (1)

Let: $y = sinh^{-1}\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)$, then:

$$(a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2 = sinhy$$

But:

$$sinhy = \frac{e^y - e^{-y}}{2}$$

$$(a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2 = \frac{e^y - e^{-y}}{2}$$

$$e^y - e^{-y} = 2\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)$$

$$e^y - e^{-y} - 2\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big) = 0$$

Multiplying by e^y, we get:

$$e^{2y} - 2\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)e^y - 1 = 0$$

solving by the quadratic formula, we find:

$$\Delta = 4\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)^2 + 4$$

$$= 4\left[\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)^2 + 1\right]$$

$$e^y = \frac{2\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big) \pm \sqrt{4\left[\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)^2 + 1\right]}}{2}$$

$$e^y = (a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2 \pm 2\sqrt{\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)^2 + 1}$$

Since $e^y > 0$, then:

$$e^y = (a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2 + 2\sqrt{\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)^2 + 1}$$

$$y = ln\left((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2 + 2\sqrt{\left((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\right)^2 + 1}\right)$$

Hence:

$$sinh^{-1}\left((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\right)$$

$$= ln\left((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2 + 2\sqrt{\left((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\right)^2 + 1}\right)$$

Note 2
Similarly, we can prove the rest of the formula.

3.2 Integrals of the inverse 2- refined neutrosophic hyperbolic functions

1) $\int \frac{1}{\sqrt{(a_1 + b_1I_1 + c_1I_2)^2 + x^2}}dx$

$$= sinh^{-1}\left(\frac{x}{a_1 + b_1I_1 + c_1I_2}\right) + C \quad , or: ln\left(x + \sqrt{(a_1 + b_1I_1 + c_1I_2)^2 + x^2}\right) + C$$

2) $\int \frac{1}{\sqrt{x^2-(a_1 + b_1I_1 + c_1I_2)^2}}dx$

$$= cosh^{-1}\left(\frac{x}{a_1 + b_1I_1 + c_1I_2}\right) + C \quad , or: ln\left(x + \sqrt{x^2-(a_1 + b_1I_1 + c_1I_2)^2}\right) + C \; ; x > a_1 + b_1I_1 + c_1I_2$$

3) $\int \frac{1}{(a_1 + b_1I_1 + c_1I_2)^2 - x^2}dx$

$$= \begin{cases} \left(\frac{1}{a_1} + \left[\frac{-b_1}{(a_1 + c_1)(a_1 + b_1 + c_1)}\right]I_1 - \left[\frac{c_1}{a_1(a_1 + c_1)}\right]I_2\right) tanh^{-1}\left(\frac{x}{a_1 + b_1I_1 + c_1I_2}\right) + C \; ; |x| < a_1 + b_1I_1 + c_1I_2 \\ \left(\frac{1}{a_1} + \left[\frac{-b_1}{(a_1 + c_1)(a_1 + b_1 + c_1)}\right]I_1 - \left[\frac{c_1}{a_1(a_1 + c_1)}\right]I_2\right) coth^{-1}\left(\frac{x}{a_1 + b_1I_1 + c_1I_2}\right) + C \; ; |x| > a_1 + b_1I_1 + c_1I_2 \end{cases}$$

$$or: \quad \frac{1}{2(a_1 + b_1I_1 + c_1I_2)} ln\left|\frac{a_1 + b_1I_1 + c_1I_2 + x}{a_1 + b_1I_1 + c_1I_2 - x}\right| + C \; ; |x| \neq a_1 + b_1I_1 + c_1I_2$$

4) $$\int \frac{1}{x\sqrt{(a_1 + b_1I_1 + c_1I_2)^2 - x^2}}dx = \frac{-1}{a_1 + b_1I_1 + c_1I_2} sech^{-1}\left|\frac{x}{a_1 + b_1I_1 + c_1I_2}\right| + C$$

$$or: \frac{-1}{a_1 + b_1I_1 + c_1I_2} ln\left(\frac{a_1 + b_1I_1 + c_1I_2 + \sqrt{(a_1 + b_1I_1 + c_1I_2)^2 - x^2}}{|x|}\right) + C$$

$$; 0 < |x| < a_1 + b_1I_1 + c_1I_2$$

5) $$\int \frac{1}{x\sqrt{(a_1 + b_1I_1 + c_1I_2)^2 + x^2}}dx = \frac{-1}{a_1 + b_1I_1 + c_1I_2} csch^{-1}\left|\frac{x}{a_1 + b_1I_1 + c_1I_2}\right| + C$$

$$or: \quad \frac{-1}{a_1+b_1I_1+c_1I_2} ln\left(\frac{a_1+b_1I_1+c_1I_2+\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}}{|x|}\right)+C \ ; x\neq 0$$

Note 3

$$\frac{1}{a_1+b_1I_1+c_1I_2}=\frac{1}{a_1}+\left[\frac{-b_1}{(a_1+c_1)(a_1+b_1+c_1)}\right]I_1-\left[\frac{c_1}{a_1(a_1+c_1)}\right]I_2$$

for all the previous rules.

Proof (1)

$$\int \frac{1}{\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}}dx = sinh^{-1}\left(\frac{x}{a_1+b_1I_1+c_1I_2}\right)+C$$

$$or: \quad = ln\left(x+\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}\right)+C$$

Method 1

$$\frac{d}{dx}\left[sinh^{-1}\left(\frac{x}{a_1+b_1I_1+c_1I_2}\right)+C\right]=\frac{\frac{1}{a_1+b_1I_1+c_1I_2}}{\sqrt{1+\left(\frac{x}{a_1+b_1I_1+c_1I_2}\right)^2}}=\frac{\frac{1}{a_1+b_1I_1+c_1I_2}}{\sqrt{1+\frac{x^2}{(a_1+b_1I_1+c_1I_2)^2}}}$$

$$=\frac{\frac{1}{a_1+b_1I_1+c_1I_2}}{\sqrt{\frac{(a_1+b_1I_1+c_1I_2)^2+x^2}{(a_1+b_1I_1+c_1I_2)^2}}}=\frac{\frac{1}{a_1+b_1I_1+c_1I_2}}{\frac{1}{a_1+b_1I_1+c_1I_2}\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}}$$

$$=\frac{1}{\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}}$$

Method 2

$$\frac{d}{dx}\left[ln\left(x+\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}\right)+C\right]=\frac{1+\frac{2x}{2\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}}}{x+\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}}$$

$$=\frac{\frac{2\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}+2x}{2\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}}}{x+\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}}=\frac{\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}+x}{\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}\left(x+\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}\right)}$$

$$=\frac{1}{\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}}$$

Method 3

Let:

$$y=sinh^{-1}\left(\frac{x}{a_1+b_1I_1+c_1I_2}\right) \quad \Longrightarrow \quad \frac{x}{a_1+b_1I_1+c_1I_2}=sinhy$$

by differentiate implicitly with respect to x, we get:

$$\frac{1}{a_1+b_1I_1+c_1I_2}=coshy\ \frac{dy}{dx}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{1}{(a_1 + b_1 I_1 + c_1 I_2) coshy} \qquad (*)$$

Since: $cosh^2 y - sinh^2 = 1 \quad \Rightarrow \quad cosh^2 y = 1 + sinh^2 \quad \Rightarrow \quad coshy = \sqrt{1 + sinh^2}$

By substitution in $(*)$, we get:

$$\frac{dy}{dx} = \frac{1}{(a_1 + b_1 I_1 + c_1 I_2)\sqrt{1 + sinh^2}}$$

But: $\frac{x}{a_1 + b_1 I_1 + c_1 I_2} = sinhy$, then:

$$\frac{dy}{dx} = \frac{1}{(a_1 + b_1 I_1 + c_1 I_2)\sqrt{1 + \left(\frac{x}{a_1 + b_1 I_1 + c_1 I_2}\right)^2}} = \frac{1}{(a_1 + b_1 I_1 + c_1 I_2)\sqrt{\frac{(a_1 + b_1 I_1 + c_1 I_2)^2 + x^2}{(a_1 + b_1 I_1 + c_1 I_2)^2}}}$$

$$= \frac{1}{(a_1 + b_1 I_1 + c_1 I_2)\frac{1}{(a_1 + b_1 I_1 + c_1 I_2)}\sqrt{(a_1 + b_1 I_1 + c_1 I_2)^2 + x^2}}$$

$$= \frac{1}{\sqrt{(a_1 + b_1 I_1 + c_1 I_2)^2 + x^2}}$$

Example 4

1) $\int \frac{1}{\sqrt{(2 + 3I_1 + 5I_2)^2 + x^2}} dx = sinh^{-1}\left(\frac{x}{2 + 3I_1 + 5I_2}\right) + C$

$$= sinh^{-1}\left(\frac{1}{2} - \frac{3}{70} I_1 - \frac{5}{14} I_2\right) x + C \ , or: \quad ln\left(x + \sqrt{(2 + 3I_1 + 5I_2)^2 + x^2}\right) + C$$

2) $\int \frac{1}{\sqrt{x^2 - (3 + I_1 + I_2)^2}} dx = \int \frac{1}{\sqrt{x^2 - (3 + I_1 + I_2)^2}} dx$

$$= cosh^{-1}\left(\frac{x}{3 + I_1 + I_2}\right) + C$$

$$= cosh^{-1}\left(\frac{1}{3} - \frac{1}{20} I_1 - \frac{1}{12} I_2\right) x + C \quad , or: \ ln\left(x + \sqrt{x^2 - (3 + I_1 + I_2)^2}\right) + C$$

4. Conclusions

The studies we presented in the area of neutrosophic derivatives and integrals are expanded upon in this work. Since they make many mathematical processes in our everyday lives easier, derivatives and integrals are crucial concepts in life. This is what led us to introducing the concept the 2- refined neutrosophic hyperbolic functions with its differential and integrals. In addition to proving most of the rules, we presented in this paper in several ways.

Acknowledgments "This study is supported via funding from Prince sattam bin Abdulaziz University project number (PSAU/2024/R/1445)".

References

[1] Smarandache, F; "(T,I,F)- Neutrosophic Structures", Neutrosophic Sets and Systems, Volume8, PP:3-10, 2015.

[2] Agboola,A.A.A; On Refined Neutrosophic Algebraic Structures. Neutrosophic Sets and Systems, Volume10, PP: 99-101, 2015.

[3] Adeleke, E.O., Agboola, A.A.A.,and Smarandache, F; Refined Neutrosophic Rings I. International Journal of Neutrosophic Science, Volume 2, Issue 2, PP:77-81, 2020.

[4] Smarandache, F; n-Valued Refined Neutrosophic Logic and Its Applications in Physics. Progress in Physics, USA, Volume 4, PP: 143-146, 2013.

[5] Vasantha Kandasamy,W.B; Smarandache,F; Neutrosophic Rings" Hexis, Phoenix, Arizona, http://fs.gallup.unm.edu/NeutrosophicRings.pdf, 2006.

[6] Zeina, M., Abobala, M; On The Refined Neutrosophic Real Analysis Based on Refined Neutrosophic Algebraic AH-Isometry", Neutrosophic Sets and Systems, Volume 54, PP: 158-168, 2023.

[7] Agboola, A.A.A.; Akinola, A.D; Oyebola, O.Y; Neutrosophic Rings I. Int. J. of Math. Comb, Volume 4, PP:1-14, 2011.

[8] Celik, M., and Hatip, A; On the Refined AH-Isometry and Its Applications in Refined Neutrosophic Surfaces. Galoitica Journal of Mathematical Structures and Applications, 2022.

[9] Alhasan, Y; The neutrosophic integrals and integration methods. Neutrosophic Sets and Systems, Volume 43, PP: 290-301, 2021.

[10] Alhasan, Y; The definite neutrosophic integrals and its applications. Neutrosophic Sets and Systems, Volume 49, PP: 277-293, 2022.

[11] Smarandache, F; Neutrosophic Precalculus and Neutrosophic Calculus. Book, 2015.

[12] Celik, M., and Hatip, A; On the Refined AH-Isometry and Its Applications in Refined Neutrosophic Surfaces. International Journal of Neutrosophic Science, Volume 2, Issue 1, PP: 21-28, 2022.

[13] Alhasan, Y; Ahmed, M; Abdulfatah, R; Sheen, S; The refined indefinite neutrosophic integral, Neutrosophic Sets and Systems, Vol. 67, 127- 134, 2024.

[14] Alhasan, Y; Mohamed, M; Abdulfatah, R; The limits of 2- refined neutrosophic, Neutrosophic Sets and Systems, Vol. 68, 39-49, 2024.

[15] Smarandache, F; Abobala, M; Operations with n-Refined Literal Neutrosophic Numbers using the Identification Method and the n-Refined AH-Isometry, Neutrosophic Sets and Systems, Vol. 70, 350-358, 2024.

Received: June 25, 2024. Accepted: Oct 10, 2024

Neutrosophic Sets and Systems, Vol. 75, 2025
University of New Mexico

Triangular fuzzy quadripartitioned neutrosophic set and its properties

Tanmoy Chatterjee[1], Surapati Pramanik[2*]

[1]Department of Mathematics, Swami Vivekananda University, Barrackpore, India, KOL- 700121

[1]email: ctanmoy1977@gmail.com

[2]Nandalal Ghosh B.T. College, Panpur, Narayanpur, Dist.-24 Pgs(N), West Bengal, India, PIN-743126

[2*]email: sura_pati@yahoo.co.in

Correspondence: sura_pati@yahoo.co.in, Tel:+919477035544

Abstract: The main objective of the paper is to hybridize the triangular fuzzy number and the quadripartitioned neutrosophic set and develop the triangular fuzzy quadripartitioned neutrosophic set. The triangular fuzzy numbers have great potential to express uncertainty systematically. So, the combination of the triangular fuzzy numbers and quadripartitioned neutrosophic sets is an intelligent mathematical tool that will be a helpful mathematical tool for decision-making. We define some operations on the triangular fuzzy quadripartitioned neutrosophic sets such as union, intersection, and complement. We establish some important theorems on operations like union, intersection in between three triangular fuzzy quadripartitioned neutrosophic sets in an elaborate way. We have defined unity and null triangular fuzzy quadripartitioned neutrosophic sets and established some mathematical operations on them. We establish some fundamental properties of the developed triangular fuzzy quadripartitioned neutrosophic sets.

Keywords: Fuzzy set; neutrosophic set; single-valued neutrosophic set; quadripartitioned neutrosophic set; triangular fuzzy neutrosophic set

1. Introduction

Neutrosophic Set (NS) was first introduced by Smarandache [1] by investigating the properties of Fuzzy Set (FS) [2] and Intuitionistic FS (IFS) [3]) by introducing indeterminacy and falsity as independent membership components. Single- Valued NS (SVNS) was introduced by Wang et al. [4] in 2010 by confining the "truth", "indeterminacy'' and "falsity'' membership degrees. Several detailed analyses of theories and different applications of NSs and their necessary important

extensions were studied by several authors [5-12]. Chatterjee et al. [13] defined Quadripartitioned SVNS (QSVNS) with the introduction of "truth", "falsity", "unknown "and "contradiction" as four independent component membership functions by utilizing four-valued logic [14] and multi-valued neutrosophic refined logic [15]. Interval Quadripartitioned NS (IQNS) was developed by Pramanik [16] by exploring the Interval NS (INS) [17] and QSVNS. Mallick and Pramanik [18] developed the theory of Pentapartitioned NS (PNS) by utilizing multi-valued refined neutrosophic logic [15] where the indeterminacy membership component is split into three independent components, namely "contradiction", "ignorance", and "unknown". Pramanik [19] defined and established the basic properties of Interval PNSs (IPNSs). Triangular Fuzzy Number (TFN) [20] is an important mathematical tool that is very useful for decision-making. Biswas et al. [21] combined the TFN and SVNS and developed the Triangular Fuzzy Neutrosophic Set (TFNS). Sinha & Majumdar (22) investigated the problems of Rural healthcare challenges through quadripartitioned single valued neutrosophic Z-numbers. TFN is not explored in the QSVNS environment. Complications appearing in decision making problems mainly because of uncertain nature of decision makers or attributes can be quiet significantly handled by exploring TFN in QSVNS environment and this approach has not been adapted in the previous studies.

Research gap: No study combining the TFN and QSVNS has been reported in the literature.

Motivation: The research gap motivates us to study by combining the concepts of TFN and QSVNS and develop the theory of Triangular Fuzzy Quadripartitioned Neutrosophic Set (TFQNS).

The TFQNS is a new notion in the field of NS. Since the TFQNS is a hybrid structure, it is capable of expressing uncertainty comprehensively. TFQNS has more advantages for dealing with uncertainty as it can utilize the advantages of TFN and QSVNS. The computational techniques based on TFN or QSVNS alone may not always produce the best results but the hybrid structure TFQNS may yield the best result.

We also investigate some fundamental properties of the newly introduced set.

The paper has four sections given as follows: Section 2 is dedicated to presenting some existing preliminary concepts of NSs. Section 3 represents the concept of TFQNS and some important mathematical operations on TFQNS. Section 4 presents a discussion. Section 5 concludes the chapter by indicating some future scope of research.

2. Preliminaries

Definition 2.1. Let a set Ω be fixed. An NS [1] X over Ω is defined as:

$X = \{\omega, (T_X(\omega), I_X(\omega), F_X(\omega)) : \omega \in \Omega\}$ where $T_X(\omega), I_X(\omega), F_X(\omega) : \Omega \to]^{-}0,1^{+}[$,

$^{-}0 \le T_X(\omega) + I_X(\omega) + F_X(\omega) \le 3^{+}$. T_X, I_X, F_X represents the truth Membership Function (MF), the contradiction MF, the unknown MF, and the falsity MF respectively.

Definition 2.2. Let a set Ω be fixed. An SVNS X over Ω is defined as:

$X = \{\omega, (T_X(\omega), I_X(\omega), F_X(\omega)) : \omega \in \Omega\}$ where $T_X, I_X, F_X : W \to [0,1]$ and $0 \le T_X(\omega) + I_X(\omega) + F_X(\omega) \le 3$.

T_X, I_X, F_X represents the truth MF, the contradiction MF, the unknown MF, and the falsity MF respectively.

Definition 2.3. Assume that Ω is the finite universe of discourse and $\phi[0,1]$ is the set of all TFNs on [0, 1] . A TFNS [21] X in Ω is represented as: $X = \{\omega, (T_X(\omega), I_X(\omega), F_X(\omega)) : \omega \in \Omega\}$
where $T_X(\omega) : \Omega \to \phi[0,1], I_X(\omega) : \Omega \to \phi[0,1], F_X(\omega) : \Omega \to \phi[0,1]$
The TFN $T_X(\omega) = (T_X^1(\omega), T_X^2(\omega), T_X^3(\omega)), I_X(\omega) = (I_X^1(\omega), I_X^2(\omega), I_X^3(\omega)), F_X(\omega) = (F_X^1(\omega), F_X^2(\omega), F_X^3(\omega))$ respectively presents the truth MF, the contradiction MF, the unknown MF, and the falsity MF and for $\forall w \in \Omega$, $0 \le T_X^3(\omega) + I_X^3(\omega) + F_X^3(\omega) \le 3$

Definition 2.4. A QSVNS [13] X in the universe of discourse Ω (a fixed set) is expressed as:
$X = \{\omega, (T_X(\omega), C_X(\omega), U_X(\omega), F_X(\omega)) : \omega \in \Omega\}$
where $T_X(\omega), C_X(\omega), U_X(\omega), F_X(\omega) : \Omega \to [0,1]$
$T_X(\omega), C_X(\omega), U_X(\omega), F_X(\omega)$ expresses the truth MF, the contradiction MF, the unknown MF, and the falsity MF respectively and with, $0 \le T_X(\omega) + C_X(\omega) + U_X(\omega) + F_X(\omega) \le 4$.

3. The Fundamental Theories of TFQNSs

Definition 3.1. TFQNS
Assume that $\bar{\bar{\chi}}$ is the finite universe of discourse and $\phi[0,1]$ is the set of all TFNs on [0, 1] .

We define a TFQNS $fy\hat{\rho}$ over $\bar{\bar{\chi}}$ and $fy\hat{\rho}$ is presented as:

$$fy\hat{\rho} = \{\omega, (fy\bar{\bar{\xi}}^{T}_{fy\hat{\rho}}(\omega), fy\bar{\bar{\mu}}^{C}_{fy\hat{\rho}}(\omega), fy\bar{\bar{\delta}}^{U}_{fy\hat{\rho}}(\omega), fy\bar{\bar{\psi}}^{F}_{fy\hat{\rho}}(\omega)) : \omega \in \bar{\bar{\chi}}\} \quad \ldots(1)$$

where $fy\bar{\bar{\xi}}^{T}_{fy\hat{\rho}}(\omega) : \bar{\bar{\chi}} \to \phi[0,1], fy\bar{\bar{\mu}}^{C}_{fy\hat{\rho}}(\omega) : \bar{\bar{\chi}} \to \phi[0,1], fy\bar{\bar{\delta}}^{U}_{fy\hat{\rho}}(\omega) : \bar{\bar{\chi}} \to \phi[0,1], fy\bar{\bar{\psi}}^{F}_{fy\hat{\rho}}(\omega) : \bar{\bar{\chi}} \to \phi[0,1],$

$$fy\hat{\rho} = \{(\omega, (fy\bar{\bar{\xi}}^{T(l)}_{fy\hat{\rho}}(\omega), fy\bar{\bar{\xi}}^{T(m)}_{fy\hat{\rho}}(\omega), fy\bar{\bar{\xi}}^{T(u)}_{fy\hat{\rho}}(\omega)), (fy\bar{\bar{\mu}}^{C(l)}_{fy\hat{\rho}}(\omega), fy\bar{\bar{\mu}}^{C(m)}_{fy\hat{\rho}}(\omega), fy\bar{\bar{\mu}}^{C(u)}_{fy\hat{\rho}}(\omega)),$$
$$(fy\bar{\bar{\delta}}^{U(l)}_{fy\hat{\rho}}(\omega), fy\bar{\bar{\delta}}^{U(m)}_{fy\hat{\rho}}(\omega), fy\bar{\bar{\delta}}^{U(u)}_{fy\hat{\rho}}(\omega)), (fy\bar{\bar{\psi}}^{F(l)}_{fy\hat{\rho}}(\omega), fy\bar{\bar{\psi}}^{F(m)}_{fy\hat{\rho}}(\omega), fy\bar{\bar{\psi}}^{F(u)}_{fy\hat{\rho}}(\omega)) : \omega \in \bar{\bar{\chi}}\} \quad \ldots(2)$$
$$\text{with } 0 \le fy\bar{\bar{\xi}}^{T(l)}_{fy\hat{\rho}}(\omega) \le fy\bar{\bar{\xi}}^{T(m)}_{fy\hat{\rho}}(\omega) \le fy\bar{\bar{\xi}}^{T(u)}_{fy\hat{\rho}}(\omega)) \le 1$$

where the TFN,

$$fy\bar{\bar{\xi}}^{T}_{fy\hat{\rho}}(\omega) = (fy\bar{\bar{\xi}}^{T(l)}_{fy\hat{\rho}}(\omega), fy\bar{\bar{\xi}}^{T(m)}_{fy\hat{\rho}}(\omega), fy\bar{\bar{\xi}}^{T(u)}_{fy\hat{\rho}}(\omega)), fy\bar{\bar{\mu}}^{C}_{fy\hat{\rho}}(\omega) = (fy\bar{\bar{\mu}}^{C(l)}_{fy\hat{\rho}}(\omega), fy\bar{\bar{\mu}}^{C(m)}_{fy\hat{\rho}}(\omega), fy\bar{\bar{\mu}}^{C(u)}_{fy\hat{\rho}}(\omega)),$$

$fy\bar{\bar{\delta}}^{U}_{fy\hat{\rho}} = (fy\bar{\bar{\delta}}^{U(l)}_{fy\hat{\rho}}(\omega), fy\bar{\bar{\delta}}^{U(m)}_{fy\hat{\rho}}(\omega), fy\bar{\bar{\delta}}^{U(u)}_{fy\hat{\rho}}(\omega)), fy\bar{\bar{\psi}}^{F}_{fy\hat{\rho}} = (fy\bar{\bar{\psi}}^{F(l)}_{fy\hat{\rho}}(\omega), fy\bar{\bar{\psi}}^{F(m)}_{fy\hat{\rho}}(\omega), fy\bar{\bar{\psi}}^{F(u)}_{fy\hat{\rho}}(\omega))$ presents

respectively the truth MF, the contradiction MF, the unknown MF, and the falsity MF and

$$\forall \hat{\theta} \in \bar{\bar{\chi}}, 0 \le fy\bar{\bar{\xi}}^{T(u)}_{fy\hat{\rho}}(\omega) + fy\bar{\bar{\mu}}^{C(u)}_{fy\hat{\rho}}(\omega) + fy\bar{\bar{\delta}}^{U(u)}_{fy\hat{\rho}}(\omega) + fy\bar{\bar{\psi}}^{F(u)}_{fy\hat{\rho}}(\omega) \le 4.$$

In general, $\forall \omega \in \bar{\bar{\chi}}, 0 \le fy\bar{\bar{\xi}}^{T(\zeta)}_{fy\hat{\rho}}(\omega) + fy\bar{\bar{\mu}}^{C(\zeta)}_{fy\hat{\rho}}(\omega) + fy\bar{\bar{\delta}}^{U(\zeta)}_{fy\hat{\rho}}(\omega) + fy\bar{\bar{\psi}}^{F(\zeta)}_{fy\hat{\rho}}(\omega) \le 4$, for, $\forall \zeta = u, l, m.$

Example 3.1 Consider a real-world problem that is associated with the completion time of a project in the neutrosophic environment.
Assume that there are three executive engineers of a construction company who are present in a meeting called by the managing director of the company to discuss a time frame that should be required for the

completion of a new project. The managing director has raised a question before the three engineers, what should be the time frame for an important construction project, that the company will undertake? The objective of the company is to provide the minimum time of completion for the project with assured quality of work. The company aims to complete the project within the shortest possible time. Three engineers constitute the universe of discourse.

Now, consider about first engineer's ($\overline{\delta}_1$) assessment regarding the expected time of project completion. According to him, the time frame of completion as expected by the company is correct. He is optimistic about the time frame where everything goes smoothly. This constitutes the truth membership function. He is quite confident in completing the project within the desired time and as per him, the time of completion is 6-8 years. On a 0-1 scale, the truth membership degree can be presented as (0.6, 0.6, 0.8) which is a TFN rating. He is quite confident that the project will be completed mostly within 6 years. But at the same time, he has some contradictions about whether the project can be completed in between 4-6 years taking into account potential delays and uncertainties that may come into play. On a 0-1 scale, the contradiction membership function may be expressed as a TFN rating. He is completely unaware that the project can be completed within 2 years. This constitutes the unknown membership function. On a 0-1 scale, the unknown membership degree may be expressed as (0.2, 0.2, 0.2) which is a TFN rating. He never relies upon the fact that the project can never be completed within 3 years. This constitutes the falsity membership function. On a 0-1 scale, the falsity membership function may be expressed as (0.3, 0.3, 0.3) which is a TFN rating. So, his overall rating is expressed as a Triangular Fuzzy Quadripartitioned Neutrosophic Number (TFQNN) as: $\langle(0.6,0.6,0.8),(0.4,0.5,0.6),(0.2,0.2,0.2),(0.3,0.3,0.3)\rangle_{\overline{\delta}_1}$.

Similarly, the second engineer's ($\overline{\delta}_2$) assessment regarding the time frame of completion is presented by TFQNN as $\langle(0.8,0.7,0.6),(0.5,0.6,0.7),(0.3,0.3,0.3),(0.2,0.2,0.2)\rangle_{\overline{\delta}_2}$. The third engineer ($\overline{\delta}_3$)presents his assessment rating regarding the probable time of completion of the project by a TFQNN rating represented as $\langle(0.7,0.8,0.9),(0.6,0.7,0.8),(0.4,0.5,0.6),(0.3,0.4,0.5)\rangle_{\overline{\delta}_3}$. All these three TFQNN ratings are elements of TFQNS represented as:

$\overline{\Theta}_1 = \langle(0.6,0.6,0.8),(0.4,0.5.0.6),(0.2,0.2,0.2),(0.3,0.3,0.3)\rangle_{\overline{\delta}_1} + \langle(0.8,0.7,0.6),(0.5,0.6,0.7),$
$(0.3,0.3,0.3),(0.2,0.2,0.2)\rangle_{\overline{\delta}_2} + \langle(0.7,0.8,0.9),(0.6,0.7,0.8),(0.4,0.5,0.6),(0.3,0.4,0.5)\rangle_{\overline{\delta}_3}$.

Definition 3.2. We introduce the notion $\hat{0}$ and $\hat{1}$as follows:

$\hat{0} = \{(0,0,0),(0,0,0),(1,1,1),(1,1,1)\}$ and $\hat{1} = \{(1,1,1),(1,1,1),(0,0,0),(0,0,0)\}$

Definition 3.3. Union of any two TFQNSs $\text{fy}\hat{\ddot{\rho}}_1, \text{fy}\hat{\ddot{\rho}}_2$ is a TFQNS $\text{fy}\hat{\ddot{\rho}}_3$, written as $\text{fy}\hat{\ddot{\rho}}_3 = \text{fy}\hat{\ddot{\rho}}_1 \cup \text{fy}\hat{\ddot{\rho}}_2$,where the ruth MF , the contradiction MF , the unknown MF of unknown, the falsity MF are presented as:

$$\text{fy}\bar{\bar{\xi}}_{\text{fn}\bar{\bar{\rho}}_3}(\omega) = (\max(\text{fy}\bar{\bar{\xi}}^{\text{T(l)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\xi}}^{\text{T(l)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)), \max(\text{fy}\bar{\bar{\xi}}^{\text{T(m)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\xi}}^{\text{T(m)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)), \max(\text{fy}\bar{\bar{\xi}}^{\text{T(m)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fn}\bar{\bar{\xi}}^{\text{T(m)}}_{\text{fn}\bar{\bar{\rho}}_2}(\omega)),$$
$$\text{fy}\bar{\bar{\mu}}_{\text{fy}\bar{\bar{\rho}}_3}(\omega) = (\max(\text{fy}\bar{\bar{\mu}}^{\text{C(l)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\mu}}^{\text{C(l)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)), \max(\text{fy}\bar{\bar{\mu}}^{\text{C(m)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\mu}}^{\text{C(m)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)), \max(\text{fy}\bar{\bar{\mu}}^{\text{C(n)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\mu}}^{\text{C(n)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)),$$
$$\text{fy}\bar{\bar{\delta}}_{\text{fy}\bar{\bar{\rho}}_3}(\omega) = \min(\text{fy}\bar{\bar{\delta}}^{\text{U(l)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\delta}}^{\text{U(l)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)), \min(\text{fy}\bar{\bar{\delta}}^{\text{U(m)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\delta}}^{\text{U(m)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)), \min(\text{fy}\bar{\bar{\delta}}^{\text{U(m)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\delta}}^{\text{U(m)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)),$$
$$\text{fy}\bar{\bar{\psi}}_{\text{fy}\bar{\bar{\rho}}_3}(\omega) = \min(\text{fy}\bar{\bar{\psi}}^{\text{F(l)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fn}\bar{\bar{\psi}}^{\text{F(l)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)), \min(\text{fy}\bar{\bar{\psi}}^{\text{F(m)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\psi}}^{\text{F(m)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)), \min(\text{fy}\bar{\bar{\psi}}^{\text{F(m)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\psi}}^{\text{F(m)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)), \forall\omega \in \bar{\bar{\chi}}.$$

$$\text{fy}\hat{\bar{\rho}}_3 = \text{fy}\hat{\bar{\rho}}_1 \cup \text{fy}\hat{\bar{\rho}}_2$$

$$= \{\omega, (\text{fy}\bar{\bar{\xi}}_{\text{fy}\bar{\bar{\rho}}_3}(\omega), \text{fy}\bar{\bar{\mu}}_{\text{fy}\bar{\bar{\rho}}_3}(\omega), \text{fy}\bar{\bar{\delta}}_{\text{fy}\bar{\bar{\rho}}_3}(\omega), \text{fy}\bar{\bar{\psi}}_{\text{fy}\bar{\bar{\rho}}_3}(\omega)) : \omega \in \bar{\bar{\chi}}\}$$

$$= \{\omega, [(\max(\text{fy}\bar{\bar{\xi}}^{\text{T(l)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\xi}}^{\text{T(l)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)), \max(\text{fy}\bar{\bar{\xi}}^{\text{T(m)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\xi}}^{\text{T(m)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)), \max(\text{fy}\bar{\bar{\xi}}^{\text{T(u)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\xi}}^{\text{T(u)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega))],$$
$$[\max(\text{fy}\bar{\bar{\mu}}^{\text{C(l)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\mu}}^{\text{C(l)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)), \max(\text{fy}\bar{\bar{\mu}}^{\text{C(m)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\mu}}^{\text{C(m)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)), \max(\text{fy}\bar{\bar{\mu}}^{\text{C(u)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\mu}}^{\text{C(u)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega))],$$
$$[\min(\text{fy}\bar{\bar{\delta}}^{\text{U(l)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\delta}}^{\text{U(l)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)), \min(\text{fy}\bar{\bar{\delta}}^{\text{U(m)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\delta}}^{\text{U(m)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)), \min(\text{fy}\bar{\bar{\delta}}^{\text{U(u)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\delta}}^{\text{U(u)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega))], \quad \ldots(3)$$
$$[\min(\text{fy}\bar{\bar{\psi}}^{\text{F(l)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\psi}}^{\text{F(l)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)), \min(\text{fy}\bar{\bar{\psi}}^{\text{F(m)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\psi}}^{\text{F(m)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)), \min(\text{fy}\bar{\bar{\psi}}^{\text{F(u)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\psi}}^{\text{F(u)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega))] : \omega \in \bar{\bar{\chi}}\}$$

Example 3. 1. Consider two TFQNSs

$$\bar{\Theta}_1 = \langle(0.6, 0.6, 0.8), (0.4, 0.5.0.6), (0.2, 0.2, 0.2), (0.3, 0.3, 0.3)\rangle_{\bar{\delta}_1}$$
$$+\langle(0.8, 0.7, 0.6), (0.5, 0.6, 0.7), (0.3, 0.3, 0.3), (0.2, 0.2, 0.2)\rangle_{\bar{\delta}_2}$$
$$+\langle(0.7, 0.8, 0.9), (0.6, 0.7, 0.8), (0.4, 0.5, 0.6), (0.3, 0.4, 0.5)\rangle_{\bar{\delta}_3},$$

$$\bar{\Theta}_2 = \langle(0.4, 0.5, 0.6), (0.3, 0.4, 0.5), (0.4, 0.5, 0.6), (0.6, 0.7, 0.8)\rangle_{\bar{\delta}_1} +$$
$$\langle(0.3, 0.4, 0.5), (0.4, 0.5, 0.6), (0.4, 0.5, 0.6), (0.7, 0.8, 0.9)\rangle_{\bar{\delta}_2}$$
$$+\langle(0.3, 0.4, 0.5), (0.2, 0.3, 0.4), (0.5, 0.6, 0.7), (0.6, 0.7, 0.8)\rangle_{\bar{\delta}_3}$$

Then, $\bar{\Theta}_1 \cup \bar{\Theta}_2 = \langle(0.6, 0.6, 0.8), (0.4, 0.5, 0.6), (0.2, 0.2, 0.2), (0.3, 0.3, 0.3)\rangle_{\bar{\delta}_1} +$

$$\langle(0.8, 0.7, 0.6), (0.5, 0.6, 0.7), (0.3, 0.3, 0.3), (0.2, 0.2, 0.2)\rangle_{\bar{\delta}_2} +$$
$$\langle(0.7, 0.8, 0.9), (0.6, 0.7, 0.8), (0.4, 0.5, 0.6), (0.3, 0.4, 0.5)\rangle_{\bar{\delta}_3}$$

Definition 3.4. Intersection of any two TFQNSs $\text{fy}\hat{\bar{\rho}}_1$, $\text{fy}\hat{\bar{\rho}}_2$ is represented as $\text{fy}\hat{\bar{\rho}}_4$ and is expressed as

$$\text{fy}\hat{\bar{\rho}}_4 = \text{fy}\hat{\bar{\rho}}_1 \cap \text{fy}\hat{\bar{\rho}}_2 \quad = \{\omega, (\text{fy}\bar{\bar{\xi}}_{\text{fy}\bar{\bar{\rho}}_4}(\omega), \text{fy}\bar{\bar{\mu}}_{\text{fy}\bar{\bar{\rho}}_4}(\omega), \text{fy}\bar{\bar{\delta}}_{\text{fy}\bar{\bar{\rho}}_4}(\omega), \text{fy}\bar{\bar{\psi}}_{\text{fy}\bar{\bar{\rho}}_4}(\omega)) : \omega \in \bar{\bar{\chi}}\} \ldots(4)$$

such that

$$\text{fy}\bar{\bar{\xi}}_{\text{fy}\bar{\bar{\rho}}_4}(\omega) = (\min(\text{fy}\bar{\bar{\xi}}^{\text{T(l)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\xi}}^{\text{T(l)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)), \min(\text{fy}\bar{\bar{\xi}}^{\text{T(m)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\xi}}^{\text{T(m)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)), \min(\text{fy}\bar{\bar{\xi}}^{\text{T(u)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\xi}}^{\text{T(m)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega))$$
$$\text{fy}\bar{\bar{\mu}}_{\text{fy}\bar{\bar{\rho}}_4}(\omega) = (\min(\text{fy}\bar{\bar{\mu}}^{\text{C(l)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\mu}}^{\text{C(l)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)), \min(\text{fy}\bar{\bar{\mu}}^{\text{C(m)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\mu}}^{\text{C(m)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)), \min(\text{fy}\bar{\bar{\mu}}^{\text{C(u)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\mu}}^{\text{C(u)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)),$$
$$\text{fy}\bar{\bar{\delta}}_{\text{fy}\bar{\bar{\rho}}_4}(\omega) = (\max(\text{fy}\bar{\bar{\delta}}^{\text{U(l)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\delta}}^{\text{U(l)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)), \max(\text{fy}\bar{\bar{\delta}}^{\text{U(m)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\delta}}^{\text{U(m)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)), \max(\text{fy}\bar{\bar{\delta}}^{\text{U(u)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\delta}}^{\text{U(u)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)),$$
$$\text{fy}\bar{\bar{\psi}}_{\text{fy}\bar{\bar{\rho}}_4}(\hat{\theta}) = (\max(\text{fy}\bar{\bar{\psi}}^{\text{F(l)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\psi}}^{\text{F(l)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)), \max(\text{fy}\bar{\bar{\psi}}^{\text{F(m)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\psi}}^{\text{F(2)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)), \max(\text{fy}\bar{\bar{\psi}}^{\text{F(u)}}_{\text{fy}\bar{\bar{\rho}}_1}(\omega), \text{fy}\bar{\bar{\psi}}^{\text{F(u)}}_{\text{fy}\bar{\bar{\rho}}_2}(\omega)), \forall\omega \in \bar{\bar{\chi}}.$$

Example 3. 2. Consider two TFQNSs as:

$$\bar{\Theta}_1 = \langle(0.6, 0.6, 0.8), (0.4, 0.5.0.6), (0.2, 0.2, 0.2), (0.3, 0.3, 0.3)\rangle_{\bar{\delta}_1}$$
$$+\langle(0.8, 0.7, 0.6), (0.5, 0.6, 0.7), (0.3, 0.3, 0.3), (0.2, 0.2, 0.2)\rangle_{\bar{\delta}_2}$$
$$+\langle(0.7, 0.8, 0.9), (0.6, 0.7, 0.8), (0.4, 0.5, 0.6), (0.3, 0.4, 0.5)\rangle_{\bar{\delta}_3}$$

$$\bar{\Theta}_2 = \langle(0.4, 0.5, 0.6), (0.3, 0.4, 0.5), (0.4, 0.5, 0.6), (0.6, 0.7, 0.8)\rangle_{\bar{\delta}_1} +$$
$$\langle(0.3, 0.4, 0.5), (0.4, 0.5, 0.6), (0.4, 0.5, 0.6), (0.7, 0.8, 0.9)\rangle_{\bar{\delta}_2}$$
$$+\langle(0.3, 0.4, 0.5), (0.2, 0.3, 0.4), (0.5, 0.6, 0.7), (0.6, 0.7, 0.8)\rangle_{\bar{\delta}_3}$$

Then, $\bar{\Theta}_1 \cap \bar{\Theta}_2 = \langle((0.4,0.5,0.6),(0.3,0.4,0.5),(0.4,0.5,0.6),(0.6,0.7,0.8)\rangle_{\bar{\delta}_1} +$
$\langle((0.3,0.4,0.5),(0.4,0.5,0.6),(0.4,0.5,0.6),(0.7,0.8,0.9)\rangle_{\bar{\delta}_2} +$
$\langle((0.3,0.4,0.5),(0.2,0.3,0.4),(0.5,0.6,0.7),(0.6,0.7,0.8)\rangle_{\bar{\delta}_3}$

Definition 3.5. Complement of a TFQNS

The complement of a TFQNS $\text{fy}\hat{\bar{\rho}}$ is expressed as $(\text{fy}\hat{\bar{\rho}})^c$ and is represented as:

$$(\text{fy}\hat{\bar{\rho}})^c = \{\omega, (\text{fy}\bar{\bar{\psi}}^{F(l)}_{\text{fy}\hat{\bar{\rho}}}(\omega), \text{fy}\bar{\bar{\psi}}^{F(m)}_{\text{fy}\hat{\bar{\rho}}}(\omega), \text{fy}\bar{\bar{\psi}}^{F(u)}_{\text{fy}\hat{\bar{\rho}}}(\omega)), (\bar{\bar{\delta}}^{U(l)}_{\text{fy}\hat{\bar{\rho}}}(\omega), \bar{\bar{\delta}}^{U(m)}_{\text{fy}\hat{\bar{\rho}}}(\omega), \bar{\bar{\delta}}^{U(u)}_{\text{fy}\hat{\bar{\rho}}}(\omega)),$$
$$, (\text{fy}\bar{\bar{\mu}}^{C(l)}_{\text{fy}\hat{\bar{\rho}}}(\omega), \text{fy}\bar{\bar{\mu}}^{C(m)}_{\text{fy}\hat{\bar{\rho}}}(\omega), \text{fy}\bar{\bar{\mu}}^{C(u)}_{\text{fy}\hat{\bar{\rho}}}(\omega)), (\bar{\bar{\xi}}^{T(l)}_{\text{fy}\hat{\bar{\rho}}}(\omega), \bar{\bar{\xi}}^{T(m)}_{\text{fy}\hat{\bar{\rho}}}(\omega), \bar{\bar{\xi}}^{T(u)}_{\text{fy}\hat{\bar{\rho}}}(\omega)) : \omega \in \bar{\bar{\chi}}\} \quad \ldots(5)$$

Example 3.3 Assume a TFQNS of the form:

$\bar{\Theta}_1 = \langle((0.6,0.6,0.8),(0.4,0.5.0.6),(0.2,0.2,0.2),(0.3,0.3,0.3)\rangle_{\bar{\delta}_1}$
$+\langle((0.8,0.7,0.6),(0.5,0.6,0.7),(0.3,0.3,0.3),(0.2,0.2,0.2)\rangle_{\bar{\delta}_2}$
$+\langle((0.7,0.8,0.9),(0.6,0.7,0.8),(0.4,0.5,0.6),(0.3,0.4,0.5)\rangle_{\bar{\delta}_3}$,

Accordingly, $(\bar{\Theta}_1)^c$

$= \langle((0.3,0.3,0.3),(0.2,0.2,0.2),(0.4,0.5,0.6),(0.6,0.6,0.8)\rangle|_{\bar{\delta}_1}$
$+\langle((0.2,0.2,0.2),(0.3,0.3,0.3),(0.5,0.6,0.7),(0.8,0.7,0.6)\rangle|_{\bar{\delta}_2}$
$+\langle((0.3,0.4,0.5),(0.4,0.5,0.6),(0.6,0.7,0.8),(0.7,0.8,0.9)\rangle|_{\bar{\delta}_3}$

Definition 3.6. Containment

A TFQNS $\text{fy}\hat{\bar{\rho}}_1$ can be defined to be contained in another TFQNS $\text{fy}\hat{\bar{\rho}}_2$ and is denoted by

$\text{fy}\hat{\bar{\rho}}_1 \subseteq \text{fy}\hat{\bar{\rho}}_2$ if and only if,

$\text{fy}\bar{\bar{\xi}}^{T(l)}_{\text{fy}\bar{\rho}_1}(\omega) \le \text{fy}\bar{\bar{\xi}}^{T(l)}_{\text{fy}\bar{\rho}_2}(\omega), \text{fy}\bar{\bar{\xi}}^{T(m)}_{\text{fy}\bar{\rho}_1}(\omega) \le \text{fy}\bar{\bar{\xi}}^{T(m)}_{\text{fy}\bar{\rho}_2}(\omega), \text{fy}\bar{\bar{\xi}}^{T(u)}_{\text{fy}\bar{\rho}_1}(\omega) \le \text{fy}\bar{\bar{\xi}}^{T(u)}_{\text{fy}\bar{\rho}_2}(\omega);$
$\text{fy}\bar{\bar{\mu}}^{C(l)}_{\text{fy}\bar{\rho}_1}(\omega) \le \text{fy}\bar{\bar{\mu}}^{C(l)}_{\text{fy}\bar{\rho}_2}(\omega), \text{fy}\bar{\bar{\mu}}^{C(m)}_{\text{fy}\bar{\rho}_1}(\omega) \le \text{fy}\bar{\bar{\mu}}^{C(m)}_{\text{fy}\bar{\rho}_2}(\omega), \text{fy}\bar{\bar{\mu}}^{C(u)}_{\text{fy}\bar{\rho}_1}(\omega) \le \text{fy}\bar{\bar{\mu}}^{C(u)}_{\text{fy}\bar{\rho}_2}(\omega),$
$\text{fy}\bar{\bar{\delta}}^{U(l)}_{\text{fy}\bar{\rho}_1}(\omega) \ge \text{fy}\bar{\bar{\delta}}^{U(l)}_{\text{fy}\bar{\rho}_2}(\omega), \text{fy}\bar{\bar{\delta}}^{U(m)}_{\text{fy}\bar{\rho}_1}(\omega) \ge \text{fy}\bar{\bar{\delta}}^{U(2)}_{\text{fy}\bar{\rho}_2}(\omega), \text{fy}\bar{\bar{\delta}}^{U(u)}_{\text{fy}\bar{\rho}_1}(\omega) \ge \text{fy}\bar{\bar{\delta}}^{U(u)}_{\text{fy}\bar{\rho}_2}(\omega),$
$\text{fy}\bar{\bar{\psi}}^{F(l)}_{\text{fy}\bar{\rho}_1}(\omega) \ge \text{fy}\bar{\bar{\psi}}^{F(l)}_{\text{fy}\bar{\rho}_2}(\omega), \text{fy}\bar{\bar{\psi}}^{F(m)}_{\text{fy}\bar{\rho}_1}(\omega) \ge \text{fy}\bar{\bar{\psi}}^{F(m)}_{\text{fy}\bar{\rho}_2}(\omega), \text{fy}\bar{\bar{\psi}}^{F(u)}_{\text{fy}\bar{\rho}_1}(\omega) \ge \text{fy}\bar{\bar{\psi}}^{F(u)}_{\text{fy}\bar{\rho}_2}(\omega), \forall\omega \in \bar{\bar{\chi}}$

Theorem 3.1 Assume that $\text{fy}\hat{\bar{\rho}}$ is a TFQNS. Then

a) $\text{fy}\hat{\bar{\rho}} \cup \text{fy}\hat{\bar{\rho}} = \text{fy}\hat{\bar{\rho}}$ b) $\text{fy}\hat{\bar{\rho}} \cap \text{fy}\hat{\bar{\rho}} = \text{fy}\hat{\bar{\rho}}$

Proof : Now $\text{fy}\hat{\bar{\rho}} \cup \text{fy}\hat{\bar{\rho}}$

$= \{\omega, [(\max(\text{fy}\bar{\bar{\xi}}^{T(l)}_{\text{fy}\bar{\rho}}(\omega), \text{fy}\bar{\bar{\xi}}^{T(l)}_{\text{fy}\bar{\rho}}(\omega)), \max(\text{fy}\bar{\bar{\xi}}^{T(m)}_{\text{fy}\bar{\rho}}(\omega), \bar{\bar{\xi}}^{T(m)}_{\text{fy}\bar{\rho}}(\omega)), \max(\text{fy}\bar{\bar{\xi}}^{T(u)}_{\text{fy}\bar{\rho}}(\omega), \text{fy}\bar{\bar{\xi}}^{T(u)}_{\text{fy}\bar{\rho}}(\omega))],$
$[\max(\text{fy}\bar{\bar{\mu}}^{C(l)}_{\text{fy}\bar{\rho}}(\omega), \text{fy}\bar{\bar{\mu}}^{C(l)}_{\text{fy}\bar{\rho}}(\omega)), \max(\text{fy}\bar{\bar{\mu}}^{C(m)}_{\text{fy}\bar{\rho}}(\omega), \text{fy}\bar{\bar{\mu}}^{C(m)}_{\text{fy}\bar{\rho}}(\omega)), \max(\text{fy}\bar{\bar{\mu}}^{C(u)}_{\text{fy}\bar{\rho}}(\omega), \text{fy}\bar{\bar{\mu}}^{C(m)}_{\text{fy}\bar{\rho}}(\omega))],$
$, [\min(\text{fy}\bar{\bar{\delta}}^{U(l)}_{\text{fy}\bar{\rho}}(\omega), \text{fy}\bar{\bar{\delta}}^{U(l)}_{\text{fy}\bar{\rho}}(\omega)), \min(\text{fy}\bar{\bar{\delta}}^{U(m)}_{\text{fy}\bar{\rho}}(\omega), \text{fy}\bar{\bar{\delta}}^{U(m)}_{\text{fy}\bar{\rho}}(\omega)), \min(\text{fy}\bar{\bar{\delta}}^{U(u)}_{\text{fy}\bar{\rho}}(\omega), \text{fy}\bar{\bar{\delta}}^{U(u)}_{\text{fy}\bar{\rho}}(\omega))],$
$[\min(\text{fy}\bar{\bar{\psi}}^{F(l)}_{\text{fy}\bar{\rho}}(\omega), \text{fy}\bar{\bar{\psi}}^{F(l)}_{\text{fy}\bar{\rho}}(\hat{\theta})), \min(\text{fy}\bar{\bar{\psi}}^{F(m)}_{\text{fy}\bar{\rho}}(\omega), \text{fy}\bar{\bar{\psi}}^{F(m)}_{\text{fy}\bar{\rho}}(\omega)), \min(\text{fy}\bar{\bar{\psi}}^{F(u)}_{\text{fy}\bar{\rho}}(\omega), \text{fy}\bar{\bar{\psi}}^{F(u)}_{\text{fy}\bar{\rho}}(\omega))] : \omega \in \bar{\bar{\chi}}\}$

$= \{(\omega, (\text{fy}\bar{\bar{\xi}}^{T(l)}_{\text{fy}\bar{\rho}}(\omega), \text{fy}\bar{\bar{\xi}}^{T(m)}_{\text{fy}\bar{\rho}}(\omega), \text{fy}\bar{\bar{\xi}}^{T(u)}_{\text{fy}\bar{\rho}}(\omega)), (\text{fy}\bar{\bar{\mu}}^{C(l)}_{\text{fy}\bar{\rho}}(\omega), \text{fy}\bar{\bar{\mu}}^{C(m)}_{\text{fy}\bar{\rho}}(\omega), \text{fy}\bar{\bar{\mu}}^{C(u)}_{\text{fy}\bar{\rho}}(\omega)),$
$(\text{fy}\bar{\bar{\delta}}^{U(l)}_{\text{fy}\bar{\rho}}(\omega), \text{fy}\bar{\bar{\delta}}^{U(m)}_{\text{fy}\bar{\rho}}(\omega), \text{fy}\bar{\bar{\delta}}^{U(u)}_{\text{fy}\bar{\rho}}(\omega)), (\text{fy}\bar{\bar{\psi}}^{F(l)}_{\text{fy}\bar{\rho}}(\omega), \text{fy}\bar{\bar{\psi}}^{F(m)}_{\text{fy}\bar{\rho}}(\omega), \text{fy}\bar{\bar{\psi}}^{F(u)}_{\text{fy}\bar{\rho}}(\omega)) : \omega \in \bar{\bar{\chi}}\}$
$= \text{fy}\hat{\bar{\rho}}$

b) $\text{fy}\hat{\bar{\rho}} \cap \text{fy}\hat{\bar{\rho}}$

$= \{\omega,[(\min(fy\ddot{\xi}^{T(l)}_{fy\ddot{\rho}}(\omega), fy\ddot{\xi}^{T(l)}_{fy\ddot{\rho}}(\omega)), \min(fy\ddot{\xi}^{T(m)}_{fy\ddot{\rho}}(\omega), fy\ddot{\xi}^{T(m)}_{fy\ddot{\rho}}(\omega)), \min(fy\ddot{\xi}^{T(u)}_{fy\ddot{\rho}}(\omega), \ddot{\xi}^{T(u)}_{fy\ddot{\rho}}(\omega))],$
$[\min(fy\ddot{\mu}^{C(l)}_{fy\ddot{\rho}}(\omega), fy\ddot{\mu}^{C(l)}_{fy\ddot{\rho}}(\omega)), \min(fy\ddot{\mu}^{C(m)}_{fy\ddot{\rho}}(\omega), fy\ddot{\mu}^{C(m)}_{fy\ddot{\rho}}(\omega)), \min(fy\ddot{\mu}^{C(u)}_{fy\ddot{\rho}}(\omega), fy\ddot{\mu}^{C(u)}_{fy\ddot{\rho}}(\omega))],$
$,[\max(fy\ddot{\delta}^{U(l)}_{fy\ddot{\rho}}(\omega), fy\ddot{\delta}^{U(l)}_{fy\ddot{\rho}}(\omega)), \max(fy\ddot{\delta}^{U(m)}_{fy\ddot{\rho}}(\omega), fy\ddot{\delta}^{U(m)}_{fy\ddot{\rho}}(\omega)), \max(fy\ddot{\delta}^{U(u)}_{fy\ddot{\rho}}(\omega), fy\ddot{\delta}^{U(u)}_{fy\ddot{\rho}}(\omega))],$
$[\max(fy\ddot{\psi}^{F(l)}_{fy\ddot{\rho}}(\omega), fy\ddot{\psi}^{F(l)}_{fy\ddot{\rho}}(\omega)), \max(fy\ddot{\psi}^{F(m)}_{fy\ddot{\rho}}(\omega), fy\ddot{\psi}^{F(m)}_{fy\ddot{\rho}}(\omega)), \max(fy\ddot{\psi}^{F(u)}_{fy\ddot{\rho}}(\omega), fy\ddot{\psi}^{F(u)}_{fy\ddot{\rho}}(\omega))] : \omega \in \ddot{\chi}\}$

$= \{(\omega,(fy\ddot{\xi}^{T(l)}_{fy\ddot{\rho}}(\omega), fy\ddot{\xi}^{T(m)}_{fy\ddot{\rho}}(\omega), fy\ddot{\xi}^{T(u)}_{fy\ddot{\rho}}(\omega)),(fy\ddot{\mu}^{C(l)}_{fy\ddot{\rho}}(\omega), fy\ddot{\mu}^{C(m)}_{fy\ddot{\rho}}(\omega), fy\ddot{\mu}^{C(u)}_{fy\ddot{\rho}}(\omega)),$
$(fy\ddot{\delta}^{U(l)}_{fy\ddot{\rho}}(\omega), fy\ddot{\delta}^{U(m)}_{fy\ddot{\rho}}(\omega), fy\ddot{\delta}^{U(u)}_{fy\ddot{\rho}}(\omega)),(fy\ddot{\psi}^{F(l)}_{fy\ddot{\rho}}(\omega), fy\ddot{\psi}^{F(m)}_{fy\ddot{\rho}}(\omega), fy\ddot{\psi}^{F(u)}_{fy\ddot{\rho}}(\omega)) : \omega \in \ddot{\chi}\}$
$= fy\ddot{\rho}$

Theorem 3.2. For any two TFQNSs $fy\ddot{\rho}_1$ and $fy\ddot{\rho}_2$,

Law of commutation

a) $fy\ddot{\rho}_1 \cup fy\ddot{\rho}_2 = fy\ddot{\rho}_2 \cup fy\ddot{\rho}_1$

b) $fy\ddot{\rho}_1 \cap fy\ddot{\rho}_2 = fy\ddot{\rho}_2 \cap fy\ddot{\rho}_1$

a) Proof: $fy\ddot{\rho}_1 \cup fy\ddot{\rho}_2$

$= \{\omega,[(\max(fy\ddot{\xi}^{T(l)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\xi}^{T(l)}_{fy\ddot{\rho}_2}(\omega)), \max(fy\ddot{\xi}^{T(m)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\xi}^{T(m)}_{fy\ddot{\rho}_2}(\omega)), \max(fy\ddot{\xi}^{T(u)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\xi}^{T(u)}_{fy\ddot{\rho}_2}(\omega))],$
$[\max(fy\ddot{\mu}^{C(l)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\mu}^{C(l)}_{fy\ddot{\rho}_2}(\omega)), \max(fy\ddot{\mu}^{C(m)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\mu}^{C(m)}_{fy\ddot{\rho}_2}(\omega)), \max(fy\ddot{\mu}^{C(u)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\mu}^{C(u)}_{fy\ddot{\rho}_2}(\omega))],$
$[\min(fy\ddot{\delta}^{U(l)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\delta}^{U(l)}_{fy\ddot{\rho}_2}(\omega)), \min(fy\ddot{\delta}^{U(m)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\delta}^{U(m)}_{fy\ddot{\rho}_2}(\omega)), \min(fy\ddot{\delta}^{U(u)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\delta}^{U(u)}_{fy\ddot{\rho}_2}(\omega))],$
$[\min(fy\ddot{\psi}^{F(l)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\psi}^{F(l)}_{fy\ddot{\rho}_2}(\omega)), \min(fy\ddot{\psi}^{F(m)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\psi}^{F(m)}_{fy\ddot{\rho}_2}(\omega)), \min(fy\ddot{\psi}^{F(u)}_{fy\ddot{\rho}_1}(\widehat{\theta}), fy\ddot{\psi}^{F(u)}_{fy\ddot{\rho}_2}(\omega))] : \omega \in \ddot{\chi}\}$

$= \{\omega,[(\max(fy\ddot{\xi}^{T(l)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\xi}^{T(l)}_{fy\ddot{\rho}_1}(\omega)), \max(fy\ddot{\xi}^{T(m)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\xi}^{T(m)}_{fy\ddot{\rho}_1}(\omega)), \max(fy\ddot{\xi}^{T(u)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\xi}^{T(u)}_{fy\ddot{\rho}_1}(\omega))],$
$[\max(fy\ddot{\mu}^{C(l)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\mu}^{C(l)}_{fy\ddot{\rho}_1}(\omega)), \max(fy\ddot{\mu}^{C(m)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\mu}^{C(m)}_{fy\ddot{\rho}_1}(\omega)), \max(fy\ddot{\mu}^{C(u)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\mu}^{C(u)}_{fy\ddot{\rho}_1}(\omega))],$
$[\min(fy\ddot{\delta}^{U(l)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\delta}^{U(l)}_{fy\ddot{\rho}_1}(\omega)), \min(fy\ddot{\delta}^{U(m)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\delta}^{U(2)}_{fy\ddot{\rho}_1}(\omega)),$
$\min(fy\ddot{\delta}^{U(u)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\delta}^{U(u)}_{fy\ddot{\rho}_1}(\omega))],[\min(fy\ddot{\psi}^{F(l)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\psi}^{F(l)}_{fy\ddot{\rho}_1}(\omega)),$
$\min(fy\ddot{\psi}^{F(m)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\psi}^{F(m)}_{fy\ddot{\rho}_1}(\omega)), \min(fy\ddot{\psi}^{F(m)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\psi}^{F(m)}_{fy\ddot{\rho}_1}(\omega)) : \omega \in \ddot{\chi}\}$

$= fy\ddot{\rho}_2 \cup fy\ddot{\rho}_1$

Proof b. $fy\ddot{\rho}_1 \cap fy\ddot{\rho}_2 = \{\omega,[(\min(fy\ddot{\xi}^{T(l)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\xi}^{T(l)}_{fy\ddot{\rho}_2}(\omega)), \min(fy\ddot{\xi}^{T(m)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\xi}^{T(m)}_{fy\ddot{\rho}_2}(\omega)), \min(fy\ddot{\xi}^{T(u)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\xi}^{T(u)}_{fy\ddot{\rho}_2}(\omega))],$

$[\min(fy\ddot{\mu}^{C(l)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\mu}^{C(l)}_{fy\ddot{\rho}_2}(\omega)), \min(fy\ddot{\mu}^{C(m)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\mu}^{C(m)}_{fy\ddot{\rho}_2}(\omega)), \min(fy\ddot{\mu}^{C(u)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\mu}^{C(m)}_{fy\ddot{\rho}_2}(\omega))],$
$[\max(fy\ddot{\delta}^{U(l)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\delta}^{U(l)}_{fy\ddot{\rho}_2}(\omega)), \max(fy\ddot{\delta}^{U(m)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\delta}^{U(m)}_{fy\ddot{\rho}_2}(\omega)), \max(fy\ddot{\delta}^{U(m)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\delta}^{U(m)}_{fy\ddot{\rho}_2}(\omega))],$
$[\max(\ddot{\psi}^{F(l)}_{fy\ddot{\rho}_1}(\omega), \ddot{\psi}^{F(l)}_{fy\ddot{\rho}_2}(\omega)), \max(fy\ddot{\psi}^{F(m)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\psi}^{F(m)}_{fy\ddot{\rho}_2}(\omega)), \max(fy\ddot{\psi}^{F(u)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\psi}^{F(u)}_{fy\ddot{\rho}_2}(\omega))] : \omega \in \ddot{\chi}\}$

$= \{\omega,[(\min(fy\ddot{\xi}^{T(l)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\xi}^{T(l)}_{fy\ddot{\rho}_1}(\omega)), \min(fy\ddot{\xi}^{T(m)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\xi}^{T(m)}_{fy\ddot{\rho}_1}(\omega)), \min(fy\ddot{\xi}^{T(u)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\xi}^{T(u)}_{fy\ddot{\rho}_1}(\omega))],$

$[\min(fy\ddot{\mu}^{C(l)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\mu}^{C(l)}_{fy\ddot{\rho}_1}(\omega)), \min(fy\ddot{\mu}^{C(m)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\mu}^{C(m)}_{fy\ddot{\rho}_1}(\omega)), \min(fy\ddot{\mu}^{C(u)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\mu}^{C(u)}_{fy\ddot{\rho}_1}(\omega))],$
$[\max(fy\ddot{\delta}^{U(1)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\delta}^{U(l)}_{fy\ddot{\rho}_1}(\omega)), \max(fy\ddot{\delta}^{U(m)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\delta}^{U(m)}_{fy\ddot{\rho}_1}(\omega)), \max(fy\ddot{\delta}^{U(u)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\delta}^{U(u)}_{fy\ddot{\rho}_1}(\omega))],$
$[\max(fy\ddot{\psi}^{F(1)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\psi}^{F(l)}_{fy\ddot{\rho}_1}(\omega)), \max(fy\ddot{\psi}^{F(m)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\psi}^{F(m)}_{fy\ddot{\rho}_1}(\omega)), \max(fy\ddot{\psi}^{F(u)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\psi}^{F(3)}_{fy\ddot{\rho}_1}(\omega))] : \omega \in \ddot{\chi}\}$

$= fy\ddot{\rho}_2 \cap fy\ddot{\rho}_1$

Theorem 3.3. For any three TFQNSs $fy\ddot{\rho}_1, fy\ddot{\rho}_2, fy\ddot{\rho}_3$, $fy\ddot{\rho}_1 \cup (fy\ddot{\rho}_2 \cup fy\ddot{\rho}_3) = (fy\ddot{\rho}_1 \cup fy\ddot{\rho}_2) \cup fy\ddot{\rho}_3$

Proof: $fy\ddot{\rho}_1 \cup (fy\ddot{\rho}_2 \cup fy\ddot{\rho}_3)$

$$= \begin{matrix}\{\omega, (fy\ddot{\xi}^{T(l)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\xi}^{T(m)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\xi}^{T(u)}_{fy\ddot{\rho}_1}(\omega)), (fy\ddot{\mu}^{C(l)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\mu}^{C(m)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\mu}^{C(u)}_{fy\ddot{\rho}_1}(\omega)),\\ (fy\ddot{\delta}^{U(l)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\delta}^{U(l)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\delta}^{U(1)}_{fy\ddot{\rho}_1}(\omega)), (fy\ddot{\psi}^{F(l)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\psi}^{F(l)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\psi}^{F(1)}_{fy\ddot{\rho}_1}(\omega)) : \omega \in \ddot{\chi}\}\end{matrix}$$

$$\begin{matrix}\{\omega, [\max(fy\ddot{\xi}^{T(l)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\xi}^{T(l)}_{fy\ddot{\rho}_3}(\omega), \max(fy\ddot{\xi}^{T(m)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\xi}^{T(m)}_{fy\ddot{\rho}_3}(\omega), \max(fy\ddot{\xi}^{T(u)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\xi}^{T(u)}_{fy\ddot{\rho}_3}(\omega)],\\ \cup\ [\max(fy\ddot{\mu}^{C(l)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\mu}^{C(l)}_{fy\ddot{\rho}_3}(\omega), \max(fy\ddot{\mu}^{C(m)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\mu}^{C(m)}_{fy\ddot{\rho}_3}(\omega), \max(fy\ddot{\mu}^{C(u)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\mu}^{C(u)}_{fy\ddot{\rho}_3}(\omega)],\\ [\min(fy\ddot{\delta}^{U(l)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\delta}^{U(l)}_{fy\ddot{\rho}_3}(\omega), \min(fy\ddot{\delta}^{U(m)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\delta}^{U(m)}_{fy\ddot{\rho}_3}(\omega), \min(fy\ddot{\delta}^{U(u)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\delta}^{U(u)}_{fy\ddot{\rho}_3}(\omega)] : \omega \in \ddot{\chi}\}\end{matrix}$$

$$\begin{matrix}= \{\omega, [\max(\ fy\ddot{\xi}^{T(l)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\xi}^{T(l)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\xi}^{T(l)}_{fy\ddot{\rho}_3}(\omega)), \max(\ fy\ddot{\xi}^{T(m)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\xi}^{T(m)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\xi}^{T(m)}_{fy\ddot{\rho}_3}(\omega)),\\ \max(\ fy\ddot{\xi}^{T(u)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\xi}^{T(u)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\xi}^{T(u)}_{fy\ddot{\rho}_3}(\omega))],\\ [\max(fy\ddot{\mu}^{C(l)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\mu}^{C(l)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\mu}^{C(l)}_{fy\ddot{\rho}_3}(\omega)), \max(fy\ddot{\mu}^{C(m)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\mu}^{C(m)}_{fy\ddot{\rho}_2}(\omega), \ddot{\mu}^{C(m)}_{fy\ddot{\rho}_3}(\omega)),\\ \max(fy\ddot{\mu}^{C(u)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\mu}^{C(u)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\mu}^{C(u)}_{fy\ddot{\rho}_3}(\omega))],\\ [\min(fy\ddot{\delta}^{U(l)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\delta}^{U(l)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\delta}^{U(l)}_{fy\ddot{\rho}_3}(\omega)), \min(fy\ddot{\delta}^{U(m)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\delta}^{U(m)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\delta}^{U(m)}_{fy\ddot{\rho}_3}(\omega)),\\ \min(fy\ddot{\delta}^{U(u)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\delta}^{U(u)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\delta}^{U(u)}_{fy\ddot{\rho}_3}(\omega))],\\ [\min(fy\ddot{\psi}^{F(l)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\psi}^{F(l)}_{fy\ddot{\rho}_2}(\omega), \ddot{\psi}^{F(l)}_{fy\ddot{\rho}_3}(\omega)), \min(fy\ddot{\psi}^{F(m)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\psi}^{F(m)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\psi}^{F(m)}_{fy\ddot{\rho}_3}(\omega)),\\ \min(fy\ddot{\psi}^{F(u)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\psi}^{F(u)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\psi}^{F(u)}_{fy\ddot{\rho}_3}(\omega))] : \omega \in \ddot{\chi}\}\end{matrix}$$

$$\begin{matrix}= \{\omega, [\max(fy\ddot{\xi}^{T(l)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\xi}^{T(l)}_{fy\ddot{\rho}_2}(\omega)), \max(fy\ddot{\xi}^{T(m)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\xi}^{T(m)}_{fy\ddot{\rho}_2}(\omega)), \max(fy\ddot{\xi}^{T(u)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\xi}^{T(u)}_{fy\ddot{\rho}_2}(\omega))],\\ [\max(fy\ddot{\mu}^{C(l)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\mu}^{C(l)}_{fy\ddot{\rho}_2}(\omega)),\\ \max(fy\ddot{\mu}^{C(m)}_{\ddot{\rho}_1}(\omega), fy\ddot{\mu}^{C(m)}_{fy\ddot{\rho}_2}(\omega)), \max(fy\ddot{\mu}^{C(u)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\mu}^{C(u)}_{fy\ddot{\rho}_2}(\omega))], [\min(fy\ddot{\delta}^{U(l)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\delta}^{U(l)}_{fy\ddot{\rho}_2}(\omega)),\\ \min(fy\ddot{\delta}^{U(m)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\delta}^{U(2)}_{fy\ddot{\rho}_2}(\omega)),\\ \min(fy\ddot{\delta}^{U(u)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\delta}^{U(u)}_{fy\ddot{\rho}_2}(\omega))], [\min(fy\ddot{\psi}^{F(l)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\psi}^{F(l)}_{fy\ddot{\rho}_2}(\omega)), \min(fy\ddot{\psi}^{F(m)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\psi}^{F(m)}_{fy\ddot{\rho}_2}(\omega)),\\ \min(fy\ddot{\psi}^{F(u)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\psi}^{F(3)}_{fy\ddot{\rho}_2}(\omega))] : \omega \in \ddot{\chi}\}\\ \cup\{\omega, (fy\ddot{\xi}^{T(l)}_{fy\ddot{\rho}_3}(\omega), fy\ddot{\xi}^{T(m)}_{fy\ddot{\rho}_3}(\omega), fy\ddot{\xi}^{T(u)}_{fy\ddot{\rho}_3}(\omega)), (fy\ddot{\mu}^{C(l)}_{fy\ddot{\rho}_3}(\omega), fy\ddot{\mu}^{C(m)}_{fy\ddot{\rho}_3}(\omega), fy\ddot{\mu}^{C(u)}_{fy\ddot{\rho}_3}(\omega)), (fy\ddot{\delta}^{U(l)}_{fy\ddot{\rho}_3}(\omega), fy\ddot{\delta}^{U(m)}_{fy\ddot{\rho}_3}(\omega), fy\ddot{\delta}^{U(u)}_{fy\ddot{\rho}_3}(\omega)),\\ (fy\ddot{\psi}^{F(l)}_{fy\ddot{\rho}_3}(\omega), fy\ddot{\psi}^{F(m)}_{fy\ddot{\rho}_3}(\omega), fy\ddot{\psi}^{F(u)}_{fy\ddot{\rho}_3}(\omega)) : \omega \in \ddot{\chi}\}\end{matrix}$$

$$= (fy\ddot{\rho}_1 \cup fy\ddot{\rho}_2) \cup fy\ddot{\rho}_3$$

Theorem 3.4. For any three TFQNSs $fy\ddot{\rho}_1, fy\ddot{\rho}_2, fy\ddot{\rho}_3$, $fy\ddot{\rho}_1 \cup (fy\ddot{\rho}_2 \cap fy\ddot{\rho}_3) = (fy\ddot{\rho}_1 \cup fy\ddot{\rho}_2) \cap (fy\ddot{\rho}_1 \cup fy\ddot{\rho}_3)$

Proof: $fy\ddot{\rho}_1 \cup (fy\ddot{\rho}_2 \cap fy\ddot{\rho}_3) =$

$$\begin{matrix}\{\omega, (fy\ddot{\xi}^{T(l)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\xi}^{T(m)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\xi}^{T(u)}_{fy\ddot{\rho}_1}(\omega)), (fy\ddot{\mu}^{C(l)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\mu}^{C(m)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\mu}^{C(u)}_{fy\ddot{\rho}_1}(\omega)),\\ (fy\ddot{\delta}^{U(l)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\delta}^{U(l)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\delta}^{U(l)}_{fy\ddot{\rho}_1}(\omega)), (fy\ddot{\psi}^{F(l)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\psi}^{F(l)}_{fy\ddot{\rho}_1}(\omega), fy\ddot{\psi}^{F(l)}_{fy\ddot{\rho}_1}(\omega)) : \omega \in \ddot{\chi}\}\end{matrix}$$

$$\begin{matrix}\cup\{\omega, [(\min(fy\ddot{\xi}^{T(l)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\xi}^{T(l)}_{fy\ddot{\rho}_3}(\omega)), \min(fy\ddot{\xi}^{T(m)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\xi}^{T(m)}_{fy\ddot{\rho}_3}(\omega)), \min(fy\ddot{\xi}^{T(u)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\xi}^{T(u)}_{fy\ddot{\rho}_3}(\omega))],\\ [\min(fy\ddot{\mu}^{C(l)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\mu}^{C(l)}_{fy\ddot{\rho}_3}(\omega)), \min(fy\ddot{\mu}^{C(m)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\mu}^{C(m)}_{fy\ddot{\rho}_3}(\omega)), \min(fy\ddot{\mu}^{C(u)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\mu}^{C(u)}_{fy\ddot{\rho}_3}(\omega))],\\ [\max(fy\ddot{\delta}^{U(l)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\delta}^{U(l)}_{fy\ddot{\rho}_3}(\omega)), \max(fy\ddot{\delta}^{U(m)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\delta}^{U(m)}_{fy\ddot{\rho}_3}(\omega)), \max(fy\ddot{\delta}^{U(u)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\delta}^{U(u)}_{fy\ddot{\rho}_3}(\omega))],\\ [\max(fy\ddot{\psi}^{F(l)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\psi}^{F(l)}_{fy\ddot{\rho}_3}(\omega)), \max(fy\ddot{\psi}^{F(m)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\psi}^{F(m)}_{fy\ddot{\rho}_3}(\omega)), \max(fy\ddot{\psi}^{F(u)}_{fy\ddot{\rho}_2}(\omega), fy\ddot{\psi}^{F(u)}_{fy\ddot{\rho}_3}(\omega))] : \omega \in \ddot{\chi}\}\end{matrix}$$

$= \{\omega, [\max(fy\ddot{\bar{\xi}}^{T(l)}_{fy\ddot{\bar{\rho}}_1}(\omega), \min(fy\ddot{\bar{\xi}}^{T(l)}_{fy\ddot{\bar{\rho}}_2}(\omega), fy\ddot{\bar{\xi}}^{T(l)}_{fy\ddot{\bar{\rho}}_3}(\omega))), \max(fy\ddot{\bar{\xi}}^{T(m)}_{fy\ddot{\bar{\rho}}_1}(\omega), \min(fy\ddot{\bar{\xi}}^{T(m)}_{fy\ddot{\bar{\rho}}_2}(\omega), fy\ddot{\bar{\xi}}^{T(m)}_{fy\ddot{\bar{\rho}}_3}(\omega))),$
$\max(fy\ddot{\bar{\xi}}^{T(u)}_{fy\ddot{\bar{\rho}}_1}(\omega), \min(fy\ddot{\bar{\xi}}^{T(u)}_{fy\ddot{\bar{\rho}}_2}(\omega), fy\ddot{\bar{\xi}}^{(u)}_{fy\ddot{\bar{\rho}}_3}(\omega)))], [\max(fy\ddot{\bar{\mu}}^{C(l)}_{fy\ddot{\bar{\rho}}_1}(\omega), \min(fy\ddot{\bar{\mu}}^{C(l)}_{fy\ddot{\bar{\rho}}_2}(\omega), fy\ddot{\bar{\mu}}^{C(l)}_{fy\ddot{\bar{\rho}}_3}(\omega))),$
$\max(fy\ddot{\bar{\mu}}^{C(m)}_{fy\ddot{\bar{\rho}}_1}(\omega), \min(fy\ddot{\bar{\mu}}^{C(m)}_{fy\ddot{\bar{\rho}}_2}(\omega), fy\ddot{\bar{\mu}}^{C(m)}_{fy\ddot{\bar{\rho}}_3}(\omega))), \max(fy\ddot{\bar{\mu}}^{C(u)}_{fy\ddot{\bar{\rho}}_1}(\omega), \min(fy\ddot{\bar{\mu}}^{C(u)}_{fy\ddot{\bar{\rho}}_2}(\omega), fy\ddot{\bar{\mu}}^{C(u)}_{fy\ddot{\bar{\rho}}_3}(\omega)))],$
$[\min(fy\ddot{\bar{\delta}}^{U(l)}_{fy\ddot{\bar{\rho}}_1}(\omega), \max(fy\ddot{\bar{\delta}}^{U(l)}_{fy\ddot{\bar{\rho}}_2}(\omega), fy\ddot{\bar{\delta}}^{U(l)}_{fy\ddot{\bar{\rho}}_3}(\omega))),$
$\min(fy\ddot{\bar{\delta}}^{U(m)}_{fy\ddot{\bar{\rho}}_1}(\omega), \max(fy\ddot{\bar{\delta}}^{U(m)}_{fy\ddot{\bar{\rho}}_2}(\omega), fy\ddot{\bar{\delta}}^{U(m)}_{fy\ddot{\bar{\rho}}_3}(\omega))), \min(fy\ddot{\bar{\delta}}^{U(u)}_{fy\ddot{\bar{\rho}}_1}(\omega), \max(fy\ddot{\bar{\delta}}^{U(u)}_{fy\ddot{\bar{\rho}}_2}(\omega), fy\ddot{\bar{\delta}}^{U(u)}_{fy\ddot{\bar{\rho}}_3}(\omega)))],$
$[\min(fy\ddot{\bar{\psi}}^{F(l)}_{fy\ddot{\bar{\rho}}_1}(\omega), \max(fy\ddot{\bar{\psi}}^{F(l)}_{fy\ddot{\bar{\rho}}_2}(\omega), fy\ddot{\bar{\psi}}^{F(l)}_{fy\ddot{\bar{\rho}}_3}(\omega))),$
$\min(fy\ddot{\bar{\psi}}^{F(m)}_{fy\ddot{\bar{\rho}}_1}(\omega), \max(fy\ddot{\bar{\psi}}^{F(m)}_{fy\ddot{\bar{\rho}}_2}(\omega), fy\ddot{\bar{\psi}}^{F(m)}_{fy\ddot{\bar{\rho}}_3}(\omega))), \min(fy\ddot{\bar{\psi}}^{F(u)}_{fy\ddot{\bar{\rho}}_1}(\omega),$
$\max(fy\ddot{\bar{\psi}}^{F(u)}_{fy\ddot{\bar{\rho}}_2}(\omega), fy\ddot{\bar{\psi}}^{F(u)}_{fy\ddot{\bar{\rho}}_3}(\omega))) : \omega \in \ddot{\bar{\chi}}\}$

$= \{\omega, [(\max(fy\ddot{\bar{\xi}}^{T(l)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\xi}}^{T(l)}_{fy\ddot{\bar{\rho}}_2}(\omega)), \max(fy\ddot{\bar{\xi}}^{T(m)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\xi}}^{T(m)}_{fy\ddot{\bar{\rho}}_2}(\omega)), \max(fy\ddot{\bar{\xi}}^{T(u)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\xi}}^{T(u)}_{fy\ddot{\bar{\rho}}_2}(\omega)))],$
$[\max(fy\ddot{\bar{\mu}}^{C(l)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\mu}}^{C(l)}_{fy\ddot{\bar{\rho}}_2}(\omega)), \max(fy\ddot{\bar{\mu}}^{C(m)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\mu}}^{C(m)}_{fy\ddot{\bar{\rho}}_2}(\omega)), \max(fy\ddot{\bar{\mu}}^{C(u)}_{fy\ddot{\bar{\rho}}_1}(\widehat{\theta}), fy\ddot{\bar{\mu}}^{C(u)}_{fy\ddot{\bar{\rho}}_2}(\widehat{\theta}))),$
$[\min(fy\ddot{\bar{\delta}}^{U(l)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\delta}}^{U(l)}_{fy\ddot{\bar{\rho}}_2}(\omega)), \min(fy\ddot{\bar{\delta}}^{U(m)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\delta}}^{U(m)}_{fy\ddot{\bar{\rho}}_2}(\omega)), \min(fy\ddot{\bar{\delta}}^{U(u)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\delta}}^{U(u)}_{fy\ddot{\bar{\rho}}_2}(\omega))),$
$[\min(fy\ddot{\bar{\psi}}^{F(l)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\psi}}^{F(l)}_{fy\ddot{\bar{\rho}}_2}(\omega)), \min(fy\ddot{\bar{\psi}}^{F(m)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\psi}}^{F(m)}_{fy\ddot{\bar{\rho}}_2}(\omega)), \min(fy\ddot{\bar{\psi}}^{F(u)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\psi}}^{F(u)}_{fy\ddot{\bar{\rho}}_2}(\omega))] : \omega \in \ddot{\bar{\chi}}\}$
$\cap \{\omega, [\max(fy\ddot{\bar{\xi}}^{T(l)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\xi}}^{T(l)}_{fy\ddot{\bar{\rho}}_3}(\omega)), \max(fy\ddot{\bar{\xi}}^{T(m)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\xi}}^{T(m)}_{fy\ddot{\bar{\rho}}_3}(\omega)), \max(fy\ddot{\bar{\xi}}^{T(u)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\xi}}^{T(u)}_{fy\ddot{\bar{\rho}}_3}(\omega))],$

$[\max(fy\ddot{\bar{\mu}}^{C(l)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\mu}}^{C(l)}_{fy\ddot{\bar{\rho}}_3}(\omega)), \max(fy\ddot{\bar{\mu}}^{C(m)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\mu}}^{C(m)}_{fy\ddot{\bar{\rho}}_3}(\omega)), \max(fy\ddot{\bar{\mu}}^{C(u)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\mu}}^{C(u)}_{fy\ddot{\bar{\rho}}_3}(\omega))],$
$[\min(fy\ddot{\bar{\delta}}^{U(l)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\delta}}^{U(l)}_{fy\ddot{\bar{\rho}}_3}(\omega)), \min(fy\ddot{\bar{\delta}}^{U(m)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\delta}}^{U(m)}_{fy\ddot{\bar{\rho}}_3}(\omega)),$
$\min(fy\ddot{\bar{\delta}}^{U(u)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\delta}}^{U(u)}_{fy\ddot{\bar{\rho}}_3}(\omega))], [\min(fy\ddot{\bar{\psi}}^{F(l)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\psi}}^{F(l)}_{fy\ddot{\bar{\rho}}_3}(\omega)), \min(fy\ddot{\bar{\psi}}^{F(m)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\psi}}^{F(m)}_{fy\ddot{\bar{\rho}}_3}(\omega)),$
$\min(fy\ddot{\bar{\psi}}^{F(u)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\psi}}^{F(u)}_{fy\ddot{\bar{\rho}}_3}(\omega))] : \omega \in \ddot{\bar{\chi}}\}$

$= (fy\widehat{\bar{\rho}}_1 \cup fy\widehat{\bar{\rho}}_2) \cap (fy\widehat{\bar{\rho}}_1 \cup fy\widehat{\bar{\rho}}_3)$

Theorem 3.5. For any two TFQNSs $fy\widehat{\bar{\rho}}_1$ and $fy\widehat{\bar{\rho}}_2$, a) $fy\widehat{\bar{\rho}}_1 \cup (fy\widehat{\bar{\rho}}_1 \cap fy\widehat{\bar{\rho}}_2) = fy\widehat{\bar{\rho}}_1$

Proof:

$fy\widehat{\bar{\rho}}_1 \cup (fy\widehat{\bar{\rho}}_1 \cap fy\widehat{\bar{\rho}}_2) = \{\omega, [\max(fy\ddot{\bar{\xi}}^{T(l)}_{fy\ddot{\bar{\rho}}_1}(\omega), \min(fy\ddot{\bar{\xi}}^{T(l)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\xi}}^{T(l)}_{fy\ddot{\bar{\rho}}_2}(\omega)), \max(fy\ddot{\bar{\xi}}^{T(m)}_{fy\ddot{\bar{\rho}}_1}(\omega), \min(fy\ddot{\bar{\xi}}^{T(m)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\xi}}^{T(m)}_{fy\ddot{\bar{\rho}}_2}(\omega)),$
$\max(fy\ddot{\bar{\xi}}^{T(u)}_{fy\ddot{\bar{\rho}}_1}(\omega), \min(fy\ddot{\bar{\xi}}^{T(u)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\xi}}^{T(u)}_{fy\ddot{\bar{\rho}}_2}(\omega)))], [\max(fy\ddot{\bar{\mu}}^{C(l)}_{fy\ddot{\bar{\rho}}_1}(\omega), \min(fy\ddot{\bar{\mu}}^{C(l)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\mu}}^{C(l)}_{fy\ddot{\bar{\rho}}_2}(\omega)),$
$\max(fy\ddot{\bar{\mu}}^{C(m)}_{fy\ddot{\bar{\rho}}_1}(\omega), \min(fy\ddot{\bar{\mu}}^{C(m)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\mu}}^{C(m)}_{fy\ddot{\bar{\rho}}_2}(\omega)), \max(fy\ddot{\bar{\mu}}^{C(u)}_{fy\ddot{\bar{\rho}}_1}(\omega), \min(fy\ddot{\bar{\mu}}^{C(u)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\mu}}^{C(m)}_{fy\ddot{\bar{\rho}}_2}(\omega))],$
$,[\min(fy\ddot{\bar{\delta}}^{U(l)}_{fy\ddot{\bar{\rho}}_1}(\omega), \max(fy\ddot{\bar{\delta}}^{U(l)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\delta}}^{U(l)}_{fy\ddot{\bar{\rho}}_2}(\omega)), \min(fy\ddot{\bar{\delta}}^{U(m)}_{fy\ddot{\bar{\rho}}_1}(\omega),$
$\max(fy\ddot{\bar{\delta}}^{U(m)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\delta}}^{U(m)}_{fy\ddot{\bar{\rho}}_2}(\omega)), \min(fy\ddot{\bar{\delta}}^{U(u)}_{fy\ddot{\bar{\rho}}_1}(\omega), \max(fy\ddot{\bar{\delta}}^{U(u)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\delta}}^{U(u)}_{fy\ddot{\bar{\rho}}_2}(\omega))],$
$[\min(fy\ddot{\bar{\psi}}^{F(l)}_{fy\ddot{\bar{\rho}}_1}(\omega), \max(fy\ddot{\bar{\psi}}^{F(l)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\psi}}^{F(l)}_{fy\ddot{\bar{\rho}}_2}(\omega)), \min(fy\ddot{\bar{\psi}}^{F(m)}_{fy\ddot{\bar{\rho}}_1}(\omega), \max(fy\ddot{\bar{\psi}}^{F(m)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\psi}}^{F(m)}_{fy\ddot{\bar{\rho}}_2}(\omega)),$
$\min(fy\ddot{\bar{\psi}}^{F(u)}_{fy\ddot{\bar{\rho}}_1}(\omega), \max(fy\ddot{\bar{\psi}}^{F(u)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\psi}}^{F(u)}_{fy\ddot{\bar{\rho}}_2}(\omega))] : \omega \in \ddot{\bar{\chi}}\}$

$= \{\omega, (fy\ddot{\bar{\xi}}^{T(l)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\xi}}^{T(m)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\xi}}^{T(u)}_{fy\ddot{\bar{\rho}}_1}(\omega)), (fy\ddot{\bar{\mu}}^{C(l)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\mu}}^{C(m)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\mu}}^{C(u)}_{fy\ddot{\bar{\rho}}_1}(\omega)),$
$(fy\ddot{\bar{\delta}}^{U(l)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\delta}}^{U(m)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\delta}}^{U(u)}_{fy\ddot{\bar{\rho}}_1}(\omega)), (fy\ddot{\bar{\psi}}^{F(l)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\psi}}^{F(m)}_{fy\ddot{\bar{\rho}}_1}(\omega), fy\ddot{\bar{\psi}}^{F(u)}_{fy\ddot{\bar{\rho}}_1}(\omega)) : \omega \in \ddot{\bar{\chi}}\}$

$= fy\widehat{\ddot{\bar{\rho}}}_1$

Similarly, b) $fy\widehat{\bar{\rho}}_1 \cap (fy\widehat{\bar{\rho}}_1 \cup fy\widehat{\bar{\rho}}_2) = fy\widehat{\bar{\rho}}_1$

Proof: $fy\widehat{\bar{\rho}}_1 \cap (fy\widehat{\bar{\rho}}_1 \cup fy\widehat{\bar{\rho}}_2)$

$$= \{\omega, [\min(fy\bar{\bar{\xi}}^{T(l)}_{fy\bar{\rho}_1}(\omega), \max(fy\bar{\bar{\xi}}^{T(l)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\xi}}^{T(l)}_{fy\bar{\rho}_2}(\hat{\theta})), \min(fy\bar{\bar{\xi}}^{T(m)}_{fy\bar{\rho}_1}(\omega), \max(fy\bar{\bar{\xi}}^{T(m)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\xi}}^{T(m)}_{fy\bar{\rho}_2}(\omega)),$$
$$\min(fy\bar{\bar{\xi}}^{T(u)}_{fy\bar{\rho}_1}(\omega), \max(fy\bar{\bar{\xi}}^{T(u)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\xi}}^{T(u)}_{fy\bar{\rho}_2}(\omega))], [\min(fy\bar{\bar{\mu}}^{C(l)}_{fy\bar{\rho}_1}(\omega), \max(fy\bar{\bar{\mu}}^{C(l)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\mu}}^{C(l)}_{fy\bar{\rho}_2}(\omega)),$$
$$\min(fy\bar{\bar{\mu}}^{C(m)}_{fy\bar{\rho}_1}(\omega), \max(fy\bar{\bar{\mu}}^{C(m)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\mu}}^{C(m)}_{fy\bar{\rho}_2}(\omega)), \min(fy\bar{\bar{\mu}}^{C(u)}_{fy\bar{\rho}_1}(\omega), \max(fy\bar{\bar{\mu}}^{C(u)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\mu}}^{C(u)}_{fy\bar{\rho}_2}(\omega)),$$
$$[\max(fy\bar{\bar{\delta}}^{U(l)}_{fy\bar{\rho}_1}(\omega), \min(fy\bar{\bar{\delta}}^{U(l)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\delta}}^{U(l)}_{fy\bar{\rho}_2}(\omega)),$$
$$\max(fy\bar{\bar{\delta}}^{U(m)}_{fy\bar{\rho}_1}(\omega), \min(fy\bar{\bar{\delta}}^{U(m)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\delta}}^{U(m)}_{fy\bar{\rho}_2}(\omega)), \max(fy\bar{\bar{\delta}}^{U(u)}_{fy\bar{\rho}_1}(\omega), \min(fy\bar{\bar{\delta}}^{U(u)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\delta}}^{U(u)}_{fy\bar{\rho}_2}(\omega)],$$
$$[\max(fy\bar{\bar{\psi}}^{F(l)}_{fy\bar{\rho}_1}(\omega), \min(fy\bar{\bar{\psi}}^{F(l)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\psi}}^{F(l)}_{fy\bar{\rho}_2}(\omega)), \max(fy\bar{\bar{\psi}}^{F(m)}_{fy\bar{\rho}_1}(\omega), \min(fy\bar{\bar{\psi}}^{F(m)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\psi}}^{F(m)}_{fy\bar{\rho}_2}(\omega)),$$
$$\max(fy\bar{\bar{\psi}}^{F(u)}_{fy\bar{\rho}_1}(\omega), \min(fy\bar{\bar{\psi}}^{F(u)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\psi}}^{F(u)}_{fy\bar{\rho}_2}(\omega)] : \omega \in \bar{\bar{\chi}}\}$$
$$= \{\omega, (fy\bar{\bar{\xi}}^{T(l)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\xi}}^{T(m)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\xi}}^{T(u)}_{fy\bar{\rho}_1}(\omega)), (fy\bar{\bar{\mu}}^{C(l)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\mu}}^{C(m)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\mu}}^{C(u)}_{fy\bar{\rho}_1}(\omega)),$$
$$(fy\bar{\bar{\delta}}^{U(l)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\delta}}^{U(m)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\delta}}^{U(u)}_{fy\bar{\rho}_1}(\omega)), (fy\bar{\bar{\psi}}^{F(l)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\psi}}^{F(m)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\psi}}^{F(u)}_{fy\bar{\rho}_1}(\omega)) : \omega \in \bar{\bar{\chi}}\}$$

$$= fy\hat{\bar{\rho}}_1$$

Theorem 3.6. For any TFQNS $fy\hat{\bar{\rho}}_1$, $(fy\hat{\bar{\rho}}_1^{\,C})^C = fy\hat{\bar{\rho}}_1$

Proof: Now, $\hat{\bar{\rho}}_1 = \{\omega, (fy\bar{\bar{\xi}}^{T(l)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\xi}}^{T(m)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\xi}}^{T(u)}_{fy\bar{\rho}_1}(\omega)), (fy\bar{\bar{\mu}}^{C(l)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\mu}}^{C(m)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\mu}}^{C(u)}_{fy\bar{\rho}_1}(\omega)), (fy\bar{\bar{\delta}}^{U(l)}_{fy\bar{\rho}_1}(\omega),$

$fy\bar{\bar{\delta}}^{U(m)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\delta}}^{U(u)}_{fy\bar{\rho}_1}(\omega)), (fy\bar{\bar{\psi}}^{F(l)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\psi}}^{F(m)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\psi}}^{F(u)}_{fy\bar{\rho}_1}(\omega)) : \omega \in \bar{\bar{\chi}}\}$

So, $(fy\hat{\bar{\rho}})^c = \{\omega, (fy\bar{\bar{\psi}}^{F(l)}_{fy\bar{\rho}}(\omega), fy\bar{\bar{\psi}}^{F(m)}_{fy\bar{\rho}}(\omega), fy\bar{\bar{\psi}}^{F(u)}_{fy\bar{\rho}}(\omega)), (fy\bar{\bar{\delta}}^{U(l)}_{fy\bar{\rho}}(\omega), fy\bar{\bar{\delta}}^{U(m)}_{fy\bar{\rho}}(\omega), fy\bar{\bar{\delta}}^{U(u)}_{fy\bar{\rho}}(\omega)),$

$(fy\bar{\bar{\mu}}^{C(l)}_{fy\bar{\rho}}(\omega), fy\bar{\bar{\mu}}^{C(m)}_{fy\bar{\rho}}(\omega), fy\bar{\bar{\mu}}^{C(u)}_{fy\bar{\rho}}(\omega)), (fy\bar{\bar{\xi}}^{T(l)}_{fy\bar{\rho}}(\omega), fy\bar{\bar{\xi}}^{T(m)}_{fy\bar{\rho}}(\omega), fy\bar{\bar{\xi}}^{T(u)}_{fy\bar{\rho}}(\omega)) : \omega \in \bar{\bar{\chi}}\}$

So, $(fy\hat{\bar{\rho}}_1^{\,C})^C = \{\omega, (fy\bar{\bar{\xi}}^{T(l)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\xi}}^{T(m)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\xi}}^{T(u)}_{fy\bar{\rho}_1}(\omega)), (fy\bar{\bar{\mu}}^{C(l)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\mu}}^{C(m)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\mu}}^{C(u)}_{fy\bar{\rho}_1}(\omega)),$

$(fy\bar{\bar{\delta}}^{U(l)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\delta}}^{U(m)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\delta}}^{U(u)}_{fy\bar{\rho}_1}(\omega)), (fy\bar{\bar{\psi}}^{F(l)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\psi}}^{F(m)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\psi}}^{F(u)}_{fy\bar{\rho}_1}(\omega)) : \omega \in \bar{\bar{\chi}}\}$

$= fy\hat{\bar{\rho}}_1$

Theorem 3.7. For any TFQNS $fy\hat{\bar{\rho}}_1$,(a) $fy\hat{\bar{\rho}}_1 \cap \hat{0} = \hat{0}$, (b) $fy\hat{\bar{\rho}}_1 \cup \hat{1} = \hat{1}$

Proof. (a) $fy\hat{\bar{\rho}}_1 \cap \hat{0}$

$$= \{\omega, (fy\bar{\bar{\xi}}^{T(l)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\xi}}^{T(m)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\xi}}^{T(u)}_{fy\bar{\rho}_1}(\omega)), (fy\bar{\bar{\mu}}^{C(l)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\mu}}^{C(m)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\mu}}^{C(3)}_{fy\bar{\rho}_1}(\omega)),$$
$$(fy\bar{\bar{\delta}}^{U(l)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\delta}}^{U(m)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\delta}}^{U(u)}_{fy\bar{\rho}_1}(\omega)), (fy\bar{\bar{\psi}}^{F(l)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\psi}}^{F(m)}_{fy\bar{\rho}_1}(\omega), fy\bar{\bar{\psi}}^{F(u)}_{fy\bar{\rho}_1}(\omega)) : \omega \in \bar{\bar{\chi}}\} \cap$$
$$\{(0,0,0), (0,0,0), (1,1,1), (1,1,1), (1,1,1)\}$$

$$= \{\omega, [\min(fy\bar{\bar{\xi}}^{T(l)}_{fy\bar{\rho}_1}(\omega), 0), \min(fy\bar{\bar{\xi}}^{T(m)}_{fy\bar{\rho}_1}(\omega), 0), \min(fy\bar{\bar{\xi}}^{T(u)}_{fy\bar{\rho}_1}(\omega), 0)], [\min(fy\bar{\bar{\mu}}^{C(l)}_{fy\bar{\rho}_1}(\omega), 0), \min(fy\bar{\bar{\mu}}^{C(m)}_{fy\bar{\rho}_1}(\omega), 0),$$
$$\min(fy\bar{\bar{\mu}}^{C(u)}_{fy\bar{\rho}_1}(\omega), 0)], [\max(fy\bar{\bar{\delta}}^{U(l)}_{fy\bar{\rho}_1}(\omega), 1), \max(fy\bar{\bar{\delta}}^{U(m)}_{fy\bar{\rho}_1}(\omega), 1),$$
$$\max(fy\bar{\bar{\delta}}^{U(u)}_{fy\bar{\rho}_1}(\omega), 1)], [\max(fy\bar{\bar{\psi}}^{F(l)}_{fy\bar{\rho}_1}(\omega), 1), \max(fy\bar{\bar{\psi}}^{F(m)}_{fy\bar{\rho}_1}(\omega), 1), \max(fy\bar{\bar{\psi}}^{F(u)}_{fy\bar{\rho}_1}(\omega), 1)] : \omega \in \bar{\bar{\chi}}\}$$
$$= \{\omega, (0,0,0), (0.0.0), (1,1,1), (1,1,1) : \omega \in \bar{\bar{\chi}}\}$$
$$= \hat{0}$$

Proof. (b)

$$\begin{aligned}&fy\ddot{\tilde{\rho}}_1 \cup \hat{1} = \{\omega, (fy\ddot{\tilde{\xi}}^{T(l)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\xi}}^{T(m)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\xi}}^{T(u)}_{fy\ddot{\tilde{\rho}}_1}(\omega)), (fy\ddot{\tilde{\mu}}^{C(l)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\mu}}^{C(m)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\mu}}^{C(u)}_{fy\ddot{\tilde{\rho}}_1}(\omega)), (fy\ddot{\tilde{\delta}}^{U(l)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\delta}}^{U(m)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\delta}}^{U(u)}_{fy\ddot{\tilde{\rho}}_1}(\omega)),\\ &(fy\ddot{\tilde{\psi}}^{F(l)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\psi}}^{F(m)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\psi}}^{F(u)}_{fy\ddot{\tilde{\rho}}_1}(\omega)) : \omega \in \ddot{\tilde{\chi}}\} \cup\\ &\{(1,1,1),(1,1,1),(0,0,0),(0,0,0),(0,0,0)\}\end{aligned}$$

$$\begin{aligned}&= \{\omega, [\max(fy\ddot{\tilde{\xi}}^{T(l)}_{fy\ddot{\tilde{\rho}}_1}(\omega),1), \max(fy\ddot{\tilde{\xi}}^{T(m)}_{fy\ddot{\tilde{\rho}}_1}(\omega),1), \min(fy\ddot{\tilde{\xi}}^{T(u)}_{fy\ddot{\tilde{\rho}}_1}(\omega),1)], [\max(fy\ddot{\tilde{\mu}}^{C(l)}_{fy\ddot{\tilde{\rho}}_1}(\omega),1), \max(fy\ddot{\tilde{\mu}}^{C(m)}_{fy\ddot{\tilde{\rho}}_1}(\omega),1),\\ &\max(fy\ddot{\tilde{\mu}}^{C(u)}_{fy\ddot{\tilde{\rho}}_1}(\omega),1)], [\min(fy\ddot{\tilde{\delta}}^{U(l)}_{fy\ddot{\tilde{\rho}}_1}(\omega),0), \min(fy\ddot{\tilde{\delta}}^{U(m)}_{fy\ddot{\tilde{\rho}}_1}(\omega),0),\\ &\min(fy\ddot{\tilde{\delta}}^{U(u)}_{fy\ddot{\tilde{\rho}}_1}(\omega),0)], [\min(fy\ddot{\tilde{\psi}}^{F(l)}_{fy\ddot{\tilde{\rho}}_1}(\omega),0), \min(fy\ddot{\tilde{\psi}}^{F(m)}_{fy\ddot{\tilde{\rho}}_1}(\omega),0), \min(fy\ddot{\tilde{\psi}}^{F(u)}_{fy\ddot{\tilde{\rho}}_1}(\omega),0)] : \omega \in \ddot{\tilde{\chi}}\}\end{aligned}$$

$$\begin{aligned}&= \{\omega, (1,1,1), (1,1,1), (0,0,0), (0,0,0), (0,0,0) : \omega \in \ddot{\tilde{\chi}}\}\\ &= \hat{1}\end{aligned}$$

Theorem 3.8. For any two TFQNSs $fy\ddot{\tilde{\rho}}_1$ & $fy\ddot{\tilde{\rho}}_2$, $(fy\ddot{\tilde{\rho}}_1 \cap fy\ddot{\tilde{\rho}}_2)^C = fy\ddot{\tilde{\rho}}_1^{\,C} \cup fy\ddot{\tilde{\rho}}_2^{\,C}$

Proof. $(fy\ddot{\tilde{\rho}}_1 \cap fy\ddot{\tilde{\rho}}_2)$ =

$$\begin{aligned}&\{\omega, [(\min(fy\ddot{\tilde{\xi}}^{T(l)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\xi}}^{T(l)}_{fy\ddot{\tilde{\rho}}_2}(\omega)), \min(fy\ddot{\tilde{\xi}}^{T(m)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\xi}}^{T(m)}_{fy\ddot{\tilde{\rho}}_2}(\omega)), \min(fy\ddot{\tilde{\xi}}^{T(u)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\xi}}^{T(u)}_{fy\ddot{\tilde{\rho}}_2}(\omega))],\\ &[\min(fy\ddot{\tilde{\mu}}^{C(l)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\mu}}^{C(l)}_{fy\ddot{\tilde{\rho}}_2}(\omega)), \min(fy\ddot{\tilde{\mu}}^{C(m)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\mu}}^{C(m)}_{fy\ddot{\tilde{\rho}}_2}(\omega)), \min(fy\ddot{\tilde{\mu}}^{C(u)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\mu}}^{C(u)}_{fy\ddot{\tilde{\rho}}_2}(\omega))],\\ &[\max(fy\ddot{\tilde{\delta}}^{U(l)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\delta}}^{U(l)}_{fy\ddot{\tilde{\rho}}_2}(\omega)), \max(fy\ddot{\tilde{\delta}}^{U(m)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\delta}}^{U(m)}_{fy\ddot{\tilde{\rho}}_2}(\omega)),\\ &\max(fy\ddot{\tilde{\delta}}^{U(u)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\delta}}^{U(u)}_{fy\ddot{\tilde{\rho}}_2}(\omega))], [\max(fy\ddot{\tilde{\psi}}^{F(l)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\psi}}^{F(l)}_{fy\ddot{\tilde{\rho}}_2}(\omega)), \max(fy\ddot{\tilde{\psi}}^{F(m)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\psi}}^{F(m)}_{fy\ddot{\tilde{\rho}}_2}(\omega)),\\ &\max(fy\ddot{\tilde{\psi}}^{F(u)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\psi}}^{F(u)}_{fy\ddot{\tilde{\rho}}_2}(\omega))] : \omega \in \ddot{\tilde{\chi}}\}\end{aligned}$$

So,

$$\begin{aligned}&(fy\ddot{\tilde{\rho}}_1 \cap fy\ddot{\tilde{\rho}}_2)^C = \{\omega, [\max(fy\ddot{\tilde{\psi}}^{F(l)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\psi}}^{F(l)}_{fy\ddot{\tilde{\rho}}_2}(\omega)), \max(fy\ddot{\tilde{\psi}}^{F(m)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\psi}}^{F(m)}_{fy\ddot{\tilde{\rho}}_2}(\omega)), \max(fy\ddot{\tilde{\psi}}^{F(u)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\psi}}^{F(u)}_{fy\ddot{\tilde{\rho}}_2}(\omega))],\\ &[\max(fy\ddot{\tilde{\delta}}^{U(l)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\delta}}^{U(l)}_{fy\ddot{\tilde{\rho}}_2}(\omega)), \max(fy\ddot{\tilde{\delta}}^{U(m)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\delta}}^{U(m)}_{fy\ddot{\tilde{\rho}}_2}(\omega)), \max(fy\ddot{\tilde{\delta}}^{U(u)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\delta}}^{U(u)}_{fy\ddot{\tilde{\rho}}_2}(\omega))],\\ &[\min(fy\ddot{\tilde{\mu}}^{C(l)}_{\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\mu}}^{C(l)}_{fy\ddot{\tilde{\rho}}_2}(\omega)), \min(fy\ddot{\tilde{\mu}}^{C(m)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\mu}}^{C(m)}_{fy\ddot{\tilde{\rho}}_2}(\omega)),\\ &\min(fy\ddot{\tilde{\mu}}^{C(u)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\mu}}^{C(u)}_{fy\ddot{\tilde{\rho}}_2}(\omega))], [\min(fy\ddot{\tilde{\xi}}^{T(l)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\xi}}^{T(l)}_{fy\ddot{\tilde{\rho}}_2}(\omega)), \min(fy\ddot{\tilde{\xi}}^{T(m)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\xi}}^{T(m)}_{fy\ddot{\tilde{\rho}}_2}(\omega)),\\ &\min(fy\ddot{\tilde{\xi}}^{T(u)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\xi}}^{T(u)}_{fy\ddot{\tilde{\rho}}_2}(\omega))] : \omega \in \ddot{\tilde{\chi}}\} \qquad (1)\end{aligned}$$

Again,

$fy\ddot{\tilde{\rho}}_1^{\,C} \cup fy\ddot{\tilde{\rho}}_2^{\,C}$ =

$$\begin{aligned}&\{\omega, (fy\ddot{\tilde{\psi}}^{F(l)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\psi}}^{F(m)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\psi}}^{F(u)}_{fy\ddot{\tilde{\rho}}_1}(\omega)), (fy\ddot{\tilde{\delta}}^{U(l)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\delta}}^{U(m)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\delta}}^{U(u)}_{fy\ddot{\tilde{\rho}}_1}(\omega)),\\ &(fy\ddot{\tilde{\mu}}^{C(l)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\mu}}^{C(m)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\mu}}^{C(u)}_{fy\ddot{\tilde{\rho}}_1}(\omega)), (fy\ddot{\tilde{\xi}}^{T(l)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\xi}}^{T(m)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\xi}}^{T(u)}_{fy\ddot{\tilde{\rho}}_1}(\omega)) : \omega \in \ddot{\tilde{\chi}}\} \cup \{\omega, (fy\ddot{\tilde{\psi}}^{F(l)}_{fy\ddot{\tilde{\rho}}_2}(\omega), fy\ddot{\tilde{\psi}}^{F(m)}_{fy\ddot{\tilde{\rho}}_2}(\omega), fy\ddot{\tilde{\psi}}^{F(u)}_{fy\ddot{\tilde{\rho}}_2}(\omega)),\\ &(fy\ddot{\tilde{\delta}}^{U(l)}_{fy\ddot{\tilde{\rho}}_2}(\omega), fy\ddot{\tilde{\delta}}^{U(m)}_{fy\ddot{\tilde{\rho}}_2}(\omega), fy\ddot{\tilde{\delta}}^{U(u)}_{fy\ddot{\tilde{\rho}}_2}(\omega)), (fy\ddot{\tilde{\mu}}^{C(l)}_{fy\ddot{\tilde{\rho}}_2}(\omega), fy\ddot{\tilde{\mu}}^{C(m)}_{fy\ddot{\tilde{\rho}}_2}(\omega), fy\ddot{\tilde{\mu}}^{C(u)}_{fy\ddot{\tilde{\rho}}_2}(\omega)), (fy\ddot{\tilde{\xi}}^{T(l)}_{fy\ddot{\tilde{\rho}}_2}(\omega), fy\ddot{\tilde{\xi}}^{T(m)}_{fy\ddot{\tilde{\rho}}_2}(\omega), fy\ddot{\tilde{\xi}}^{T(u)}_{fy\ddot{\tilde{\rho}}_2}(\omega)) : \omega \in \ddot{\tilde{\chi}}\}\end{aligned}$$

=

$$\begin{aligned}&\{\omega, [\max(fy\ddot{\tilde{\psi}}^{F(l)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\psi}}^{F(l)}_{fy\ddot{\tilde{\rho}}_2}(\omega)), \max(fy\ddot{\tilde{\psi}}^{F(2)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\psi}}^{F(m)}_{fy\ddot{\tilde{\rho}}_2}(\omega)), \max(fy\ddot{\tilde{\psi}}^{F(u)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\psi}}^{F(u)}_{fy\ddot{\tilde{\rho}}_2}(\omega))],\\ &[\max(fy\ddot{\tilde{\delta}}^{U(l)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\delta}}^{U(l)}_{fy\ddot{\tilde{\rho}}_2}(\omega)), \max(fy\ddot{\tilde{\delta}}^{U(m)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\delta}}^{U(m)}_{fy\ddot{\tilde{\rho}}_2}(\omega)), \max(fy\ddot{\tilde{\delta}}^{U(u)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\delta}}^{U(u)}_{fy\ddot{\tilde{\rho}}_2}(\omega))],\\ &[\min(fy\ddot{\tilde{\mu}}^{C(l)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\mu}}^{C(l)}_{fy\ddot{\tilde{\rho}}_2}(\omega)), \min(fy\ddot{\tilde{\mu}}^{C(m)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\mu}}^{C(m)}_{fy\ddot{\tilde{\rho}}_2}(\omega)),\\ &\min(fy\ddot{\tilde{\mu}}^{C(u)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\mu}}^{C(u)}_{fy\ddot{\tilde{\rho}}_2}(\omega))], [\min(fy\ddot{\tilde{\xi}}^{T(l)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\xi}}^{T(l)}_{fy\ddot{\tilde{\rho}}_2}(\omega)), \min(fy\ddot{\tilde{\xi}}^{T(m)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\xi}}^{T(m)}_{fy\ddot{\tilde{\rho}}_2}(\omega)),\\ &\min(fy\ddot{\tilde{\xi}}^{T(u)}_{fy\ddot{\tilde{\rho}}_1}(\omega), fy\ddot{\tilde{\xi}}^{T(u)}_{fy\ddot{\tilde{\rho}}_2}(\omega))] : \omega \in \ddot{\tilde{\chi}}\} \qquad (2)\end{aligned}$$

Comparing eqn (1) and eqn (2), we have $(fy\ddot{\tilde{\rho}}_1 \cap fy\ddot{\tilde{\rho}}_2)^C = fy\ddot{\tilde{\rho}}_1^{\,C} \cup fy\ddot{\tilde{\rho}}_2^{\,C}$

4. Discussion

In this paper, the notion of TFQNS is introduced by combining the TFN and the QSVNS to utilize the advantages of TFN and QSVNS. The significance of introducing the hybrid set structure TFQNS is that the computational techniques based on TFN or QSVNS alone may not always produce the best results. But a fusion of them may produce better results. We have presented a real-world example of TFQNS in section 3.1 which is elegant to express uncertainty by utilizing triangular fuzzy number which was not possible using QSVNS alone. So, TFQNS is more advantageous than QSVNS. Limitation of the study: Only a few properties have been established. Many more results are to be explored for TFQNS. In the future, various operators can be defined in TFQNS environment which can be utilized to deal with multi criteria decision making problems. and utilize the theory to deal with practical applications in the areas such as information fusion, information retrieval etc.

5. Conclusions

In this paper, we have introduced a new concept of TFQNS and verified its important properties like unification, intersection, complement etc. We hope that this treatise will enlighten a future scope of development of logical system in information science and some innovative operators can be developed which will be helpful in decision making [23-31], information retrieval system [32-33], cyber security control selection [34], graph theory [35-36], etc.

Ethical approval: NA.
Funding: Not applicable (NA)
Authors Contribution: All the author's contributed equally.

References

1. Smarandache, F. (1998). A unifying field of logics. Neutrosophy: neutrosophic probability, set and logic. Rehoboth: American Research Press.
2. Zadeh, L. A. (1965). Fuzzy sets. Information and Control, 8(3), 338-353.
3. Atanassov, K. T. (1986). Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 20(1), 87-96.
4. Wang, H., Smarandache, F., Zhang, Y., Sunderraman, R. (2010). Single valued neutrosophic sets. Multispace Multistructure, 4, 410–413.
5. Smarandache, F. & Pramanik, S. (Eds). (2016). New trends in neutrosophic theory and applications. Brussels: Pons Editions.
6. Smarandache, F. & Pramanik, S. (Eds). (2018). New trends in neutrosophic theory and applications (Vol. 2). Brussels: Pons Editions.
7. Broumi, S., Bakali, A., Talea, M., Smarandache, F., Uluçay, V., Sahin, S., Dey, A., Dhar, M., Tan, R. P., de Oliveira, A., & Pramanik, S. (2018). Neutrosophic sets: An overview. In F. Smarandache, & S. Pramanik (Eds., vol.2), New trends in neutrosophic theory and applications (pp. 403-434). Brussels: Pons Editions.

8. Pramanik, S., Mallick, R., & Dasgupta, A. (2018). Contributions of selected Indian researchers to multi-attribute decision making in neutrosophic environment. Neutrosophic Sets and Systems, 20, 108-131. http://doi.org/10.5281/zenodo.1284870
9. Khan, M., Son, L. H., Ali, M., Chau, H. T. M., Na, N. T. N., & Smarandache, F. (2018). Systematic review of decision making algorithms in extended neutrosophic sets. Symmetry, 10(8), 314.
10. Peng, X., & Dai, J. (2020). A bibliometric analysis of neutrosophic set: Two decades review from 1998 to 2017. Artificial Intelligence Review, 53(1), 199-255.
11. Pramanik, S. (2020). Rough neutrosophic set: an overview. In F. Smarandache, & S. Broumi, Eds.), Neutrosophic theories in communication, management and information technology (pp.275-311). New York. Nova Science Publishers.
12. Pramanik, S. (2022). Single-valued neutrosophic set: An overview. In: N. Rezaei (Eds) Transdisciplinarity. Integrated Sciene, vol 5(pp.563-608). Springer, Cham. https://doi.org/10.1007/978-3-030-94651-7_26
13. Chatterjee, R., Majumdar, P., & Samanta, S. K. (2016). On some similarity measures and entropy on quadripartitioned single valued neutrosophic sets. Journal of Intelligent & Fuzzy Systems, 30(4), 2475-2485.
14. Belnap N.D. (1977). A useful four-valued logic. In: Dunn J.M., Epstein G. (Eds.) Modern uses of multiple-valued logic. Episteme (A series in the foundational, methodological, philosophical, psychological, sociological, and political aspects of the sciences, pure and applied, (pp.5-37), vol 2. Dordrecht: Springer. https://doi.org/10.1007/978-94-010-1161-7_2.0
15. Smarandache, F. (2013). n-Valued refined neutrosophic logic and its applications to physics. Progress in Physics, 4, 143-146.
16. Pramanik, S. (2022). Interval quadripartitioned neutrosophic sets. Neutrosophic Sets and Systems, 51, 2022, 146-156.
17. Wang, H., Madiraju, P., Zhang, Y. Q., & Sunderraman, R. (2005). Interval neutrosophic sets. International Journal of Applied Mathematics & Statistics, 3 (5), 1-18.
18. Mallick, R., & Pramanik, S. (2020). Pentapartitioned neutrosophic set and its properties. Neutrosophic Sets and Systems, 36, 184-192.
19. Pramanik, S. (2023). Interval pentapartitioned neutrosophic sets. *Neutrosophic Sets and Systems*, *55*, 232-246.
20. Pramanik, S., & Roy, T. K. (2007). Fuzzy goal programming approach to multilevel programming problems. *European Journal of Operational Research, 176* (2) 1151-1166. doi:10.1016/j.ejor.2005.08.024
21. Biswas, P., Pramanik, S., & Giri, B. C. (2016). Aggregation of triangular fuzzy neutrosophic set information and its application to multi-attribute decision making. Neutrosophic Sets and Systems, 12, 20-40.
22. Sinha, K., & Majumdar, P. (2024). Rural healthcare challenges through quadripartitioned single valued neutrosophic Z-numbers. Nonlinear Studies, 31(1), 211-219.
23. Pramanik, S., Dey, P. P., Giri, B. C., & Smarandache, F. (2017). An extended TOPSIS for multi-attribute decision making problems with neutrosophic cubic information. *Neutrosophic Sets and Systems*, *17*, 20-28.

24. Pramanik, S., Dalapati, S., & Roy, T. K. (2018). Neutrosophic multi-attribute group decision making strategy for logistics center location selection. In F. Smarandache, M. A. Basset, & V. Chang (Eds), Neutrosophic operational research, volume III, (pp.13-32). Brussels: Pons Publishing House.
25. Majumder, P., Paul, A., & Pramanik, S. (2023). Single-valued pentapartitioned neutrosophic weighted hyperbolic tangent similarity measure to determine the most significant environmental risks during the COVID-19 pandemic. Neutrosophic Sets and Systems, 57, 57-75.
26. Pramanik, S., Das, S., Das, R., Tripathy, B. C. (2023). Neutrosophic BWM-TOPSIS strategy under SVNS environment. Neutrosophic Sets and Systems, 56, 178-189.
27. Gamal, A., Mohamed, R., Abdel-Basset, M., Hezam, I. M., & Smarandache, F. (2023). Consideration of disruptive technologies and supply chain sustainability through α-discounting AHP–VIKOR: calibration, validation, analysis, and methods. Soft Computing. https://doi.org/10.1007/s00500-023-08819-0
28. Kamran, M., Ismail, R., Al-Sabri, E. H. A., Salamat, N., Farman, M., & Ashraf, S. (2023). An optimization strategy for MADM framework with confidence level aggregation operators under probabilistic neutrosophic hesitant fuzzy rough environment. *Symmetry*, *15*(3), 578.
29. Borah, G, & Dutta, P. (2023). Aggregation operators of quadripartitioned single-valued neutrosophic Z-numbers with applications to diverse COVID-19 scenarios. Engineering Applications of Artificial Intelligence, 2023, 119, 105748. doi: https://doi.org/10.1016/j.engappai.2022.105748
30. Sinha, K., Majumdar, P., & Broumi, S. (2022). MADM technique under QSVN environment using different prioritized operator. *Neutrosophic Sets and Systems*, *48*, 443-456
31. Sinha, K., Majumdar, P., & Broumi, S. (2022). Vaccine distribution technique under QSVN environment using different aggregation operator. Neutrosophic Sets and Systems, 48, 356-367.
32. Sahoo, S., Panigrahi, P., & Pramanik, S. (2023). Assign weights to ranking factors under group popularity for library and information system based on the CRITIC strategy. In D. C, Kar, S.Z. Khan, A. Durrany, & P. K. Jain (Eds), New and Innovative Libraries in Digital Era: Services and Practices (pp. 390-397). Bookwell, Delhi.
33. Sahoo, S., Pramanik, S., & Panigrahi, P., (2023). SVNN-Entropy weighting strategy (SVNN-EWS) for popularity ranking factors in library and information system: a neutrosophic framework. College Library, 38 (1), 88-102.
34. Kamran, M., Ashraf, S., Salamat, N., Naeem, M., & Botmart, T. (2023). Cyber security control selection based decision support algorithm under single valued neutrosophic hesitant fuzzy Einstein aggregation information. *AIMS Mathematics*, *8*(3), 5551-5573.doi: 10.3934/math.2023280
35. Mahapatra, T., Sahoo, S., Ghorai, G., & Pal, M. (2021). Interval valued m-polar fuzzy planar graph and its application. Artificial Intelligence Review, 54,1649–1675.
36. Das, S., Das, R., & Pramanik, S. (2022). Single valued pentapartitioned neutrosophic graphs. *Neutrosophic Sets and Systems*,*50*, 225-238.

Received: June 27, 2024. Accepted: Oct 11, 2024

NSS

Neutrosophic Sets and Systems, Vol. 75, 2025

University of New Mexico

A Study on n-th Derivative of Neutrosophic Function and Neutrosophic Differential Equation

Gugu Narzary[1], Bhimraj Basumatary[2,*]

[1]Department of Mathematical Sciences, Bodoland University, Kokrajhar; gugunarzary99@gmail.com

[2]Department of Mathematical Sciences, Bodoland University, Kokrajhar; brbasumatary14@gmail.com

*Correspondence: brbasumatary14@gmail.com.

Abstract. Neutrosophic sets play an important role in addressing uncertainty, vagueness, and indeterminacy in problem-solving. In this research article, we deal with the concepts of the derivative of neutrosophic real functions. This article aims to understand the n-th order derivative of neutrosophic real functions. In addition, the n-th order neutrosophic differential equation is defined, and at the same time, we have established both the existence and uniqueness of the solution to the n-th order neutrosophic differential equation.

Keywords: neutrosophic set; neutrosophic real number; derivative of neutrosophic function; n-th order neutrosophic differential equation

List of Abbreviation

Complete Metric Space : CMS

Initial Value Problem : IVP

Lattice Problem : LP

Neutrosophic Complex Number : NCN

Neutrosophic Function : NF

Neutrosophic Mereo Continuity : NMC

Neutrosophic Mereo Derivative : NMD

Neutrosophic Mereo Integral : NMI

Neutrosophic Mereo Limit : NML

Neutrosophic Real and Complex Number : NRCN

Neutrosophic Real Number : NRN

Neutrosophic Real Numbers : NRNs

Pentagonal Neutrosophic Number : PNN
Shortest Path Problem : SPP
Single Valued Neutrosophic Polygroups : SVNPs
Triangular Neutrosophic Number : TNN

1. **Introduction**

The American scientist and philosopher F. Smarandache introduced neutrosophic logic, and this logic is a generalization of fuzzy logic invented by Zadeh in 1965 [1]. Neutrosophic sets were introduced into the literature by Smarandache, supporting incomplete, undefined, and inconsistent information. In neutrosophic sets, the vagueness is explicitly measured by a new parameter I. True membership (T), ambiguity of membership (I), and false membership (F) are three independent parameters that allow us to characterize a Neutrosophic number. Smarandache proposed neutrosophic logic to represent a mathematical model of uncertainty, inaccuracy, ambiguity, imprecision, vagueness, unknown, incompleteness, inconsistency, redundancy, and contradiction, the concept of neutrosophy being a new branch of philosophy introduced by Smarandache [2–15]. He introduced the meaning of the standard form of an NRN and the circumstances for the division of two NRNs, characterized the standard form of an NCN, and found the root $n \geq 2$ of an NRCN [2–5]. While studying the concept of neutrosophic probability and neutrosophic statistics, Professor Smarandache [2–9] entered the concept of provisional calculus, where he first introduced the concepts of the NML, NMC, NMD, and NMI. Al-Tahan [11] presented the results on SVNPs. Edalatpanah [12] proposed a new simple algorithm for solving linear neutrosophic programming, in which the variables and the RHS represent the TNNs. Chakraborty [13,14] applied the PNN to the LP and the SPP. Mondal et al. [16] described the application of the neutrosophic differential equation on Mine safety via a single-valued neutrosophic number. Sumathi et al. [17] discussed the differential equation in a neutrosophic environment and the solution of a second-order linear differential equation with trapezoidal neutrosophic numbers as boundary conditions. Acharya et al. [18] used the differential equations in a neutrosophic environment under Hukuhara differentiability to model the amount of glucose distribution and absorption rates in blood. Lathamaheswari et al. [19] solved the neutrosophic differential equation by using bipolar trapezoidal neutrosophic number and applied this concept in predicting bacterial reproduction over separate bodies. Parikh et al. [20] describe the solution of a first-order linear non-homogeneous fuzzy differential equation with initial conditions in a neutrosophic environment. He also introduced the neutrosophic analytical method and the fourth-order Runge-Kutta numerical method by using triangular neutrosophic numbers. Recently Alhasan [21–23] introduced the differential and integral of

neutrosophic real functions. Salamah et al. [24] used the concepts of continuity, differentiability, and integrability from real analysis to study the derivative and integration of a neutrosophic real function with one variable depending on the geometry isometry(AH-Isometry), also they studied the neutrosophic differential equation by using One-Dimensional geometry AH-Isometry, where they discuss the methods of finding the solution of neutrosophic identical linear differential equation and neutrosophic non-homogeneous linear differential equation [25].

In this article, the main contributions are:

- An n-th order derivative of neutrosophic functions.
- An n-th order neutrosophic differential equation.
- Existence and uniqueness to the solution of n-th order neutrosophic differential equation.

1.1. **Motivation**

In our literature review, we have seen that few works have been done on neutrosophic differential equations. However, there are almost no work has been done on nth-order differential equations for neutrosophic numbers of the form $\eta_1 + \eta_2 I$, where $\eta_1, \eta_2 \in R$, and I represent literal indeterminacy (I- neutrosophic structure). Therefore, there is a lot of scope and opportunity to work in this area. So, to proceed in this direction, we must first define the nth-order derivative of a neutrosophic functions by using the first-order neutrosophic derivative, which has been previously done. Also, we must define the first-order neutrosophic differential equation to discuss the n-th order differential equation and the existence of its solution.

In our reality, many things cannot be precisely defined, and they contain an indeterminacy part. As a result, we can't find an accurate solution to such a problem by using a differential equation of classical real numbers, and this motivates us to think about similar types of development and modification in a neutrosophic environment.

This article is structured into six sections. The first section offers an introduction that provides a scientific overview of neutrosophists. It also includes information about the contributions and motivation of this article. The second section explains some basic concepts such as NRNs, the derivative of neutrosophic real functions, its rule, and the properties of the derivative. The third section introduces the n-th order neutrosophic derivative and provides examples. In the fourth section, we study the simple first and second-order neutrosophic differential equations with examples by using the concepts from classical differential equations. We also study the existence and uniqueness of a solution to the n-th order neutrosophic differential equation. In the fifth section, we mention a few applications of neutrosophic differential equations, which we plan to study in the future. Lastly, in the sixth section, we provide a conclusion to this article.

2. Preliminaries

2.1. NRN [3]

Let η be a NRN, then it takes the standard form $\eta = \eta_1 + \eta_2 I$, where $\eta_1, \eta_2 \in R$, and I represent literal indeterminacy, such that $I.0 = 0$ and $I^n = I \quad \forall n \in Z^+$.

2.2. Division of two NRNs [3]

If $\eta = \eta_1 + \eta_2 I, \quad \gamma = \gamma_1 + \gamma_2 I$ are two NRNs

Then, $\dfrac{\eta_1 + \eta_2 I}{\gamma_1 + \gamma_2 I} = \dfrac{\eta_1}{\gamma_1} + \dfrac{\gamma_1\eta_2 - \eta_1\gamma_2}{\gamma_1(\gamma_1 + \gamma_2)}.I$, provided, $\gamma_1 \neq 0$ and $\gamma_1 \neq -\gamma_2$.

2.3. The neutrosophic derivative [20]

Let $g : D_g \subseteq R \to R_g \cup \{I\}$, if

$$\lim_{h+h_0 I \to 0+0I} \frac{g(s + h + h_0 I) - g(s, I)}{h + h_0 I}$$

exist, then we say that the function $g(s, I)$ is differentiable w.r.t. s and is given by

$$g'(s, I) = \lim_{h+h_0 I \to 0+0I} \frac{g(s + h + h_0 I) - g(s, I)}{h + h_0 I}$$

Where $h + h_0 I$ is the number of small indetermined changes in s, and $h, h_0 \in R$, while I is Indeterminacy.

2.3.1. Some rules of neutrosophic derivatives [20]

By the above definition given in 2.3, we can prove the following formulas:

(1) $\dfrac{d}{ds}(\gamma_1 + \gamma_2 I) = 0 + 0I$; where $\gamma_1, \gamma_2 \in R$.

(2) $\dfrac{d}{ds}[(\eta_1 + \eta_2 I)s + (\gamma_1 + \gamma_2 I)] = \eta_1 + \eta_2 I$; where $\eta_1, \eta_2\gamma_1, \gamma_2 \in R$.

(3) $\dfrac{d}{ds}[(\eta_1 + \eta_2 I)s^n] = n(\eta_1 + \eta_2 I)s^{n-1}$; $n \in R$.

(4) $\dfrac{d}{ds}[e^{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)}] = (\eta_1 + \eta_2 I)e^{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)}$

(5) $\dfrac{d}{ds}(\eta_1+\eta_2 I)^s = (\eta_1+\eta_2 I)^s \ln(\eta_1+\eta_2 I)$; where $\gamma_1 > 0, \gamma_2 > 0$ and $I \geq 0$ or $\gamma > 0, d < 0$ and $I \leq 0$

(6) $\dfrac{d}{ds}\left[\log_{\eta_1+\eta_2 I} s\right] = \dfrac{1}{s \ln(\eta_1 + \eta_2 I)}$; where $\gamma_1 > 0, \gamma_2 > 0$ and $I \geq 0$ or $\gamma_1 > 0, \gamma_2 < 0$ and $I \leq 0$

(7) $\frac{d}{ds}[\ln((\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I))] = \frac{\eta_1+\eta_2 I}{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)}$

(8) $\frac{d}{ds}\left[\sqrt{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)}\right] = \frac{\eta_1+\eta_2 I}{2\sqrt{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)}}$

(9) $\frac{d}{ds}[\sin\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}] = (\eta_1+\eta_2 I)\cos\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}$

(10) $\frac{d}{ds}[\cos\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}] = -(\eta_1+\eta_2 I)\sin\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}$

(11) $\frac{d}{ds}[\tan\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}] = (\eta_1+\eta_2 I)\sec^2\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}$

(12) $\frac{d}{ds}[\cot\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}] = -(\eta_1+\eta_2 I)\csc^2\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}$

(13) $\frac{d}{ds}[\sec\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}] = (\eta_1+\eta_2 I)\sec\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}\tan\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}$

(14) $\frac{d}{ds}[\csc\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}] = -(\eta_1+\eta_2 I)\csc\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}\cot\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}$

2.4. **Properties of neutrosophic derivative [20]**

(1) $\frac{d}{ds}[f(s,I)\pm g(s,I)] = \frac{d}{ds}[f(s,I)]\pm\frac{d}{ds}[g(s,I)]$; where $f(s,I)$ and $g(s,I)$ are any two differentiable NF.

(2) $\frac{d}{ds}[(\gamma_1+\gamma_2 I)f(s,I)] = (\gamma_1+\gamma_2 I)\frac{d}{ds}[f(s,I)]$; where $\gamma_1,\gamma_2 \in R$.

(3) $\frac{d}{ds}[f(s,I).g(s,I)] = f(s,I)\frac{d}{ds}[g(s,I)] + [g(s,I)]\frac{d}{ds}[f(s,I)]$; where $f(s,I)$ and $g(s,I)$ are any two differentiable NF.

(4) $\frac{d}{ds}\left[\frac{f(s,I)}{g(s,I)}\right] = \frac{f(s,I)\frac{d}{ds}[g(s,I)] - [g(s,I)]\frac{d}{ds}[f(s,I)]}{[g(s,I)]^2}$; where $f(s,I)$ and $g(s,I)$ are any two differentiable NF.

(5) If $y=f(t,I)$ and $t=g(s,I)$ are two NF, then

$$\frac{dy}{ds} = \frac{dy}{dt}.\frac{dt}{ds} \Rightarrow \frac{dy}{ds} = f'(t, I).g'(s, I)$$

This is the derivative of composite NFs, which is also known as Chain rule.

Main Work

3. N-th order derivative of a NF

Let $f : D_f \subseteq R \to R_f \cup \{I\}$ and $y = f(s, I)$ be any NF. Then $\frac{dy}{ds} = f'(s, I)$ is called the first-order derivative of the NF w.r.t. s. Again if we differentiate the first-order derivative w.r.t. s, that is $\frac{d}{ds}\left(\frac{dy}{ds}\right) = f''(s, I)$ or $\frac{d^2y}{ds^2} = f''(s, I)$, then it is said to be second-order derivative of the NF. Similarly the derivative $\frac{d}{ds}\left(\frac{d^2y}{ds^2}\right) = \frac{d^3y}{ds^3}$ is called the third-order derivative of the NF and so on.

Continuing this process up to n-times, we get the nth-order derivative of the NF. That is the derivative $\frac{d^ny}{ds^n} = f^n(s, I)$ is said to be the nth order derivative of the NF.

If $y = f(s, I)$, then it's derivatives are denoted by

$$\begin{array}{cccccc} y_1 & y_2 & y_3 & \cdots & y_n & \\ \text{or } y' & y'' & y''' & \ldots & y^{(n)} & \\ \text{or } \dot{y} & \ddot{y} & \dddot{y} & \ldots & y^{(n)} & \\ \text{or } f'(s, I) & f''(s, I) & f'''(s, I) & \ldots & f^{(n)}(s, I) & \\ \text{or } \frac{dy}{ds} & \frac{d^2y}{ds^2} & \frac{d^3y}{ds^3} & \ldots & \frac{d^ny}{ds^n} & \text{and so on.} \end{array}$$

3.1. The n-th derivative of some special functions

(1) $f(s, I) = (\eta_1 + \eta_2 I)s^n$, where n is any positive integer.

Solution: We have, $f(s, I) = (\eta_1 + \eta_2 I)s^n$, then

$f'(s, I) = n.(\eta_1 + \eta_2 I).s^{n-1}$

$f''(s, I) = n.(n-1).(\eta_1 + \eta_2 I).s^{n-2}$

$f'''(s, I) = n.(n-1).(n-2).(\eta_1 + \eta_2 I).s^{n-3}$

Proceeding in this way upto n-times, we get

$f^n(s, I) = n.(n-1).(n-2).(n-(n-1)).(\eta_1 + \eta_2 I).s^{n-n}$

or $f^n(s, I) = n.(n-1).(n-2).1.(\eta_1 + \eta_2 I).1$

or $f^n(s, I) = n!.(\eta_1 + \eta_2 I).$

(2) $f(s, I) = ((\eta_1 + \eta_2 I)s + (\gamma_1 + \gamma_2 I))^m$, where m is any integer.

Solution: We have, $f(s, I) = ((\eta_1 + \eta_2 I)s + (\gamma_1 + \gamma_2 I))^m$, then

$f'(s,I) = m.(\eta_1+\eta_2 I).((\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I))^{m-1}$

$f''(s,I) = m.(m-1).(\eta_1+\eta_2 I)^2.((\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I))^{m-2}$

$f'''(s,I) = m.(m-1).(m-2).(\eta_1+\eta_2 I)^3.((\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I))^{m-3}$

Proceeding in this way upto n-times, we get

$f^n(s,I) = m.(m-1).(m-2).(m-(n-1)).(\eta_1+\eta_2 I)^n.((\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I))^{m-n}$

or $f^n(s,I) = \dfrac{m.(m-1)...(m-n+1).(m-n)...3.2.1}{(m-n)...3.2.1}.(\eta_1+\eta_2 I)^n.((\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I))^{m-n}$

or $f^n(s,I) = \dfrac{m!}{(m-n)!}.(\eta_1+\eta_2 I)^n.((\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I))^{m-n}$

(3) $f(s,I) = e^{(\eta_1+\eta_2 I)s}$

Solution: We have, $f(s,I) = e^{(\eta_1+\eta_2 I)s}$, then

$f'(s,I) = (\eta_1+\eta_2 I)e^{(\eta_1+\eta_2 I)s}$

$f''(s,I) = (\eta_1+\eta_2 I)^2 e^{(\eta_1+\eta_2 I)s}$

$f'''(s,I) = (\eta_1+\eta_2 I)^3 e^{(\eta_1+\eta_2 I)s}$

Proceeding in this way upto n-times, we get

$f^n(s,I) = (\eta_1+\eta_2 I)^n e^{(\eta_1+\eta_2 I)s}$

Note: If $f(s,I) = e^{((\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I))}$, then $f^n(s,I) = (\eta_1+\eta_2 I)^n e^{((\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I))s}$

(4) $f(s,I) = \log\left((\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\right)$

Solution: We have, $f(s,I) = \log\left((\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\right)$, then

$f'(s,I) = \dfrac{(\eta_1+\eta_2 I)}{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)} = (\eta_1+\eta_2 I)\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}^{-1}$

$f''(s,I) = (-1)(\eta_1+\eta_2 I)^2\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}^{-2}$

$f'''(s,I) = (-1)(-2)(\eta_1+\eta_2 I)^3\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}^{-3}$

or $f'''(s,I) = \dfrac{(-1)^2 2!(\eta_1+\eta_2 I)^3}{\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}^3}$

$f^{iv}(s,I) = (-1)(-2)(-3)(\eta_1+\eta_2 I)^4\{(\eta_1+\eta_2 I)s+(c+dI)\}^{-4}$

or $f^{iv}(s,I) = \dfrac{(-1)^3 3!(\eta_1+\eta_2 I)^4}{\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}^4}$

Proceeding in this way upto n-times, we get

$f^n(s,I) = \dfrac{(-1)^{n-1}(n-1)!(\eta_1+\eta_2 I)^n}{\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}^n}$

(5) $f(s,I) = \sin\left((\eta_1 + \eta_2 I)s + (\gamma_1 + \gamma_2 I)\right)$

Solution: We have, $f(s,I) = \sin\left((\eta_1 + \eta_2 I)s + (\gamma_1 + \gamma_2 I)\right)$, then

$f'(s,I) = (\eta_1 + \eta_2 I)\cos\left((\eta_1 + \eta_2 I)s + (\gamma_1 + \gamma_2 I)\right)$

or $f'(s,I) = (\eta_1 + \eta_2 I)\sin\Big(\frac{\pi}{2} + (\eta_1 + \eta_2 I)s + (\gamma_1 + \gamma_2 I)\Big)$

$f''(s,I) = (\eta_1 + \eta_2 I)^2\cos\Big(\frac{\pi}{2} + (\eta_1 + \eta_2 I)s + (\gamma_1 + \gamma_2 I)\Big)$

or $f''(s,I) = (\eta_1 + \eta_2 I)^2\sin\Big(\frac{2\pi}{2} + (\eta_1 + \eta_2 I)s + (\gamma_1 + \gamma_2 I)\Big)$

$f'''(s,I) = (\eta_1 + \eta_2 I)^3\cos\Big(\frac{2\pi}{2} + (\eta_1 + \eta_2 I)s + (\gamma_1 + \gamma_2 I)\Big)$

or $f'''(s,I) = (\eta_1 + \eta_2 I)^3\sin\Big(\frac{3\pi}{2} + (\eta_1 + \eta_2 I)s + (\gamma_1 + \gamma_2 I)\Big)$

Proceeding in this way upto n-times, we get

$f^n(s,I) = (\eta_1 + \eta_2 I)^n\sin\Big(\frac{n\pi}{2} + (\eta_1 + \eta_2 I)s + (\gamma_1 + \gamma_2 I)\Big)$

(6) $f(s,I) = \cos\left((\eta_1 + \eta_2 I)s + (\gamma_1 + \gamma_2 I)\right)$

Solution: We have, $f(s,I) = \cos\left((\eta_1 + \eta_2 I)s + (\gamma_1 + \gamma_2 I)\right)$, then

$f'(s,I) = (\eta_1 + \eta_2 I)[-\sin\left((\eta_1 + \eta_2 I)s + (\gamma_1 + \gamma_2 I)\right)]$

or $f'(s,I) = (\eta_1 + \eta_2 I)\cos\Big(\frac{\pi}{2} + (\eta_1 + \eta_2 I)s + (\gamma_1 + \gamma_2 I)\Big)$

$f''(s,I) = (\eta_1 + \eta_2 I)^2\Big[-\sin\Big(\frac{\pi}{2} + (\eta_1 + \eta_2 I)s + (\gamma_1 + \gamma_2 I)\Big)\Big]$

or $f''(s,I) = (\eta_1 + \eta_2 I)^2\cos\Big(\frac{2\pi}{2} + (\eta_1 + \eta_2 I)s + (\gamma_1 + \gamma_2 I)\Big)$

$f'''(s,I) = (\eta_1 + \eta_2 I)^3\Big[-\sin\Big(\frac{2\pi}{2} + (\eta_1 + \eta_2 I)s + (\gamma_1 + \gamma_2 I)\Big)\Big]$

or $f'''(s,I) = (\eta_1 + \eta_2 I)^3\cos\Big(\frac{3\pi}{2} + (\eta_1 + \eta_2 I)s + (\gamma_1 + \gamma_2 I)\Big)$

Proceeding in this way upto n-times, we get

$f^n(s,I) = (\eta_1 + \eta_2 I)^n\cos\Big(\frac{n\pi}{2} + (\eta_1 + \eta_2 I)s + (\gamma_1 + \gamma_2 I)\Big)$

(7) $f(s,I) = e^{(\eta_1+\eta_2 I)s}\sin\{(\gamma_1 + \gamma_2 I)s\}$

Solution: We have, $f(s,I) = e^{(\eta_1+\eta_2 I)s}\sin\{(\gamma_1 + \gamma_2 I)s\}$, then

$f'(s,I) = e^{(\eta_1+\eta_2 I)s}(\eta_1 + \eta_2 I)\sin\{(\gamma_1 + \gamma_2 I)s\} + e^{(\eta_1+\eta_2 I)s}\cos\{(\gamma_1 + \gamma_2 I)s\}(\gamma_1 + \gamma_2 I)$

$\Rightarrow f'(s,I) = e^{(\eta_1+\eta_2 I)s}[(a\eta_1 + \eta_2 I)\sin\{(\gamma_1 + \gamma_2 I)s\} + (\gamma_1 + \gamma_2 I)\cos\{(\gamma_1 + \gamma_2 I)s\}]$

Let $\eta_1 + \eta_2 I = r\cos\Theta, \quad \gamma_1 + \gamma_2 I = r\sin\Theta$, where $r = r_1 + r_2 I$ and $\Theta = \theta + I$, then

$r = \left[(\eta_1 + \eta_2 I)^2 + (\gamma_1 + \gamma_2 I)^2\right]^{\frac{1}{2}}$

$\Rightarrow r = \left[({\eta_1}^2 + {\gamma_1}^2) + \{{\eta_2}^2 + {\gamma_2}^2 + 2(\eta_1\eta_2 + \gamma_1\gamma_2)\}I\right]^{\frac{1}{2}}$

$\Rightarrow r = \left(u_1 + u_2 I\right)^{\frac{1}{2}}$, where $u_1 = ({\eta_1}^2 + {\gamma_1}^2)$ and $u_2 = {\eta_2}^2 + {\gamma_2}^2 + 2(\eta_1\eta_2 + \gamma_1\gamma_2)$

and $\Theta = tan^{-1}\dfrac{\gamma_1 + \gamma_2 I}{\eta_1 + \eta_2 I}$

$\Rightarrow \Theta = tan^{-1}\dfrac{\gamma_1}{\eta_1} + \dfrac{\eta_1\gamma_2 - \eta_2\gamma_1}{\eta_1(\eta_1 + \eta_2)}.I$

Thus, we get

$f'(s, I) = e^{(\eta_1+\eta_2 I)s}[r\cos\Theta\sin\{(\gamma_1 + \gamma_2 I)s\} + r\sin\Theta\cos\{(\gamma_1 + \gamma_2 I)s\}]$

$\Rightarrow f'(s, I) = re^{(\eta_1+\eta_2 I)s}\sin[(\gamma_1 + \gamma_2 I)s + \Theta]$

Again differentiating the above equation with respect to s, we get,

$f''(s, I) = r\Big[e^{(\eta_1+\eta_2 I)s}(\eta_1 + \eta_2 I)\sin\{(\gamma_1 + \gamma_2 I)s + \Theta\} + e^{(\eta_1+\eta_2 I)s}(\gamma_1 + \gamma_2 I)\cos\{(\gamma_1 + \gamma_2 I)s + \Theta\}\Big]$

$\Rightarrow f''(s, I) = re^{(\eta_1+\eta_2 I)s}\Big[(\eta_1+\eta_2 I)\sin\{(\gamma_1+\gamma_2 I)s+\Theta\}+(\gamma_1+\gamma_2 I)\cos\{(\gamma_1+\gamma_2 I)s+\Theta\}\Big]$

$\Rightarrow f''(s, I) = re^{(\eta_1+\eta_2 I)s}\Big[r\cos\Theta\sin\{(\gamma_1 + \gamma_2 I)s + \Theta\} + r\sin\Theta\cos\{(\gamma_1 + \gamma_2 I)s + \Theta\}\Big]$

$\Rightarrow f''(s, I) = r^2 e^{(\eta_1+\eta_2 I)s}\sin\Big[(\gamma_1 + \gamma_2 I)s + 2\Theta\Big]$

Similarly,

$f'''(s, I) = r^3 e^{(\eta_1+\eta_2 I)s}\sin\Big[(\gamma_1 + \gamma_2 I)s + 3\Theta\Big]$

Proceeding in this way upto n-times, we get

$f^n(s, I) = r^n e^{(\eta_1+\eta_2 I)s}\sin\Big[(\gamma_1 + \gamma_2 I)s + n\Theta\Big]$

$f^n(s, I) = \Big((\eta_1 + \eta_2 I)^2 + (\gamma_1 + \gamma_2 I)^2\Big)^{\frac{n}{2}}.e^{(\eta_1+\eta_2 I)s}\sin\Big[(\gamma_1 + \gamma_2 I)s + n\tan^{-1}\dfrac{\gamma_1 + \gamma_2 I}{\eta_1 + \eta_2 I}\Big]$

or, $f^n(s, I) = \left(u_1 + u_2 I\right)^{\frac{n}{2}}.e^{(\eta_1+\eta_2 I)s}\sin\Big[(\gamma_1 + \gamma_2 I)s + n\tan^{-1}\dfrac{\gamma_1}{\eta_1} + \dfrac{\eta_1\gamma_2 - \eta_2\gamma_1}{\eta_1(\eta_1 + \eta_2)}.I\Big]$

(8) If $f(s, I) = e^{(\eta_1+\eta_2 I)s}\cos\{(\gamma_1 + \gamma_2 I)s\}$, then

$f^n(s, I) = \Big((\eta_1 + \eta_2 I)^2 + (\gamma_1 + \gamma_2 I)^2\Big)^{\frac{n}{2}}e^{(\eta_1+\eta_2 I)s}\cos\Big[(\gamma_1 + \gamma_2 I)s + n\tan^{-1}\dfrac{\gamma_1 + \gamma_2 I}{\eta_1 + \eta_2 I}\Big]$

or, $f^n(s, I) = \left(u_1 + u_2 I\right)^{\frac{n}{2}}.e^{(\eta_1+\eta_2 I)s}\cos\Big[(\gamma_1 + \gamma_2 I)s + n\tan^{-1}\dfrac{\gamma_1}{\eta_1} + \dfrac{\eta_1\gamma_2 - \eta_2\gamma_1}{\eta_1(\eta_1 + \eta_2)}.I\Big]$

Where, $u_1 = ({\eta_1}^2 + {\gamma_1}^2)$ and $u_2 = {\eta_2}^2 + {\gamma_2}^2 + 2(\eta_1\eta_2 + \gamma_1\gamma_2)$

(9) If $f(s, I) = e^{(\eta_1+\eta_2 I)s}\sin\{(\gamma_1 + \gamma_2 I)s + (\zeta_1 + \zeta_2 I)\}$, then

$f^n(s, I) = \Big((\eta_1 + \eta_2 I)^2 + (\gamma_1 + \gamma_2 I)^2\Big)^{\frac{n}{2}}e^{(\eta_1+\eta_2 I)s}\sin\Big[(\gamma_1 + \gamma_2 I)s + (\zeta_1 + \zeta_2 I) +$

$n\tan^{-1}\frac{\gamma_1+\gamma_2 I}{\eta_1+\eta_2 I}\Big]$

or, $f^n(s,I) = \left(u_1+u_2 I\right)^{\frac{n}{2}}.e^{(\eta_1+\eta_2 I)s}\sin\Big[(\gamma_1+\gamma_2 I)s+(\zeta_1+\zeta_2 I)+n\tan^{-1}\Big(\frac{\gamma_1}{\eta_1}+\frac{\eta_1\gamma_2-\eta_2\gamma_1}{\eta_1(\eta_1+\eta_2)}.I\Big)\Big]$

Where, $u_1=({\eta_1}^2+{\gamma_1}^2)$ and $u_2={\eta_2}^2+{\gamma_2}^2+2(\eta_1\eta_2+\gamma_1\gamma_2)$

(10) If $f(s,I)=e^{(\eta_1+\eta_2 I)s}\cos\{(\gamma_1+\gamma_2 I)s+(\zeta_1+\zeta_2 I)\}$, then

$f^n(s,I) = \Big((\eta_1+\eta_2 I)^2+(\gamma_1+\gamma_2 I)^2\Big)^{\frac{n}{2}}e^{(\eta_1+\eta_2 I)s}\cos\Big[(\gamma_1+\gamma_2 I)s+(\zeta_1+\zeta_2 I)+n\tan^{-1}\frac{\gamma_1+\gamma_2 I}{\eta_1+\eta_2 I}\Big]$

or, $f^n(s,I) = \left(u_1+u_2 I\right)^{\frac{n}{2}}.e^{(\eta_1+\eta_2 I)s}\cos\Big[(\gamma_1+\gamma_2 I)s+(\zeta_1+\zeta_2 I)+n\tan^{-1}\Big(\frac{\gamma_1}{\eta_1}+\frac{\eta_1\gamma_2-\eta_2\gamma_1}{\eta_1(\eta_1+\eta_2)}.I\Big)\Big]$

Where, $u_1=({\eta_1}^2+{\gamma_1}^2)$ and $u_2={\eta_2}^2+{\gamma_2}^2+2(\eta_1\eta_2+\gamma_1\gamma_2)$

4. **N-th order Neutrosophic differential equation**

Definition 4.1 The equation which includes the first-order neutrosophic derivative as its highest derivative is called the first-order neutrosophic differential equation and is represented by

$$y'(s,I)=g(s,y(s,I))$$

In other way, an equation of the form $F(s,y(s,I),y'(s,I))=0$ is called a first-order neutrosophic differential equation and the function $f(s,I)$ that satisfies $F(s,y(s,I),y'(s,I))=0$ for every value of s is called its solution.

Similarly, the equation that includes the second-order neutrosophic derivative as its highest derivative is called second-order neutrosophic differential equation and is represented by

$$y''(s,I)=g(s,y(s,I),y'(s,I))$$
$$\text{or} \qquad F(s,y(s,I),y'(s,I),y''(s,I))=0$$

In the same way, we can define the n-th order neutrosophic differential equation by

$$y^n(s,I)=g(s,y(s,I),y'(s,I),y''(s,I),...,y^{n-1}(s,I))$$
$$\text{or} \qquad F(s,y(s,I),y'(s,I),y''(s,I),...,y^n(s,I))=0$$

Where $y(s,I)$ is a NF of s and I. Here, y^j denotes the j^{th} derivative of y w.r.t. s, i.e. $y^j=\frac{d^j y}{ds^j}$, j=0,1,2,...,n.

If $y(s_0, I) = l_1, \quad y'(s_0, I) = l_2, \quad y''(s_0, I) = l_3, \quad ..., \quad y^{n-1}(s_0, I) = l_n$ are the given initial values, where $l_j \quad ; \quad j = 1, 2, ..., n$ are the NRNs and are of the form $l_j = l_{j1} + l_{j2}I$, then the Cauchy problem of n-th order is given by

$$\begin{aligned} &y^n(s, I) = g(s, y(s, I), y'(s, I), y''(s, I), ..., y^{n-1}(s, I)); \\ &y(s_0, I) = l_1, \quad y'(s_0, I) = l_2, \quad y''(s_0, I) = l_3, ..., y^{n-1}(s_0, I) = l_n. \end{aligned} \tag{1}$$

In this section, we studied the existence and uniqueness of solutions to the n-th order neutrosophic differential equation under the conditions of contraction mapping. For this, we define the metric in definition 4.2.

Example 4.1.1 $y' = \sqrt{7}s = (2.6457.....)s$ is a 1st-order classical differential equation, but since $\sqrt{7}$ has infinitely many decimals, we cannot work with this exact number in our real life. Hence, we need to approximate this number, and we may write it as

$$y'(s, I) = (2 + I)s; \text{ where } I = (0.6, 0.7)$$

Which becomes first-order neutrosophic differential equation and we can find its solution by integrating both sides.

So, integrating both sides, we get

$$\begin{aligned} y(s, I) &= (2 + I)\frac{s^2}{2} + C \\ \Rightarrow y(s, I) &= \Big(\frac{2 + I}{2}\Big)s^2 + C \\ \Rightarrow y(s, I) &= \Big(\frac{2}{2} + \frac{2 \times 1 - 2 \times 0}{2(2 + 0)}.I\Big)s^2 + C \\ \Rightarrow y(s, I) &= \Big(1 + \frac{1}{2}.I\Big)s^2 + C \\ \Rightarrow y(s, I) &= (1 + 0.5I)s^2 + C \end{aligned}$$

Where C is indeterminate real constant.

Since $I = (0.6, 0.7)$, and let $C = 10 = 10 + 0.I$, then, the solution of the given neutrosophic differential equation $y'(s, I) = (2 + I)s$ is bounded by the lines marked by red and blue colors in the figure 1.

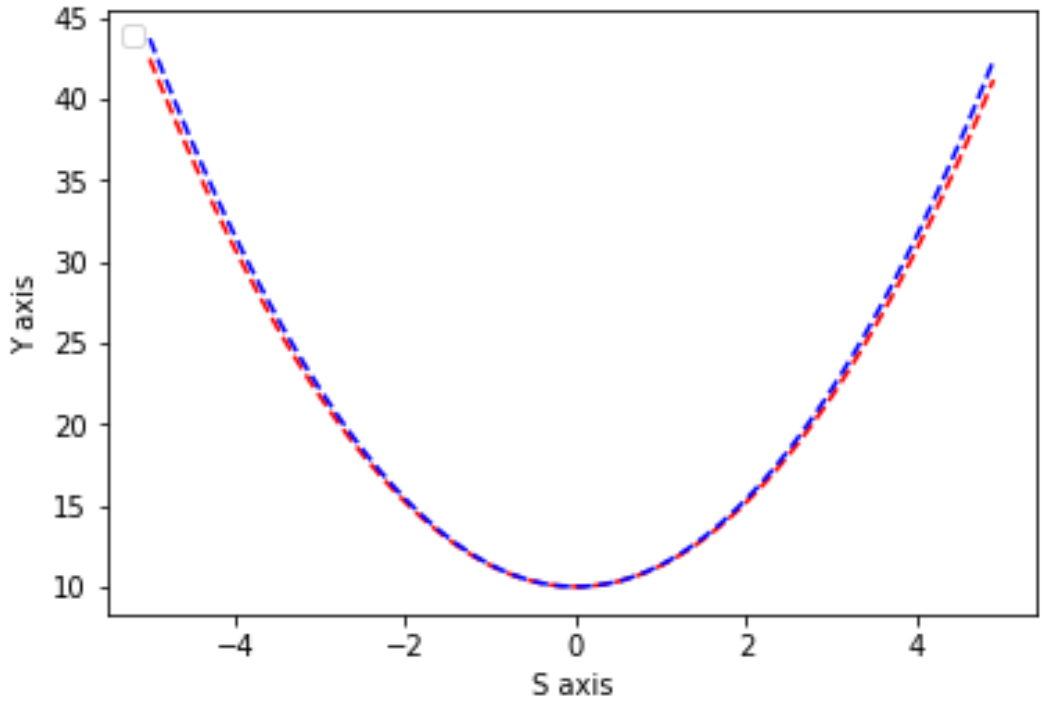


FIGURE 1. Graph of $y(s, I) = (1 + 0.5I)s^2 + C$

Example 4.1.2 Let us consider the second-order neutrosophic differential equation $y''(s, I) = 24Is$, which is directly integrable, so its solution can be found by integrating two times.

$$y'(s, I) = 24\frac{s^2}{2}I + A$$

$$\Rightarrow y'(s, I) = 12Is^2 + A$$

$$\Rightarrow y(s, I) = 4Is^3 + As + B$$

where A and B are indeterminate real constants.

Since, there is indeterminacy in the solution of the given differential equations, so, the graphical representation can only be shown approximately through dot lines, as illustrated in the figure 2 below.

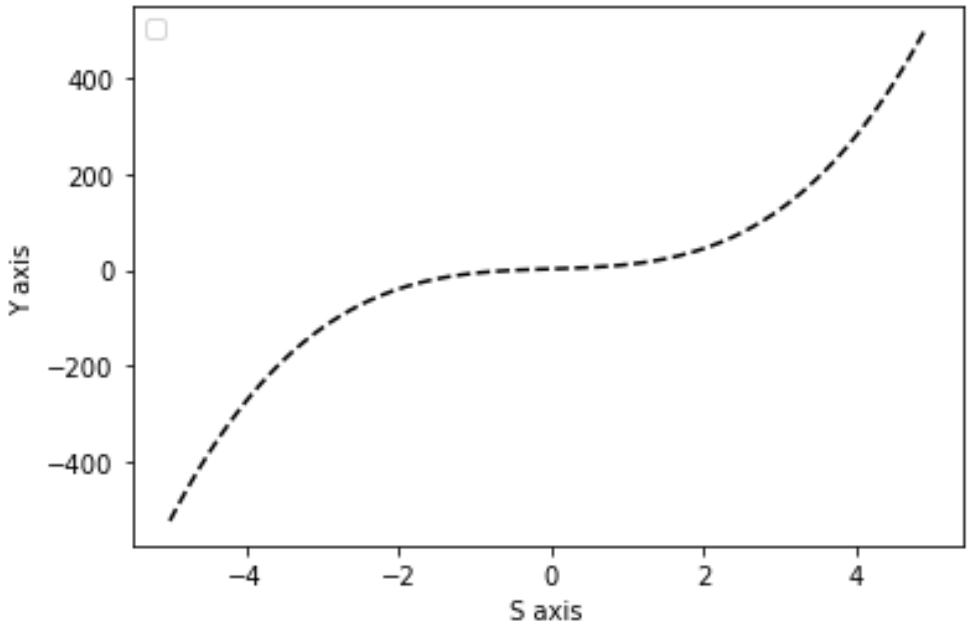


FIGURE 2. Graph of $y(s, I) = 4Is^3 + As + B$

Definition 4.2 The metric d can be defined on R_N^n (Set of all ordered n tuples of NRN) by

$$d(x_1 + x_2I, y_1 + y_2I) = \max_{1\le k\le n} |(x_{j1} + x_{j2}I) - (y_{j1} + y_{j2}I)|;$$
$$x_1 + x_2I, y_1 + y_2I \in R_N^n \quad ; \quad x_1, x_2, y_1, y_2 \in R$$

Let us denote the space of continuous function $x = x_1 + x_2I : J = [s_0, S] \to R_N^n$ by $C(J, R_N^n)$ and it is a CMS with the distance

$$D(x_1 + x_2I, y_1 + y_2I) = \sup_{s\in J} \left[d\{(x_1 + x_2I)(s), (y_1 + y_2I)(s)\}e^{-\rho s}\right]$$

Where $\rho \in R$ is fixed. For $x_1 + x_2I, y_1 + y_2I \in C^n(J, R_N^n)$, we consider the distance

$D_n(x_1 + x_2I, y_1 + y_2I) = D\left[x_1 + x_2I, y_1 + y_2I\right] + D\left[(x_1 + x_2I)', (y_1 + y_2I)'\right] + ... + D\left[(x_1 + x_2I)^n, (y_1 + y_2I)^n\right]$

or $D_n(x_1 + x_2I, y_1 + y_2I) = \sum_{j=0}^{n} D\left[(x_1 + x_2I)^j, (y_1 + y_2I)^j\right]$

Lemma 4.1 $(C^n(J, R_N^n), D_n)$ is a CMS.

Proof: Let $\{x_a\}_{a=1}^{\infty} = \{x_{a1} + x_{a2}I\}_{a=1}^{\infty} \subset C^n(J, R_N^n)$ be any cauchy sequence in $(C^n(J, R_N^n), D_n)$. Then

$D_n(x_{a1} + x_{a2}I, x_{b1} + x_{b2}I) = D(x_{a1} + x_{a2}I, x_{b1} + x_{b2}I) + D\left[(x_{a1} + x_{a2}I)', (x_{b1} + x_{b2}I)'\right] + ... + D\left[(x_{a1} + x_{a2}I)^n, (x_{b1} + x_{b2}I)^n\right] \to 0$ as $a, b \to +\infty$

This implies

$D(x_{a1}+x_{a2}I, x_{b1}+x_{b2}I) \to 0, D\big[(x_{a1}+x_{a2}I)', (x_{b1}+x_{b2}I)'\big] \to 0, ..., D\big[(x_{a1}+x_{a2}I)^n, (x_{b1}+x_{b2}I)^n\big] \to 0$

This shows that $\{x_{a1}+x_{a2}I\}_{a=1}^{\infty}, \{(x_{a1}+x_{a2}I)'\}_{a=1}^{\infty}, ..., \{(x_{a1}+x_{a2}I)^n\}_{a=1}^{\infty}$ are the cauchy sequence in the CMS $(C(J,R_N^n), D)$. So, $\exists \quad x_{01}+x_{02}I, x_{11}+x_{12}I, x_{21}+x_{22}I, ..., x_{n1}+x_{n2}I \in C(J,R_N^n)$ such that $\{x_{a1}+x_{a2}I\} \to x_{01}+x_{02}I, \{(x_{a1}+x_{a2}I)'\} \to x_{11}+x_{12}I, ..., \{(x_{a1}+x_{a2}I)^n\} \to x_{n1}+x_{n2}I$ as $a \to +\infty$.

We have to show that $x_{01}+x_{02}I \in C^n(J,R_N^n)$ and $(x_{01}+x_{02}I)' = x_{11}+x_{12}I, (x_{01}+x_{02}I)'' = x_{21}+x_{22}I, ..., (x_{01}+x_{02}I)^n = x_{n1}+x_{n2}I$.

In this case, we have

$D_n(x_{a1}+x_{a2}I, x_{01}+x_{02}I) = D(x_{a1}+x_{a2}I, x_{01}+x_{02}I) + D\big[(x_{a1}+x_{a2}I)', (x_{01}+x_{02}I)'\big] + ... + D\big[(x_{a1}+x_{a2}I)^n, (x_{01}+x_{02}I)^n\big]$.

$\Rightarrow D_n(x_{a1}+x_{a2}I, x_{01}+x_{02}I) = D(x_{a1}+x_{a2}I, x_{01}+x_{02}I) + D\big[(x_{a1}+x_{a2}I)', x_{11}+x_{12}I\big] + ... + D\big[(x_{a1}+x_{a2}I)^n, x_{n1}+x_{n2}I\big] \to 0$ as $a \to +\infty$.

This shows that $\{x_{a1}+x_{a2}I\} \to x_{01}+x_{02}I \in C^n(J,R_N^n)$ and hence $(C^n(J,R_N^n), D_n)$ is a CMS.

Theorem 4.1. *Let $f : [s_0,S] \times R_N^n \times R_N^n \times ... \times R_N^n \to R_N^n$ is continuous, then a mapping $y : [s_0,S] \to R_N^n$ is a solution of the IVP (1) iff $y \in C^n(J,R_N^n)$ satisfy the following integral equation for all $u_1 \in [s_0,S]$:*

$$y(u_1,I) = l_1 + l_2(u_1-s_0) + (l_3\int_{s_0}^{u_1}(u_2-s_0)du_2 + l_4\int_{s_0}^{u_1}\int_{s_0}^{u_2}(u_3-s_0)du_3du_2 + ... + l_n\int_{s_0}^{u_1} ... \int_{s_0}^{u_{n-2}}(u_{n-1}-s_0)du_{n-1}...du_2 + \int_{s_0}^{u_1}\int_{s_0}^{u_2}...\int_{s_0}^{u_n} f((s,I), y(s,I), ..., y^{n-1}(s,I))dsdu_n...du_2) \tag{2}$$

Proof: *We have*

$$y^n(s,I) = g(s, y(s,I), y'(s,I), y''(s,I), ..., y^{n-1}(s,I))$$

For $u_n \in [s_0,S]$, we have

$y^n(u_n,I) = g(u_n, y(u_n,I), y'(u_n,I), ...y^{n-1}(u_n,I))$

$\Rightarrow \int_{s_0}^{u_n} y^n(u_n,I) = \int_{s_0}^{u_n} g(u_n, y(u_n,I), y'(u_n,I), ...y^{n-1}(u_n,I))$

$\Rightarrow \Big[y^{n-1}(s,I)\Big]_{s_0}^{u_n} = \int_{s_0}^{u_n} g(s, y(s,I), y'(s,I), ...y^{n-1}(s,I))ds$

$\Rightarrow y^{n-1}(u_n,I) = \int_{s_0}^{u_n} g(s, y(s,I), y'(s,I), ...y^{n-1}(s,I))ds + y^{n-1}(s_0,I)$

$\Rightarrow y^{n-1}(u_n,I) = l_n + \int_{s_0}^{u_n} g(s, y(s,I), y'(s,I), ...y^{n-1}(s,I))ds$

Integrating above equality from s_0 to $u_{n-1} \in [s_0,S]$ w.r.t. u_n, we get

$$y^{n-2}(u_{n-1},I) = l_{n-1} + l_n(u_{n-1}-s_0) + \int_{s_0}^{u_{n-1}} \int_{s_0}^{u_n} g(s,y(s,I),y'(s,I),...y^{n-1}(s,I))dsdu_n$$

Proceeding in this way for $u_2 \in [s_0,S]$, *we get*

$$y'(u_2,I) = l_2 + l_3(u_2-s_0) + l_4\int_{s_0}^{u_2}(u_3-s_0)du_3 + ... + l_n\int_{s_0}^{u_2}...\int_{s_0}^{u_{n-2}}(u_{n-1}-s_0)du_{n-1}...du_2 + \int_{s_0}^{u_2}...\int_{s_0}^{u_n} f(s,y(s,I),...,y^{n-1}(s,I))dsdu_n...du_2$$

Now, integrating the above equality from s_0 *to* $u_1 \in [s_0,S]$ *w.r.t.* u_2, *we get*

$$y(u_1,I) = l_1 + l_2(u_1-s_0) + l_3\int_{s_0}^{u_1}(u_2-s_0)du_2 + l_4\int_{s_0}^{u_1}\int_{s_0}^{u_2}(u_3-s_0)du_3du_2 + ... + l_n\int_{s_0}^{u_1}...\int_{s_0}^{u_{n-2}} (u_{n-1}-s_0)du_{n-1}...du_2 + \int_{s_0}^{u_1}\int_{s_0}^{u_2}...\int_{s_0}^{u_n} f(s,y(s,I),...,y^{n-1}(s,I))dsdu_n...du_2))$$

Definition 4.3 *[17] Let us introduce the following function defined on* $[s_0,S]$ *by*

$$\Phi_{\rho,0(u_n)} = \int_{s_0}^{u_n} e^{\rho s}ds$$

$$\Phi_{\rho,1(u_{n-i})} = \int_{s_0}^{u_{n-1}}\int_{s_0}^{u_n} e^{\rho s}dsdu_n$$

Similarly, for $2 \le i \le n-1$

$$\Phi_{\rho,i(u_{n-i})} = \int_{s_0}^{u_{n-j}}\int_{s_0}^{u_{n-j+1}}...\int_{s_0}^{u_n} e^{\rho s}dsdu_n...du_{n-i+1}$$

Theorem 4.2. *Let* $f : [s_0,S] \times R_N^n \times R_N^n \times ... \times R_N^n \to R_N^n$ *be continuous and suppose that* $\exists \quad G_1,G_2,...,G_n > 0$ *such that*

$$d(f(s,x_1,x_2,...,x_n),f(s,y_1,y_2,...,y_n)) \le \sum_{j=1}^{n} G_j d_\infty(x_j,y_j) \tag{3}$$

$\forall s \in [s_0,S], \quad x_j,y_j \in R_N^n, \quad j = 1,2,...,n$. *Then the IVP* (1) *has a unique solution.*

Proof: *Let us consider the CMS* $\left(C^n(J,R_N^n),D_n\right)$ *and* $J = [s_0,S]$

Let us define the operator $B_{n-1} : C^{n-1}(I,R_N^n) \to C^{n-1}(I,R_N^n) \quad by \quad x \to B_{n-1}x$

Let $B_{n-1}x(u_1) = l_1 + l_2(u_1-s_0) + l_3\int_{s_0}^{u_1}(u_2-s_0)du_2 + l_4\int_{s_0}^{u_1}\int_{s_0}^{u_2}(u_3-s_0)du_3du_2 + ... + l_n\int_{s_0}^{u_1}...\int_{s_0}^{u_{n-2}}(u_{n-1}-s_0)du_{n-1}...du_2 + \int_{s_0}^{u_1}\int_{s_0}^{u_2}...\int_{s_0}^{u_n} f(s,x(s,I),...,x^{n-1}(s,I))dsdu_n...du_2))$

To prove the theorem, it is sufficient to show that B_{n-1} *is a contraction mapping for* $\rho > 0$.

Now, we have

$$D_{n-1}(B_{n-1}x,B_{n-1}y) = \sum_{j=0}^{n-1} D\left((B_{n-1}x)^j,(B_{n-1}y)^j\right)$$

$$= \sup_{u_1\in J}\Big[d\{(l_1 + l_2(u_1-s_0) + l_3\int_{s_0}^{u_1}(u_2-s_0)du_2 + l_4\int_{s_0}^{u_1}\int_{s_0}^{u_2}(u_3-s_0)du_3du_2 + ... + l_n\int_{s_0}^{u_1}$$

$$... \int_{s_0}^{u_{n-2}} (u_{n-1} - s_0) du_{n-1} ... du_2 + \int_{s_0}^{u_1} \int_{s_0}^{u_2} ... \int_{s_0}^{u_n} f(s, x(s, I), ..., x^{n-1}(s, I)) ds du_n ... du_2),$$
$$(l_1 + l_2(u_1 - s_0) + l_3 \int_{s_0}^{u_1} (u_2 - s_0) du_2 + l_4 \int_{s_0}^{u_1} \int_{s_0}^{u_2} (u_3 - s_0) du_3 du_2 + ... + l_n \int_{s_0}^{u_1} ... \int_{s_0}^{u_{n-2}}$$
$$(u_{n-1} - s_0) du_{n-1} ... du_2 + \int_{s_0}^{u_1} \int_{s_0}^{u_2} ... \int_{s_0}^{u_n} f(s, y(s, I), ..., y^{n-1}(s, I)) ds du_n ... du_2)\} e^{-\rho u_1} \Big]$$
$$+ \sup_{u_2 \in J} \Big[d\{(l_2 + l_3(u_2 - s_0) du_2 + l_4 \int_{s_0}^{u_2} du_3 + ... + l_n \int_{s_0}^{u_2} ... \int_{s_0}^{u_{n-2}} (u_{n-1} - s_0) du_{n-1} ... du_3 +$$
$$\int_{s_0}^{u_2} ... \int_{s_0}^{u_n} f(s, x(s, I), ..., x^{n-1}(s, I)) ds du_n ... du_3), (l_2 + l_3(u_2 - s_0) du_2 + l_4 \int_{s_0}^{u_2} du_3 + ...$$
$$+ l_n \int_{s_0}^{u_2} ... \int_{s_0}^{u_{n-2}} (u_{n-1} - s_0) du_{n-1} ... du_3 + \int_{s_0}^{u_2} ... \int_{s_0}^{u_n} f(s, y(s, I), ..., y^{n-1}(s, I)) ds du_n ... du_3)$$
$$\}e^{-\rho u_2} \Big] + ... + \sup_{u_n \in J} \Big[d\{(l_n + \int_{s_0}^{u_n} f(s, x(s, I), ..., x^{n-1}(s, I)) ds), (l_n + \int_{s_0}^{u_n} f(s, y(s, I), ...,$$
$$y^{n-1}(s, I)) ds)\} e^{-\rho u_n} \Big]$$

$$= \sup_{u_1 \in J} \Big[\Big\{ \max_{1 \le k \le n} |(l_1 + l_2(u_1 - s_0) + l_3 \int_{s_0}^{u_1} (u_2 - s_0) du_2 + l_4 \int_{s_0}^{u_1} \int_{s_0}^{u_2} (u_3 - s_0) du_3 du_2 + ... +$$
$$l_n \int_{s_0}^{u_1} ... \int_{s_0}^{u_{n-2}} (u_{n-1} - s_0) du_{n-1} ... du_2 + \int_{s_0}^{u_1} \int_{s_0}^{u_2} ... \int_{s_0}^{u_n} f(s, x(s, I), ..., x^{n-1}(s, I)) ds du_n ...$$
$$du_2)| - |(l_1 + l_2(u_1 - s_0) + l_3 \int_{s_0}^{u_1} (u_2 - s_0) du_2 + l_4 \int_{s_0}^{u_1} \int_{s_0}^{u_2} (u_3 - s_0) du_3 du_2 + ... + l_n \int_{s_0}^{u_1}$$
$$... \int_{s_0}^{u_{n-2}} (u_{n-1} - s_0) du_{n-1} ... du_2 + \int_{s_0}^{u_1} \int_{s_0}^{u_2} ... \int_{s_0}^{u_n} f(s, y(s, I), ..., y^{n-1}(s, I)) ds du_n ... du_2)| \Big\}$$
$$e^{-\rho u_1} \Big] + \sup_{u_2 \in J} \Big[\Big\{ \max_{1 \le k \le n} |(l_2 + l_3(u_2 - s_0) + l_4 \int_{s_0}^{u_2} (u_3 - s_0) du_3 + ... + l_n \int_{s_0}^{u_2} ... \int_{s_0}^{u_{n-2}} du_{n-1} ... du_3 +$$
$$\int_{s_0}^{u_2} ... \int_{s_0}^{u_n} f(s, x(s, I), ..., x^{n-1}(s, I)) ds du_n ... du_3)| - |(l_2 + l_3(u_2 - s_0) + l_4 \int_{s_0}^{u_2} (u_3 - s_0) du_3 + ... +$$
$$l_n \int_{s_0}^{u_2} ... \int_{s_0}^{u_{n-2}} du_{n-1} ... du_3 + \int_{s_0}^{u_2} ... \int_{s_0}^{u_n} f(s, y(s, I), ..., y^{n-1}(s, I)) ds du_n ... du_3)| \Big\} e^{-\rho u_2} \Big] + ... +$$
$$\sup_{u_n \in J} \Big[\Big\{ \max_{1 \le k \le n} |l_n + \int_{s_0}^{u_n} f(s, x(s, I), ..., x^{n-1}(s, I)) ds| - |l_n + \int_{s_0}^{u_n} f(s, y(s, I), ..., y^{n-1}(s, I))$$
$$ds| \Big\} e^{-\rho u_n} \Big]$$

$$= \sup_{u_1 \in J} \Big[\Big\{ \max_{1 \le k \le n} | \int_{s_0}^{u_1} \int_{s_0}^{u_2} ... \int_{s_0}^{u_n} \{f(s, x(s, I), ..., x^{n-1}(s, I)) - f(s, y(s, I), ..., y^{n-1}(s, I))\} ds$$
$$du_n ... du_2| \Big\} e^{-\rho u_1} \Big] + \sup_{u_2 \in J} \Big[\Big\{ \max_{1 \le k \le n} | \int_{s_0}^{u_2} ... \int_{s_0}^{u_n} \{f(s, x(s, I), ..., x^{n-1}(s, I)) - f(s, y(s, I), ...,$$
$$y^{n-1}(s, I))\} ds du_n ... du_3| \Big\} e^{-\rho u_2} \Big] + ... + \sup_{u_n \in J} \Big[\Big\{ \max_{1 \le k \le n} | \int_{s_0}^{u_n} \{f(s, x(s, I), ..., x^{n-1}(s, I)) -$$
$$f(s, y(s, I), ..., y^{n-1}(s, I))\} ds| \Big\} e^{-\rho u_n} \Big]$$

$$= \sup_{u_1 \in J} \Big[\int_{s_0}^{u_1} \int_{s_0}^{u_2} ... \int_{s_0}^{u_n} ds du_n ... du_2 \Big\{ \max_{1 \le k \le n} |(f(s, x(s, I), ..., x^{n-1}(s, I)) - f(s, y(s, I), ...,$$
$$y^{n-1}(s, I)))| \Big\} e^{-\rho u_1} \Big] + \sup_{u_2 \in J} \Big[\int_{s_0}^{u_2} ... \int_{s_0}^{u_n} ds du_n ... du_3 \Big\{ \max_{1 \le k \le n} |(f(s, x(s, I), ..., x^{n-1}(s, I)) -$$
$$f(s, y(s, I), ..., y^{n-1}(s, I)))| \Big\} e^{-\rho u_2} \Big] + ... + \sup_{u_n \in J} \Big[\int_{s_0}^{u_n} ds \Big\{ \max_{1 \le k \le n} |(f(s, x(s, I), ..., x^{n-1}(s, I)) -$$

$f(s,y(s,I),...,y^{n-1}(s,I)))|\Big\}e^{-\rho u_n}\Big]$

$=$

$\sup_{u_1\in J}\Big[e^{-\rho u_1}\int_{s_0}^{u_1}...\int_{s_0}^{u_n}dsdu_n...du_2d\Big\{f(s,x(s,I),...,x^{n-1}(s,I)),f(s,y(s,I),...,y^{n-1}(s,I))\Big\}\Big]+$

$\sup_{u_2\in J}\Big[e^{-\rho u_2}\int_{s_0}^{u_2}...\int_{s_0}^{u_n}dsdu_n...du_3d\Big\{f(s,x(s,I),...,x^{n-1}(s,I)),f(s,y(s,I),...,y^{n-1}(s,I))\Big\}\Big]+$

$...+\sup_{u_n\in J}\Big[e^{-\rho u_n}\int_{s_0}^{u_n}dsd\Big\{f(s,x(s,I),...,x^{n-1}(s,I)),f(s,y(s,I),...,y^{n-1}(s,I))\Big\}\Big]$

$\leq \sup_{u_1\in J}\Big[e^{-\rho u_1}\int_{s_0}^{u_1}...\int_{s_0}^{u_n}\sum_{j=0}^{n-1}G_{j+1}d(x^j(s,I),y^j(s,I))dsdu_n...du_2\Big]+\sup_{u_2\in J}\Big[e^{-\rho u_2}\int_{s_0}^{u_2}...\int_{s_0}^{u_{n-1}}$

$\int_{s_0}^{u_n}\sum_{j=0}^{n-1}G_{j+1}d(x^j(s,I),y^j(s,I))dsdu_n...du_3\Big]+...+\sup_{u_n\in J}\Big[e^{-\rho u_n}\int_{s_0}^{u_n}\sum_{j=0}^{n-1}G_{j+1}$

$d(x^j(s,I),y^j(s,I))ds\Big]$

$\leq \max\big\{G_1,G_2,...,G_n\big\}D_{n-1}(x,y)\sum_{i=0}^{n-1}\sup_{u_{n-i}\in J}\{e^{-\rho u_{n-i}}\Phi_{i(u_{n-i})}\}$

For every continuous function F on $[a,b]$, we have

$$\int_a^y dy_n\int_a^{y_n}dy_{n-1}...\int_a^{y_2}F(y_1)dy_1=\frac{1}{(n-1)!}\int_a^y(y-t)^{n-1}F(t)dt$$

Also, we have

$$\Phi_{\rho,i(u_{n-i})}=\int_{s_0}^{u_{n-i}}\int_{s_0}^{u_{n-i+1}}...\int_{s_0}^{u_n}e^{\rho s}dsdu_n...du_{n-i+1}$$

$$\Rightarrow\Phi_{\rho,i(u_{n-i})}=\frac{1}{i!}\int_{s_0}^{u_{n-i}}(u_{n-i}-s)^ie^{\rho s}ds\quad i=0,1,...,n-1$$

Now, for $i=0,1,...,n-1$, we have

$$\sup_{u_{n-i}\in J}\{e^{-\rho u_{n-i}}\Phi_{\rho,i(u_{n-i})}\}=\frac{1}{i!}\sup_{u_{n-i}\in J}\Big\{\int_{s_0}^{u_{n-i}}(u_{n-i}-s)^ie^{\rho s}dse^{-\rho u_{n-i}}\Big\}$$

$$\Rightarrow\sup_{u_{n-i}\in J}\{e^{-\rho u_{n-i}}\Phi_{\rho,i(u_{n-i})}\}\leq\frac{1}{i!}\sup_{u_{n-i}\in J}\Big\{(S-s_0)^i\int_{s_0}^{u_{n-i}}e^{-\rho(u_{n-i}-s)}ds\Big\}$$

$$\Rightarrow\sup_{u_{n-i}\in J}\{e^{-\rho u_{n-i}}\Phi_{\rho,i(u_{n-i})}\}\leq\frac{1}{i!}\sup_{u_{n-i}\in J}\Big\{(S-s_0)^i\frac{1-e^{-\rho(u_{n-i}-s_0)}}{\rho}\Big\}$$

$$\Rightarrow\sup_{u_{n-i}\in J}\{e^{-\rho u_{n-i}}\Phi_{\rho,i(u_{n-i})}\}\leq\frac{1}{i!}(S-s_0)^i\frac{1-e^{-\rho(S-s_0)}}{\rho}\to 0\ as\ \rho\to\infty$$

So, for $\rho>0$, if we choose, $\max\big\{G_1,G_2,...,G_n\big\}\sum_{i=0}^{n-1}(b-s_0)^i\frac{1-e^{-\rho(b-s_0)}}{\rho i!}\leq 1$

Then, we get

$$D_{n-1}(B_{n-1}x,B_{n-1}y)\leq D_{n-1}(x,y)\max\big\{G_1,G_2,...,G_n\big\}\sum_{i=0}^{n-1}(b-s_0)^i\frac{1-e^{-\rho(b-s_0)}}{\rho i!}$$

This shows that D_{n-1} is a contraction mapping and hence $\exists$ a unique solution for (1).

Example 4.1 *Let us consider the following neutrosophic cauchy problem:*

$$\begin{aligned} y^n(s,I) = w_1 y(s,I) + w_2 y'(s,I) + ... + w_n y^{n-1}(s,I) + B(s) \quad s \in [s_0, S] \\ y(s_0,I) = l_1, \quad y'(s_0,I) = l_2, \quad ..., \quad y^{n-1}(s_0,I) = l_n \end{aligned} \tag{4}$$

Where $w_j, l_j; \quad j = 1, 2, ..., n$ *are NRN which are of the form* $w_{j1} + w_{j2}I$, $l_{j1} + l_{j2}I$ *respectively and* $w_{j1}, w_{j2}, l_{j1}, l_{j2} \in R$; $B \in C([s_0, S], R_N^n)$.
Let us define the function f by

$$f(s, y_1, y_2, ..., y_n) = w_1 y_1 + w_2 y_2 + ... + w_n y_n + B(s)$$

which satisfies (3).
Now,

$$\begin{aligned} & d(f(s, y_1, y_2, ..., y_n), f(s, z_1, z_2, ..., z_n)) \\ & = d(w_1 y_1 + w_2 y_2 + ... + w_n y_n + B(s), w_1 z_1 + w_2 z_2 + ... + w_n z_n + B(s)) \\ & \leq \sum_{j=1}^{n} d(w_j y_j, w_j z_j) \\ & \leq \sum_{j=1}^{n} |w_j| d(y_j, z_j) \quad \forall \quad s \in [s_0, S]; \quad y_j, z_j \in R_N^n. \end{aligned}$$

Thus from the above Theorem 4.2, we can conclude that the neutrosophic cauchy problem (4) has a unique solution in $C^n(J, R_N^n)$.

5. Application

Many scientific and engineering problems require the use of neutrosophic differential equations. These problems arise in various fields, including electronics, mechanics, medicine, communication, transportation, and the industrial sector. The equations are used to analyze the behavior of phenomena that are subject to ambiguity or uncertainty. The following are a few real-world applications of nth-order neutrosophic differential equations in different disciplines where we are planning to use our study.

(1) *In electrical engineering, nth-order neutrosophic differential equations are used to describe the behavior of circuits containing resistors, inductors, and capacitors.*
(2) *In telecommunications, nth-order neutrosophic differential equations can be used to represent how signals propagate along transmission lines, particularly for high-frequency transmissions.*
(3) *To ensure the stability and safety of structures like buildings and bridges, vibrations and oscillations are studied using nth-order neutrosophic differential equations.*
(4) *We can utilize nth-order neutrosophic differential equations to represent the motion of a mass linked to a spring and a damper and to characterize the system's response to outside influences.*

(5) *To evaluate the behavior of complex pendulum systems or multi-degree-of-freedom pendulum systems, nth-order neutrosophic differential equations are used.*

(6) *Temperature distribution is studied using nth-order neutrosophic differential equations, which are used to analyze the heat transport in materials over time.*

6. Conclusions

In this article, we defined the n-th order derivative of neutrosophic real functions with suitable examples. In addition, we have discussed the integral equation of form (2) to obtain the existence and uniqueness of the solution to the n-th order neutrosophic differential equation. This form of integral equation can be used to obtain the approximate solution to the given differential equation (1).

In the future, we are going to solve some real-life problems associated with uncertain data such as differential areas in mechanical, electrical, communication, and civil engineering fields.

Acknowledgments:

The first author acknowledges the financial support received from the University Grant Commission, New Delhi under the Scheme of the "National Fellowship and Scholarship for Higher Education of ST Students" vide award letter number 202324-NFST-ASS-00490, Dated 27th February 2024 to carry out this research work.

References

[1] Zadeh, L.A. Fuzzy sets, Information and Control 1965, 8(3), 338-353.

[2] Alhasan, Y.A. Concepts of Neutrosophic Complex Numbers, International Journal of Neutrosophic Science 2020, 8(1), 9-18.

[3] Smarandache, F. Introduction to Neutrosophic statistics, Sitech-Education Publisher 2014, 34-50.

[4] Smarandache, F.; Kandasamy, W.B.V. Finite Neutrosophic Complex Numbers, Zip Publisher, Columbus, Ohio, USA, 2011, 1-16.

[5] Smarandache, F. Neutrosophy: Neutrosophic Probability, Set, and Logic, American Research Press, Rehoboth, USA, 1998.

[6] Smarandache, F. Introduction to Neutrosophic Measure, Neutrosophic Integral, and Neutrosophic Probability, Sitech-Education Publisher, Craiova – Columbus, 2013.

[7] Smarandache, F. Neutrosophic Precalculus and Neutrosophic Calculus, Book, 2015.

[8] Smarandache, F. A Unifying Field in Logics: Neutrosophic Logic, Neutrosophy, Neutrosophic Set, Neutrosophic Probability and Statistics, 4^{th} ed.; American Research Press: Rehoboth, 2001; 90-124.

[9] Smarandache, F. Proceedings of the First International Conference on Neutrosophy, Neutrosophic Set, Neutrosophic Probability and Statistics, 2^{nd} ed.; University of New Mexico, Albuquerque 2001; 5-21.

[10] Smarandache, F. Neutrosophy and Neutrosophic Logic, First International Conference on Neutrosophy, 2002.

[11] Al- Tahan, M. Some Results on Single Valued Neutrosophic (Weak) Polygroups, International Journal of Neutrosophic Science 2020, 2(1), 38-46.

[12] Edalatpanah, S. A Direct Model for Triangular Neutrosophic Linear Programming, International Journal of Neutrosophic Science 2020, 1(1), 19-28.

[13] Chakraborty, A. A New Score Function of Pentagonal Neutrosophic Number and its Application in Networking Problem, International Journal of Neutrosophic Science 2020, 1(1), 40-51.

[14] Chakraborty, A. Application of Pentagonal Neutrosophic Number in Shortest Path Problem, International Journal of Neutrosophic Science 2020, 3(1), 21-28.

[15] Alhamido, R.K.; Ismail, M.; Smarandache, F. The polar form of a Neutrosophic Complex Number, International Journal of Neutrosophic Science 2020, 10(1), 36-44.

[16] Mondal, D.; Tudu, S.; Roy, G.; Roy, T. A Model Describing the Neutrosophic Differential Equation and its Application on Mine Safety, Neutrosophic Sets ans System 2021, 46, 386-401.

[17] Sumathi, I.R.; Sweety, C. New approach on differential equation via trapezoidal neutrosophic number, Complex & Intelligent Systems 2019, 5, 417-424.

[18] Acharya, A.; Mahanta, A.; Mukherjee, S.; Biswas, M.A.; Das, K.P.; Mondal, S.K.; Roy, B. A Neutrosophic differential equation approach for modelling glucose distribution in the bloodstream using neutrosophic sets, Decision Analytic Journal 2023, 8.

[19] Lathamaheswari, M.; Sudha, S. Bipolar Trapezoidal Neutrosophic Defferential Equation and its Application, International Journal of Neutrosophic Science 2022, 18(04), 16-43.

[20] Parikh, M.; Sahni, M.; Sahni, R. Solution of First Order Initial Value Problem using Analytical and Numerical Method in Neutrosophic Environment, Neutrosophic Sets and Systems 2022, 51.

[21] Alhasan, Y.A. The neutrosophic differential calculus, Neutrosophic Sets and systems 2022, 49, 357-374.

[22] Alhasan, Y.A. The definite neutrosophic integrals and its applications, Neutrosophic Sets and Systems 2022, 49, 276-293.

[23] Alhasan, Y.A. The neutrosophic integrals and integration methods, Neutrosophic Sets and Systems 2021, 43, 291-301.

[24] Salamah, A.; Alaswad, M.; Dallah, R. A Study of Derivative and Integration a Neutrosophiv Functions, Journal of Neutrosophic and Fuzzy Systems 2023, 05(02), 33-37.

[25] Salamah, A.; Alaswad, M.; Dallah, R. A Study of Neutrosophic Differential Equation by Using the One-Dimensional Geometry AH-Isometry, Prospects for Applied Mathimatics and data Analysis 2023, 01(01), 57-62.

[26] Alhasan, Y.A. The General Exponential form of a Neutrosophic Complex Number, International Journal of Neutrosophic Science 2020, 11(2), 100-107.

[27] Salahshour, S. Nth-order Fuzzy Differential Equations Under Generalized Differentiability, Journal of Fuzzy Set Valued Analysis 2011, 2011.

[28] Ahmed, H. The Special Neutrosophic Functions, International Journal of Neutrosophic Science 2020, 4(2), 104-116.

[29] Ibrahim, M.H.; Abdullah, G.; Mohamed, A.B.; Nabeeh, N.; Smarandache, F. Optimal selection of battery recycling plant location: strategies, challenges, perspectives, and sustainability, Soft Computing 2023.

[30] Mohamed, A.B.; Abdullah, G.; Ibrahim, M.H.; Sallam, K.M. Sustainability assessment of optimal location of electric vehicle charge stations: a conceptual frameworks for green energy into smart cities, Environment, Development and Sustainability 2023

[31] Mohamed, A.B.; Abdullah, G.; Nour, M.; Askar, S.S.; Mohamed, A. A Risk Assessment Model for Cyber-Physical Water and Wastewater Systems: Towards Sustainable Development, Sustainability 2022, 14(8).

[32] Jdid, M.; Smarandache, F.; Broumi, S. Inspection Assignment Form for Product Quality Control Using Neutrosophic Logic, Neutrosophic Systems with Applications 2023, 1, 4–13.

[33] Dhar, R. Compactness and Neutrosophic Topological Space via Grills, Neutrosophic Systems with Applications 2023, 2, 1–7.

[34] Sallam, K.M.; Ali, W.M. Neutrosophic MCDM Methodology for Evaluation Onshore Wind for Electricity Generation and Sustainability Ecological, Neutrosophic Systems with Applications 2023, 4, 53–61.

Received: June 29, 2024. Accepted: Oct 11, 2024

Neutrosophic Sets and Systems, Vol. 75, 2025
University of New Mexico

The linear 2-refined neutrosophic differential equations

Yaser Ahmad Alhasan[1,*], Issam Mohammed Abdallah Abdalaziz[2], Raja Abdullah Abdulfatah[3] and Qusay Alhassan[4]

[1]Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, KSA; y.alhasan@psau.edu.sa
[2]Deanship the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj Saudi Arabia.; I.abdalaziz@psau.edu.sa
[3]Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia; ra.raj.1960@hotmail.com
[4]Faculty of Informatics Engineering, Cordoba Private University, Syria; qusayalhassan77@gmail.com

***Corresponding author:** y.alhasan@psau.edu.sa

Abstract: In this paper, we studied the linear 2-refined neutrosophic differential equations and defined the homogeneous and non-homogeneous 2-refined neutrosophic differential equations. Also to presenting 2-refined neutrosophic differential equation of Bernoulli, which turns into a linear equation and thus facilitates its solution. Apart from talking about how to solve these equations and giving enough examples to support it.

Keywords: integral; differential equation; indeterminacy; Bernoulli; neutrosophic.

1. Introduction and Preliminaries

To describe a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, and contradiction, Smarandache suggested the neutrosophic Logic as an alternative to the current logics. Smarandache made refined neutrosophic numbers available in the following form: $(a, b_1I_1, b_2I_2, \dots, b_nI_n)$ where $a, b_1, b_2, \dots, b_n \in R \text{ or } C$ [1]

Agboola introduced the concept of refined neutrosophic algebraic structures [2]. Also, the refined neutrosophic rings I was studied in paper [3], where it assumed that I splits into two indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. It then follows logically that: [3]

$$I_1I_1 = {I_1}^2 = I_1 \quad (1)$$
$$I_2I_2 = {I_2}^2 = I_2 \quad (2)$$
$$I_1I_2 = I_2I_1 = I_1 \quad (3)$$

In addition, there are many papers presenting studies on refined neutrosophic numbers [4-5-6-7-8-12-13-14-15] and Mehmet Celik and Ahmed Hatip presented a study on the refined ah-isometry and its applications in refined neutrosophic surfaces. Smarandache discussed neutrosophic indefinite integral (Refined Indeterminacy) [11]

Alhasan also presented several papers on calculus, in which he discussed neutrosophic definite and indefinite integrals. He also presented the most important applications of definite integrals in neutrosophic logic [9-10].

2. Main Discussion

2.1 The homogeneous linear 2-refined neutrosophic differential equations

Definition 1

The general form of the homogeneous linear 2-refined neutrosophic differential equation is given by:

$$\acute{y} + f(x, I_1, I_2)y = 0$$

Whereas $f: D_f \subseteq R \to R_f \cup \{I_1, I_2\}$, while I_1, I_2 are indeterminacy.

$$\acute{y} = -f(x, I_1, I_2)y$$
$$\frac{\acute{y}}{y} = -f(x, I_1, I_2)$$

By integrating the two side, we get:

$$ln\left(\frac{y}{C}\right) = \int -f(x, I_1, I_2)\, dx$$

$$\frac{y}{C} = e^{\int -f(x, I_1, I_2) dx}$$

$$y = C\, e^{\int -f(x, I_1, I_2) dx}$$

Where $C = a_0 + b_0 I_1 + c_0 I_2$ and a_0, b_0, c_0 are real numbers, while I_1, I_2 are indeterminacy.

Example 1

Find the general solution of the following linear 2-refined neutrosophic differential equation:

$$\acute{y} - ((1 - 5I_1 + 2I_2)x + 4 + 7I_2)^3 y = 0$$

Solution:

$$\acute{y} = ((1 - 5I_1 + 2I_2)x + 4 + 7I_2)^3 y$$

$$\frac{\acute{y}}{y} = ((1 - 5I_1 + 2I_2)x + 4 + 7I_2)^3$$

By integrating the two side, we get:

$$ln\left(\frac{y}{C}\right) = \int ((1 - 5I_1 + 2I_2)x + 4 + 7I_2)^3\, dx$$

$$\frac{y}{C} = e^{\int ((1-5I_1+2I_2)x+4+7I_2)^3 dx}$$

$$y = C\, e^{\left(\frac{1}{1-5I_1+2I_2}\right)\frac{((1-5I_1+2I_2)x+4+7I_2)^4}{4}}$$

$$y = C\, e^{\left(1-\frac{5}{6}I_1-\frac{2}{3}I_2\right)\frac{((1-5I_1+2I_2)x+4+7I_2)^4}{4}}$$

Where $C = a_0 + b_0 I_1 + c_0 I_2$ and a_0, b_0, c_0 are real numbers.

Example 2
Find the general solution of the following linear 2-refined neutrosophic differential equation:

$$\acute{y} + sin((4 + 7I_1 - I_2)x - 1 + I_1 - I_2)y = 0$$

Solution:

$$\acute{y} = -sin((4 + 7I_1 - I_2)x - 1 + I_1 - I_2)y$$

$$\frac{\acute{y}}{y} = -sin((4 + 7I_1 - I_2)x - 1 + I_1 - I_2)$$

By integrating the two side, we get:

$$ln\left(\frac{y}{C}\right) = \int -sin((4 + 7I_1 - I_2)x - 1 + I_1 - I_2)\, dx$$

$$\frac{y}{C} = e^{\int -sin((4+7I_1-I_2)x-1+I_1-I_2)dx}$$

$$y = C\, e^{\left(\frac{1}{4+7I_1-I_2}\right)cos((4+7I_1-I_2)x-1+I_1-I_2)}$$

$$y = C\, e^{\left(\frac{1}{4}-\frac{7}{30}I_1+\frac{1}{12}I_2\right)cos((4+7I_1-I_2)x-1+I_1-I_2)}$$

Where $C = a_0 + b_0 I_1 + c_0 I_2$ and a_0, b_0, c_0 are real numbers.

2.2 The non-homogeneous linear 2-refined neutrosophic differential equation

Definition 2
The general form of the non-homogeneous linear 2-refined neutrosophic differential equation is given as:

$$\acute{y} + f(x, I_1, I_2)y = q(x, I_1, I_2) \quad (1)$$

Where $f: D_f \subseteq R \to R_f \cup \{I_1, I_2\}$, $q: D_q \subseteq R \to R_q \cup \{I_1, I_2\}$, and I_1, I_2 are indeterminacy.

Solution steps:

1) The complement factor of the equation (1) is:

$$\mu(x) = e^{\int f(x,I_1,I_2)dx}$$

2) By multiplying equation (1) by the complement factor, we get:

$$\mu(x)\, \acute{y} + f(x, I_1, I_2)\, \mu(x)y = q(x, I_1, I_2)\, \mu(x)$$

$$\mu(x)\, \acute{y} + f(x, I_1, I_2)e^{\int f(x,I_1,I_2)dx}\, y = q(x, I_1, I_2)\, e^{\int f(x,I_1,I_2)dx}$$

$$\left(y\, \mu(x)\right)' = q(x, I_1, I_2)\, e^{\int f(x,I_1,I_2)dx}$$

By integrating the two side, we get:

$$y = \frac{1}{\mu(x)}\left(\int q(x, I_1, I_2)\, e^{\int f(x,I_1,I_2)dx}\, dx + C\right)$$

Where $C = a_0 + b_0 I_1 + c_0 I_2$ and a_0, b_0, c_0 are real numbers, while I_1, I_2 are indeterminacy.

Example 3

Find the general solution of the following non-homogeneous linear 2-refined neutrosophic differential equation:

$$\acute{y} + \frac{(3 + I_1 - I_2)x - 7I_2}{\sqrt{(3 + I_1 - I_2)x^2 - 14xI_2}}\, y = \sqrt{(3 + I_1 - I_2)x^2 - 14xI_2}$$

Solution:

$$f(x, I_1, I_2) = \frac{(3 + I_1 - I_2)x - 7I_2}{\sqrt{(3 + I_1 - I_2)x^2 - 14xI_2}}\ , q(x, I_1, I_2) = \sqrt{(3 + I_1 - I_2)x^2 - 14xI_2}$$

The complement factor of the equation is:

$$\mu(x) = e^{\int \frac{(3+I_1-I_2)x-7I_2}{\sqrt{(3+I_1-I_2)x^2-14xI_2}}dx} = \frac{1}{2}\sqrt{(3 + I_1 - I_2)x^2 - 14xI_2}$$

By multiplying equation (*) by the complement factor, we get:

$$\mu(x)\, \acute{y} + \frac{(3 + I_1 - I_2)x - 7I_2}{\sqrt{(3 + I_1 - I_2)x^2 - 14xI_2}}\, \mu(x) y = \sqrt{(3 + I_1 - I_2)x^2 - 14xI_2}\, \mu(x)$$

$$\mu(x)\, \acute{y} + \frac{(3 + I_1 - I_2)x - 7I_2}{\sqrt{(3 + I_1 - I_2)x^2 - 14xI_2}} \left(\frac{1}{2}\sqrt{(3 + I_1 - I_2)x^2 - 14xI_2}\right) y = \sqrt{(3 + I_1 - I_2)x^2 - 14xI_2}\, \left(\frac{1}{2}\sqrt{(3 + I_1 - I_2)x^2 - 14xI_2}\right)$$

$$\big(y\, \mu(x)\big)' = \frac{1}{2}\big((3 + I_1 - I_2)x^2 - 14xI_2\big)$$

By integrating the two side, we get:

$$y\, \mu(x) = \frac{1}{2}\big((3 + I_1 - I_2)x^2 - 14xI_2\big)$$

$$y = \frac{1}{\frac{1}{2}\sqrt{(3 + I_1 - I_2)x^2 - 14xI_2}}\left(\int \frac{1}{2}\big((3 + I_1 - I_2)x^2 - 14xI_2\big)\ dx\right)$$

$$y = \frac{2}{\sqrt{(3 + I_1 - I_2)x^2 - 14xI_2}}\left(\left(\frac{1}{2} + \frac{1}{6}I_1 - \frac{1}{6}I_2\right)x^3 - 7x^2 I_2 + C\right)$$

Where $C = a_0 + b_0 I_1 + c_0 I_2$ and a_0, b_0, c_0 are real numbers.

3. The 2-refined neutrosophic differential equations that translate into linear

3.1 The Bernoulli's 2-refined neutrosophic differential equation

Definition 2

We call the equation that take the form:

$$\acute{y} + f(x, I_1, I_2)\, y = q(x, I_1, I_2)\, y^n\ ;\ n \neq 0, n \neq 1$$

The 2-refined neutrosophic differential equation of Bernoulli. Let's follow the following steps to solve it:

1) we'll first divide the differential equation by y^n to get:

$$\frac{\acute{y}}{y^n}+\frac{f(x,I_1,I_2)\,y}{y^n}=\frac{q(x,I_1,I_2)\,y^n}{y^n}$$

$$y^{-n}\,\acute{y}+f(x,I_1,I_2)\,y^{1-n}\;=q(x,I_1,I_2)\quad(*)$$

2) Assume that: $w=\;y^{1-n}\;\;\Longrightarrow\;\;\acute{w}=(1-n)\,y^{-n}\acute{y}$

$$\Longrightarrow\;\;y^{-n}\acute{y}=\frac{1}{1-n}\acute{w}$$

3) by substituting into equation $(*)$, we get:

$$\frac{1}{1-n}\acute{w}+f(x,I_1,I_2)\,w=q(x,I_1,I_2)$$

$$\acute{w}+(1-n)f(x,I_1,I_2)\,w=(1-n)q(x,I_1,I_2)\quad(**)$$

4) The complement factor of the equation $(**)$ is:

$$\mu(x)=e^{(1-n)\int f(x,I_1,I_2)dx}$$

5) By multiplying equation $(**)$by the complement factor, we get:

$$\acute{w}\,\mu(x)+(1-n)f(x,I_1,I_2)\,\mu(x)\,w\;=(1-n)\,q(x,I_1,I_2)\,\mu(x)$$

$$\acute{w}\,\mu(x)+(1-n)f(x,I_1,I_2)\,e^{(1-n)\int f(x,I_1,I_2)dx}\,w\;=(1-n)\,q(x,I_1,I_2)e^{(1-n)\int f(x,I_1,I_2)dx}$$

$$\big(w\,\mu(x)\big)'=(1-n)\,q(x,I_1,I_2)e^{(1-n)\int f(x,I_1,I_2)dx}$$

By integrating the two side, we get:

$$w\;=\frac{1}{\mu(x)}\left(\int(1-n)\,q(x,I_1,I_2)e^{(1-n)\int f(x,I_1,I_2)dx}\,dx+C\right)$$

6) Back to the primary variable y, we get:

$$y^{1-n}=\frac{1}{\mu(x)}\left(\int(1-n)\,q(x,I_1,I_2)e^{(1-n)\int f(x,I_1,I_2)dx}\,dx+C\right)$$

$$\text{Or}\quad y^{n-1}=\frac{e^{(1-n)\int f(x,I_1,I_2)dx}}{\int(1-n)\,q(x,I_1,I_2)e^{(1-n)\int f(x,I_1,I_2)dx}\,dx+C}$$

Where $C=a_0+b_0I_1+c_0I_2$ and a_0,b_0,c_0 are real numbers, while I_1,I_2 are indeterminacy.

Example 4
Find the general solution of the following Bernoulli's 2-refined neutrosophic differential equation:

$$\acute{y} + csc\big((9+11I_1+13I_2)x-11+2I_1-4I_2\big)\, cot\big((9+11I_1+13I_2)x-11+2I_1-4I_2\big)\, y = e^{\left(\frac{1}{18}-\frac{1}{132}I_1-\frac{13}{396}I_2\right)csc\left((9+11I_1+13I_2)x-11+2I_1-4I_2\right)}\sqrt{y}$$

Solution:

$$\frac{\acute{y}}{\sqrt{y}} + csc\big((9+11I_1+13I_2)x-11+2I_1-4I_2\big)\, cot\big((9+11I_1+13I_2)x-11+2I_1-4I_2\big)\, \frac{y}{\sqrt{y}} = e^{\left(\frac{1}{18}-\frac{1}{132}I_1-\frac{13}{396}I_2\right)csc\left((9+11I_1+13I_2)x-11+2I_1-4I_2\right)}$$

$$y^{-1/2}\acute{y} + csc\big((9+11I_1+13I_2)x-11+2I_1-4I_2\big)\, cot\big((9+11I_1+13I_2)x-11+2I_1-4I_2\big)\, y^{1/2} = e^{\left(\frac{1}{18}-\frac{1}{132}I_1-\frac{13}{396}I_2\right)csc\left((9+11I_1+13I_2)x-11+2I_1-4I_2\right)} \quad (*)$$

$$w = y^{1/2} \implies \acute{w} = \frac{1}{2}y^{-1/2}\acute{y}$$

$$\implies y^{-1/2}\acute{y} = 2\acute{w}$$

By substitution in (*), we get:

$$2\acute{w} + csc\big((9+11I_1+13I_2)x-11+2I_1-4I_2\big)\, cot\big((9+11I_1+13I_2)x-11+2I_1-4I_2\big)\, w = e^{\left(\frac{1}{18}-\frac{1}{132}I_1-\frac{13}{396}I_2\right)csc\left((9+11I_1+13I_2)x-11+2I_1-4I_2\right)} \quad (**)$$

The complement factor is:

$$\mu(x) = e^{(1-n)\int f(x,I_1,I_2)dx}$$

$$\mu(x) = e^{\int\frac{1}{2}csc\left((9+11I_1+13I_2)x-11+2I_1-4I_2\right)\, cot\left((9+11I_1+13I_2)x-11+2I_1-4I_2\right)dx}$$

$$= e^{-\frac{1}{2}\left(\frac{1}{9+11I_1+13I_2}\right)csc\left((9+11I_1+13I_2)x-11+2I_1-4I_2\right)}$$

$$= e^{-\frac{1}{2}\left(\frac{1}{9}-\frac{1}{66}I_1-\frac{13}{198}I_2\right)csc\left((9+11I_1+13I_2)x-11+2I_1-4I_2\right)}$$

By multiplying equation (**) by the complement factor, we get:

$$2\acute{w}\,\mu(x) + csc\big((9+11I_1+13I_2)x-11+2I_1-4I_2\big)\, cot\big((9+11I_1+13I_2)x-11+2I_1-4I_2\big)\, \mu(x)\, w = e^{\left(\frac{1}{18}-\frac{1}{132}I_1-\frac{13}{396}I_2\right)csc\left((9+11I_1+13I_2)x-11+2I_1-4I_2\right)}\, \mu(x)$$

$$\acute{w}\,\mu(x) + \frac{1}{2}csc\big((9+11I_1+13I_2)x-11+2I_1-4I_2\big)\, cot\big((9+11I_1+13I_2)x-11+2I_1-4I_2\big)\, \mu(x)\, w = \frac{1}{2}e^{\left(\frac{1}{18}-\frac{1}{132}I_1-\frac{13}{396}I_2\right)csc\left((9+11I_1+13I_2)x-11+2I_1-4I_2\right)}\, \mu(x)$$

$$\begin{aligned}&\acute{w}\,\mu(x)\\ &+\frac{1}{2}csc\big((9+11I_1+13I_2)x-11+2I_1-4I_2\big)\, cot\big((9+11I_1+13I_2)x-11+2I_1\\ &-4I_2\big)\, e^{-\frac{1}{2}\left(\frac{1}{9}-\frac{1}{66}I_1-\frac{13}{198}I_2\right)csc\left((9+11I_1+13I_2)x-11+2I_1-4I_2\right)}\, w\\ &=\frac{1}{2}e^{\left(\frac{1}{18}-\frac{1}{132}I_1-\frac{13}{396}I_2\right)csc\left((9+11I_1+13I_2)x-11+2I_1-4I_2\right)}e^{-\frac{1}{2}\left(\frac{1}{9}-\frac{1}{66}I_1-\frac{13}{198}I_2\right)csc\left((9+11I_1+13I_2)x-11+2I_1-4I_2\right)}\end{aligned}$$

$$\acute{w}\,\mu(x) + \frac{1}{2}csc\big((9+11I_1+13I_2)x-11+2I_1-4I_2\big)\,cot\big((9+11I_1+13I_2)x-11+2I_1-4I_2\big)\,e^{-\frac{1}{2}\left(\frac{1}{9}-\frac{1}{66}I_1-\frac{13}{198}I_2\right)csc\left((9+11I_1+13I_2)x-11+2I_1-4I_2\right)}\,w = \frac{1}{2}e^{\left(\frac{1}{18}-\frac{1}{132}I_1-\frac{13}{396}I_2\right)csc\left((9+11I_1+13I_2)x-11+2I_1-4I_2\right)}e^{-\left(\frac{1}{18}-\frac{1}{132}I_1-\frac{13}{396}I_2\right)csc\left((9+11I_1+13I_2)x-11+2I_1-4I_2\right)}$$

$$\acute{w}\,\mu(x) + \frac{1}{2}csc\big((9+11I_1+13I_2)x-11+2I_1-4I_2\big)\,cot\big((9+11I_1+13I_2)x-11+2I_1-4I_2\big)\,e^{-\frac{1}{2}\left(\frac{1}{9}-\frac{1}{66}I_1+\frac{13}{198}I_2\right)csc\left((9+11I_1+13I_2)x-11+2I_1-4I_2\right)}\,w = \frac{1}{2}$$

$$\big(w\,\mu(x)\big)' = \frac{1}{2}$$

By integrating the two side, we get:

$$w\,\mu(x) = \int \frac{1}{2}dx$$

$$w = \frac{1}{\mu(x)}\left(\frac{1}{2}x + C\right)$$

Back to the primary variable y, we get:

$$y^{1/2} = \frac{1}{\mu(x)}\left(\frac{1}{2}x + C\right)$$

$$y^{\frac{1}{2}} = \frac{1}{e^{-\left(\frac{1}{18}-\frac{1}{132}I_1+\frac{13}{396}I_2\right)csc\left((9+11I_1+13I_2)x-11+2I_1-4I_2\right)}}\left(\frac{1}{2}x + C\right)$$

$$\Longrightarrow \qquad y^{\frac{1}{2}} = \left(\frac{1}{2}x + C\right)e^{\left(\frac{1}{18}-\frac{1}{132}I_1+\frac{13}{396}I_2\right)csc\left((9+11I_1+13I_2)x-11+2I_1-4I_2\right)}$$

Where $C = a_0 + b_0I_1 + c_0I_2$ and a_0, b_0, c_0 are real numbers.

3.2 The 2-refined neutrosophic differential equations takes the following form

$$\acute{f}(y)\frac{dy}{dx} + m(x, I_1, I_2)f(y) = q(x, I_1, I_2) \quad (3)$$

To solve this equation, we follow the following steps:

1) Assume that: $z = f(y) \quad \Longrightarrow \quad \acute{z} = (\acute{f}(y)\frac{dy}{dx}$

By substitution into equation (3), we get:

$$\acute{z} + m(x, I_1, I_2)z = q(x, I_1, I_2) \quad (4)$$

2) The complement factor of the equation (4) is:

$$\mu(x) = e^{\int m(x,I_1,I_2)dx}$$

3) By multiplying equation (4) by the complement factor, we get:

$$\acute{z}\,\mu(x) + m(x, I_1, I_2)\mu(x)\,z = q(x, I_1, I_2)\,\mu(x)$$

$$\acute{z}\,\mu(x) + m(x, I_1, I_2)e^{\int m(x,I_1,I_2)dx}\,\mathrm{z} = q(x, I_1, I_2)\,e^{\int m(x,I_1,I_2)dx}$$

$$\big(z\,\mu(x)\big)' = q(x, I_1, I_2)\; e^{\int m(x,I_1,I_2)dx}$$

$$\big(z\,\mu(x)\big)' = q(x, I_1, I_2)\; e^{\int m(x,I_1,I_2)dx}$$

By integrating the two side, we get:

$$z = \frac{1}{\mu(x)}\left(\int q(x, I_1, I_2)\; e^{\int m(x,I_1,I_2)dx} + C\right)$$

4) Back to the primary variable y, we get:

$$f(y) = \frac{1}{\mu(x)}\left(\int q(x, I_1, I_2)\; e^{\int m(x,I_1,I_2)dx} + C\right)$$

Where $C = a_0 + b_0 I_1 + c_0 I_2$ and a_0, b_0, c_0 are real numbers.

Example 5

Find the general solution of the following neutrosophic differential equation:

$$cos\, y \frac{dy}{dx} + cot(x + 1 + 2I_1 - 3I_2)\; sin\, y = cos(x + 1 + 2I_1 - 3I_2)\;\; (*)$$

Solution:

$$z = sin\, y \quad \Longrightarrow \quad \acute{z} = cosy\, \frac{dy}{dx}$$

By substitution in (*), we get:

$$\acute{z} + cot(x + 1 + 2I_1 - 3I_2)\, z = cos(x + 1 + 2I_1 - 3I_2) \qquad (\acute{*})$$

The complement factor is:

$$\mu(x) = e^{\int cot(x+1+2I_1-3I_2)\, dx}$$

$$= e^{ln\,(sin(x+1+2I_1-3I_2))} = sin(x + 1 + 2I_1 - 3I_2)$$

By multiplying equation $(\acute{*})$ by the complement factor, we get:

$$\acute{z}\,\mu(x) + cot(x + 1 + 2I_1 - 3I_2)\,\mu(x)\, z = \mu(x)\, cos(x + 1 + 2I_1 - 3I_2)$$

$$\acute{z}\,\mu(x) + cot(x + 1 + 2I_1 - 3I_2)\, sin(x + 1 + 2I_1 - 3I_2)\, z = cos(x + 1 + 2I_1 - 3I_2)\, sin(x + 1 + 2I_1 - 3I_2)$$

$$\acute{z}\,\mu(x) + cos(x + 1 + 2I_1 - 3I_2)\, z = cos(x + 1 + 2I_1 - 3I_2)\, sin(x + 1 + 2I_1 - 3I_2)$$

$$\big(z\,\mu(x)\big)' = cos(x + 1 + 2I_1 - 3I_2)\, sin(x + 1 + 2I_1 - 3I_2)$$

By integrating the two side, we get:

$$z\,\mu(x) = \int cos(x + 1 + 2I_1 - 3I_2)\, sin(x + 1 + 2I_1 - 3I_2)dx$$

$$z\,\mu(x) = \int \frac{1}{2} sin(2x + 2 + 4I_1 - 6I_2)dx$$

$$z\,\mu(x) = \frac{-1}{4} cos(2x + 2 + 4I_1 - 6I_2) + C$$

$$z = \frac{1}{sin(x+1+2I_1-3I_2)}\left(\frac{-1}{4}cos(2x+2+4I_1-6I_2)+C\right)$$

Back to the primary variable y, we get:

$$sin\, y = \frac{1}{sin(x+1+2I_1-3I_2)}\left(\frac{-1}{4}cos(2x+2+4I_1-6I_2)+C\right)$$

Where $C = a_0 + b_0I_1 + c_0I_2$ and a_0, b_0, c_0 are real numbers.

4. Conclusions

This work is regard as a seminal study in the field of 2-refined neutrosophic differential equations. It included study types of the 2-refined neutrosophic differential equations. In addition, we discussed 2-refined neutrosophic differential equations that transform to linear. The most significant of all was the 2-refined neutrosophic differential equations Bernoulli.

Acknowledgments "This study is supported via funding from Prince sattam bin Abdulaziz University project number (PSAU/2024/R/1445)".

References

[1] Smarandache, F; "(T,I,F)- Neutrosophic Structures", Neutrosophic Sets and Systems, Volume8, PP:3-10, 2015.

[2] Agboola,A.A.A; On Refined Neutrosophic Algebraic Structures. Neutrosophic Sets and Systems, Volume10, PP: 99-101, 2015.

[3] Adeleke, E.O., Agboola, A.A.A.,and Smarandache, F; Refined Neutrosophic Rings I. International Journal of Neutrosophic Science, Volume 2, Issue 2, PP:77-81, 2020.

[4] Smarandache, F; n-Valued Refined Neutrosophic Logic and Its Applications in Physics. Progress in Physics, USA, Volume 4, PP: 143-146, 2013.

[5] Vasantha Kandasamy,W.B; Smarandache,F; Neutrosophic Rings" Hexis, Phoenix, Arizona, http://fs.gallup.unm.edu/NeutrosophicRings.pdf, 2006.

[6] Zeina, M., Abobala, M; On The Refined Neutrosophic Real Analysis Based on Refined Neutrosophic Algebraic AH-Isometry", Neutrosophic Sets and Systems, Volume 54, PP: 158-168, 2023.

[7] Agboola, A.A.A.; Akinola, A.D; Oyebola, O.Y; Neutrosophic Rings I. Int. J. of Math. Comb, Volume 4, PP:1-14, 2011.

[8] Celik, M., and Hatip, A; On the Refined AH-Isometry and Its Applications in Refined Neutrosophic Surfaces. Galoitica Journal of Mathematical Structures and Applications, 2022.

[9] Alhasan, Y; The neutrosophic integrals and integration methods. Neutrosophic Sets and Systems, Volume 43, PP: 290-301, 2021.

[10] Alhasan, Y; The definite neutrosophic integrals and its applications. Neutrosophic Sets and Systems, Volume 49, PP: 277-293, 2022.

[11] Smarandache, F; Neutrosophic Precalculus and Neutrosophic Calculus. Book, 2015.

[12] Celik, M., and Hatip, A; On the Refined AH-Isometry and Its Applications in Refined Neutrosophic Surfaces. International Journal of Neutrosophic Science, Volume 2, Issue 1, PP: 21-28, 2022.

[13] Alhasan, Y; Ahmed, M; Abdulfatah, R; Sheen, S; The refined indefinite neutrosophic integral, Neutrosophic Sets and Systems, Vol. 67, 127- 134, 2024.

[14] Alhasan, Y; Mohamed, M; Abdulfatah, R; The limits of 2- refined neutrosophic, Neutrosophic Sets and Systems, Vol. 68, 39-49, 2024.

[15] Smarandache, F; Abobala, M; Operations with n-Refined Literal Neutrosophic Numbers using the Identification Method and the n-Refined AH-Isometry, Neutrosophic Sets and Systems, Vol. 70, 350-358, 2024.

Received: June 28, 2024. Accepted: Oct 10, 2024

NSS

Neutrosophic Sets and Systems, Vol. 75, 2025

University of New Mexico

A Study on some Properties of Fourier Integrals Based on Neutrosophic Function

Prasen Boro[1] and Bhimraj Basumatary[2,*]

[1]Department of Mathematical Sciences, Bodoland University, Kokrajhar-783370, INDIA; prasenboro509@gmail.com

[2]Department of Mathematical Sciences, Bodoland University, Kokrajhar-783370, INDIA; brbasumatary14@gmaill.com

*Correspondence: brbasumatary14@gmaill.com

Abstract. Fourier transforms is one of the oldest and a well-known technique in field of mathematic and engineering mathematical work. Fourier transform method represents the variable as a summation of complex exponentials. Fourier analysis has been used in signal processing and digital image processing for the analysis of a single image as a two-dimensional wave form, and many other type of form like Quantum mechanics, Signal processing, Image Processing. This analysis also represents filters, Transformation, representation, and encoding, Data Processing, Analysis and many more fields. In this article, some basics of Fourier Integrals have been discussed in terms of neutrosophic set. Dirichlet's Conditions, Fourier integral formula and it's five different forms are studied based on neutrosophic set. This article includes the F.T., F.S.T. and F.C.T. of a neutrosophic function and their inversion formulae. In this study, some properties of F.T. are discussed for a neutrosophic function. This study will help to get better results in signal processing, image processing, and in other fields also. This serves as an overview of the Fourier integral of a neutrosophic function.

Keywords: Fourier integral, F.T., neoutrosophic function.

Abreviation: F.T.= Fourier transform

F.S.T= Fourier sine transform

F.C.T.= Fourier cosine transform

N.R.N.= Neutrosophic real number

N.C.N.= Neutrosophic complex number

1. Introduction:

In place of the present logic, F. Smarandache[1,2] put forward the neutrosophic concept to represent a mathematical model of undetermined, uncertainity, vagueness, unclearness, incompleteness, inconsistency, redundancy, in which the neutrosophy is a new concept brought in by Smarandache[1,2]. He defined the N.R.N. in standard form and the considerations for existence of N.R.N.s' division. The standard form of N.C.N.s is also deduced and the root index is found as $n \geq 2$ [3,4]. Studying the concept of neutrosophic probability [1,5] and the neutrosophic statistics [4,6], Professor Smarandache , for the first time studied the concept of preliminary calculus by introducing ideas of neutrosophic mereo-limit, mereo-continuity, mereo-derivative and mereo-integral [7,8].Madeleine Al-Taha produced results on single valued neutrosophic (weak) polygon [9]. Edalatpanah purport a new direct algorithm for the solution of neutrosophic linear programming in which the variables and right hand side are expressed with triangular neutrosophic numbers[10]. Pentagonal neutrosophic number is used in Networking problem and Shortest Path problem by Chakraborty [11,12].

A. Kharal[15] presents a method of multicriteria decision making using neutrosophic sets. A. A. Salama and F. Smarandache [16] used neutrosophic set to introduce new types of neutrosophic crisp sets with three types 1, 2, 3. D. Koundal, S. Gupta and S. Singh [17] demonstrates the use of neutrosophic theory in medical image denoising and segmentation, using which the performance is observed to be much better.

The concepts of Neutrosophic set have been used in different areas of Mathematics. Here we are using this concepts in Fourier integral and Fourier transform. Fourier integral represents a certain type of non periodic functions that are defined on either $(-\infty, \infty)$ or $(0, \infty)$. Fourier transform is a mathematical tool used to decompose a signal into its constituent frequency components. It breaks down signals into a combination of sines and cosines, which can be used to analyse the frequency content of a signal. Fourier analysis has been used in digital image and processing of image and for analysis of a single image into a two-dimensional wave form, and more recently has been used for magnetic resonance imaging, angiographic assessment, automated lung segmentation and image quality assessment and Mobile stethoscope [18]. Fourier transforms which is also used in frequency domain representation. Fourier analysis used as time series analysis proved its application in Quantum mechanics; Signal processing, Image Processing and filters, representation, Data Processing and Analysis and many more.

Fourier transforms are obviously very essential to conduct of Fourier spectroscopy, and that alone would justify its importance. Fourier transforms are very vital in other pursuits as well; such as electrical signal analysis, diffraction, optical testing, optical processing, imaging, holography, and also for remote sensing [13, 14]. Thus, knowledge of Fourier transforms can be a springboard to many other fields. The main idea behind Fourier transforms is that a function of direct time can be expressed as a complex valued function of reciprocal space, that is, frequency.

In this paper we are studying the Fourier integral and Fourier transform by using the concept of Neutrosophic set. The use of Neutrosophic concept in the Fourier transform will help to get the frequencies more accurately which constitude a signal or a sound and can be filter out the unwanted frequencies in more appropriately. This study will give better results in signal processing, image processing and filtering, noise filtering, etc.

This paper consist of 6 sections. The 1st section provides an introduction, in which review of Fourier transform has been given. In 2nd section, definition of neutrosophic real number is given and discussed about division of two neutrosophic real numbers. The 3rd section gives the knowledge of Fourier integral theorem for a neutrosophic function and different forms of neutrosophic Fourier integrals. In the 4th section, we have discussed about the F.T. of a neutrosophic function which includes Fourier sine transform and Fourier cosine transform of a neutrosophic function. In the 5th section we studied about the properties of F.T. of a neutrosophic function and their proofs. The 6th section is the conclusion of the article.

2. **Preliminaries:**

In this part, definitions of N.R.N. and division of two N.R.N.s are discussed.

2.1. *Neutrosophic Real Number[4]:*

If a number that can be written in the form $p_n + q_nI$, where p_n, q_n are real numbers and I is an indeterminate number such that $I.0 = 0$ and $I^N = I$, for all natural number $\mathbf{N}$, is called N.R.N.. Here we denote the N.R.N. by w, and thus we can write $w = p_n + q_nI$ and it is knwon as the standard form of N.R.N..

2.2. *Division of two N.R.N.s[4]:*

Consider that w_1 and w_2 be two N.R.N.s where, $w_1 = p_{n1} + q_{n1}I$ and $w_2 = p_{n2} + q_{n2}I$. Then the division of these N.R.N.s, i.e. $(p_{n1} + q_{n1}I) \div (p_{n2} + q_{n2}I)$ is given by

$$\frac{p_{n1} + q_{n1}I}{p_{n2} + q_{n2}I} = \frac{p_{n1}}{p_{n2}} + \frac{p_{n2}q_{n1} - p_{n1}q_{n2}}{p_{n2}(p_{n2} + q_{n2})}.I$$

provided $p_{n2}(p_{n2} + q_{n2}) \neq 0$ or $p_{n2} \neq 0$ and $p_{n2} \neq -q_{n2}$

3. **Neutrosophic Dirichlet's Conditions:**

Dirichlet's conditions are those condition which must be satisfied by a function $f(x)$ to be expanded by using Fourier series. There are three Dirichlet's conditions, which must be satisfied by the function $f(x)$. In this article, we shall discuss the Dirichlet's conditions by using the concept of neutrosophic set.

Consider, $f^N(x, I)$ be any neutrosophic function satisfying the conditions given below:

(i) $f^N(x, I)$ is defined in the interval $-(h + I) < x < (h + I)$.

(ii) $f^N(x,I)$ and $f'^N(x,I)$ are sectionally continous in $-(h+I) < x < (h+I)$.
(iii) $f^N(x,I)$ is periodic with period $2(h+I)$.
Then the above conditions are called neutrosophic Dirichlet's conditions.
Here, I is called indeterminate number such that $I.0 = 0$ and $I^n = I, n \in \mathbf{N}$, the set of natural numbers.

4. Fourier Integral Theorem (or Formula) for a neutrosophic function. Fourier Integral representation for a neutrosophic function:

Consider, $f^N(x,I)$ be a neutrosophic function and it satisfy the following statements:
(a) $f^N(x,I)$ fulfill neutrosophic Dirichlet's conditions in each finite interval $-(h+I) \leq x \leq (h+I)$
(b) $\int_{-\infty}^{\infty} |f^N(x,I)|dx$ converges that means, $f^N(x,I)$ is absolutely integrable in $-\infty < x < \infty$.
Then Fourier integral theorem for a neutrosophic function will be

$$f^N(x,I) = \int_0^{\infty} \{A_N(s,I)coss(p_n + q_nI)x + B_N(s,I)sins(p_n + q_nI)x\}ds \tag{1}$$

$$where,\ A_N(s,I) = \frac{1}{\pi}\int_{-\infty}^{\infty} f^N(u,I)coss(p_n + q_nI)udu \tag{2}$$

$$and\ B_N(s,I) = \frac{1}{\pi}\int_{-\infty}^{\infty} f^N(u,I)sins(p_n + q_nI)udu \tag{3}$$

Thus Fourier integral theorem of a neutrosophic function is also re-written as

$$f^N(x,I) = \frac{1}{2\pi}\int_{s=-\infty}^{\infty}\int_{u=-\infty}^{\infty} f^N(u,I)coss\{(p_n + q_nI)(x-u)\}duds \tag{4}$$

5. Different forms of Fourier integral theorem of a neutrosophic function:

There are five different forms of Fourier integral theorem in classical method. These five forms are discussed in the sense of neutrosophic set as given below.

(i) General Form:

$$f^N(x,I) = \frac{1}{\pi}\int_{-\infty}^{\infty} f^N(u,I)\left\{\int_0^{\infty} coss\{(p_n + q_nI)(x-u)\}ds\right\}du \tag{5}$$

Proof: When x is a point of continuity of $f^N(x,I)$, we have

$$\begin{aligned}
f^N(x,I) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f^N(u,I)coss((p_n+q_nI)(x-u))du\right\}ds \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f^N(u,I)\left\{\int_{-\infty}^{\infty} coss((p_n+q_nI)(x-u))ds\right\}du \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f^N(u,I)\left\{2\int_{0}^{\infty} coss((p_n+q_nI)(x-u))ds\right\}du, \\
&[using\ the\ property\ of\ definite\ integral \\
&\int_{-a}^{a} f^N(x,I)dx = 2\int_{0}^{a} f(x,I)dx\ if\ f^N(-x,I) = f^N(x,I)] \\
&= \frac{1}{\pi}\int_{u=-\infty}^{\infty} f^N(u,I)\left\{\int_{s=0}^{\infty} coss((p_n+q_nI)(x-u))ds\right\}du
\end{aligned}$$

(ii) Another General Form: If $f^N(x,I)$ is continous at x, then

$$f^N(x,I) = \int_0^{\infty}\{A_N(s,I)coss(p_n+q_nI)x + B_N(s,I)sins(p_n+q_nI)x\}\,ds \tag{6}$$

$$where,\ A_N(s,I) = \frac{1}{\pi}\int_{-\infty}^{\infty} f^N(u,I)coss(p_n+q_nI)udu \tag{7}$$

$$and\ B_N(s,I) = \frac{1}{\pi}\int_{-\infty}^{\infty} f^N(u,I)sins(p_n+q_nI)udu \tag{8}$$

Proof: We have

$$\begin{aligned}
f^N(x,I) =& \frac{1}{\pi}\int_{-\infty}^{\infty} f^N(u,I)\left\{\int_{-\infty}^{\infty} coss(p_n+q_nI)(x-u)ds\right\}du \\
=& \frac{1}{\pi}\int_{0}^{\infty}\left\{\int_{-\infty}^{\infty} f^N(u,I)coss(p_n+q_nI)(x-u)du\right\}ds \\
=& \frac{1}{\pi}\int_{0}^{\infty}\left\{\int_{-\infty}^{\infty} f^N(u,I)coss((p_n+q_nI)(x-u))ds\right\}du, \\
=& \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f^N(u,I)\{coss((p_n+q_nI)xcoss(p_n+q_nI)u \\
&+ sins(p_n+q_nI)xsins(p_n+q_nI)u)du\}ds \\
=& \int_0^{\infty}[coss(p_n+q_nI)x\left\{\frac{1}{\pi}\int_{-\infty}^{\infty} f^N(u,I)coss(p_n+q_nI)udu\right\} \\
&+ sins(p_n+q_nI)x\left\{\frac{1}{\pi}\int_{-\infty}^{\infty} f^N(u,I)sins(p_n+q_nI)udu\right\}]ds \\
=& \int_0^{\infty}\{A_N(s,I)coss(p_n+q_nI)x + B_N(s,I)sins(p_n+q_nI)x\}\,ds,\ [using(7)and(8)]
\end{aligned}$$

(iii) Fourier Sine Integral formula for a Neutrosophic Function:

If $f^N(x,I)$ is an odd neutrosophic function, then

$$f^N(x,I) = \frac{2}{\pi}\int_0^{\infty} sins(p_n+q_nI)x\left\{\int_0^{\infty} f^N(u,I)sins(p_n+q_nI)udu\right\}ds \tag{9}$$

Proof: Proceeding as in form (ii), we have

$$= \frac{1}{\pi}\int_0^\infty [coss(p_n+q_nI)x \int_{-\infty}^\infty f^N(u,I)coss(p_n+q_nI)udu$$
$$+sins(p_n+q_nI)x \int_{-\infty}^\infty f^N(u,I)sins(p_n+q_nI)udu]ds \quad (10)$$

Since, $f^N(x,I)$ is an odd function, it follows that $f^N(u,I)coss(p_n+q_nI)u$ is an odd function and $f^N(u,I)sins(p_n+q_nI)u$ is an even function and so by property of definite integral, we have

$$\int_{-\infty}^\infty f^N(u,I)coss(p_n+q_nI)udu = 0$$
$$and \int_{-\infty}^\infty f^N(u,I)sins(p_n+q_nI)udu = 2\int_0^\infty f^N(u,I)sins(p_n+q_nI)udu$$

Therefore (10) implies

$$f^N(x,I) = \frac{2}{\pi}\int_0^\infty sins(p_n+q_nI)x \left\{\int_0^\infty f^N(u,I)sins(p_n+q_nI)udu\right\}ds$$

(iv) Fourier Cosine Integral formula for a Neutrosophic Function:

If $f^N(x,I)$ is an even neutrosophic function, then

$$f^N(x,I) = \frac{2}{\pi}\int_0^\infty coss(p_n+q_nI)x \left\{\int_0^\infty f^N(u,I)coss(p_n+q_nI)udu\right\}ds \quad (11)$$

Proof: Proceeding as in (ii), we get

$$= \frac{1}{\pi}\int_0^\infty [coss(p_n+q_nI)x \int_{-\infty}^\infty f^N(u,I)coss(p_n+q_nI)udu$$
$$+sins(p_n+q_nI)x \int_{-\infty}^\infty f^N(u,I)sins(p_n+q_nI)udu]ds \quad (12)$$

Since, $f^N(x,I)$ is an even function, it follows that $f^N(u,I)coss(p_n+q_nI)u$ is an even function and $f^N(u,I)sins(p_n+q_nI)u$ is an odd function and hence we have

$$\int_{-\infty}^\infty f^N(u,I)coss(p_n+q_nI)udu = 2\int_0^\infty f^N(u,I)coss(p_n+q_nI)udu$$
$$and \int_{-\infty}^\infty f^N(u,I)sins(p_n+q_nI)udu = 0$$

Therefore (13) implies that

$$f^N(x,I) = \frac{2}{\pi}\int_0^\infty coss(p_n+q_nI)x \left\{\int_0^\infty f^N(u,I)coss(p_n+q_nI)udu\right\}ds$$

(v) Complex or Exponential form of Fourier Integral Formula for a Neutrosophic Function:

$$f^N(x,I) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-is(p_n+q_nI)x}\left\{\int_{-\infty}^\infty f^N(u,I)e^{is(p_n+q_nI)u}du\right\}ds \quad (14)$$

Proof: From general form (i) of Fourier integral theorem of a neutrosophic function, we have

$$\begin{aligned}
f^N(x,I) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f^N(u,I)\left\{\int_0^{\infty} coss(p_n+q_nI)(x-u)du\right\}ds \\
&= \frac{1}{\pi}\int_0^{\infty}\left\{\int_{-\infty}^{\infty} f^N(u,I)coss(p_n+q_nI)(x-u)ds\right\}du \\
&(changing\ the\ order\ of\ integration) \\
&= \frac{1}{\pi}\int_0^{\infty}\left\{\int_{-\infty}^{\infty} f^N(u,I)\frac{e^{is(p_n+q_nI)(x-u)}+e^{-is(p_n+q_nI)(x-u)}}{2}du\right\}ds \\
&= \frac{1}{2\pi}\int_0^{\infty} e^{is(p_n+q_nI)x}\left\{\int_{-infty}^{\infty} f^N(u,I)e^{-is(p_n+q_nI)u}du\right\}ds \\
&+ \frac{1}{2\pi}\int_0^{\infty} e^{-is(p_n+q_nI)x}\left\{\int_{-\infty}^{\infty} f^N(u,I)e^{is(p_n+q_nI)u}du\right\}ds \qquad (15)
\end{aligned}$$

substituting $s=-y$ and $ds=-dy$ in the first integral only of (15), we get

$$\begin{aligned}
f^N(x,I) &= \frac{1}{2\pi}\int_0^{-\infty} e^{-iy(p_n+q_nI)x}\left\{\int_{-\infty}^{\infty} f^N(u,I)e^{iy(p_n+q_nI)u}du\right\}(-dy) \\
&+ \frac{1}{2\pi}\int_0^{\infty} e^{-is(p_n+q_nI)x}\left\{\int_{-\infty}^{\infty} f^N(u,I)e^{is(p_n+q_nI)u}du\right\}ds \\
&= \frac{1}{2\pi}\int_{-\infty}^{0} e^{-iy(p_n+q_nI)x}\left\{\int_{-\infty}^{\infty} f^N(u,I)e^{iy(p_n+q_nI)u}du\right\}(dy) \\
&+ \frac{1}{2\pi}\int_0^{\infty} e^{-is(p_n+q_nI)x}\left\{\int_{-\infty}^{\infty} f^N(u,I)e^{is(p_n+q_nI)u}du\right\}ds \\
&= \frac{1}{2\pi}\int_{-\infty}^{0} e^{-is(p_n+q_nI)x}\left\{\int_{-\infty}^{\infty} f^N(u,I)e^{iy(p_n+q_nI)u}du\right\}(ds) \\
&+ \frac{1}{2\pi}\int_0^{\infty} e^{-is(p_n+q_nI)x}\left\{\int_{-\infty}^{\infty} f^N(u,I)e^{is(p_n+q_nI)u}du\right\}ds \\
Therefore\ f^N(x,I) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-is(p_n+q_nI)x}\left\{\int_{-\infty}^{\infty} f^N(u,I)e^{is(p_n+q_nI)u}du\right\}ds
\end{aligned}$$

Note: In (15), substituting $s=-y$ and $ds=-dy$ in the second integral only in place of first integral and proceeding as before, we get

$$f^N(x,I)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{is(p_n+q_nI)x}\left[\int_{-\infty}^{\infty} f^N(u,I)e^{-is(p_n+q_nI)u}du\right]ds$$

which is another form of complex form of Fourier integral formula in terms a neutrosophic function.

Example 1: Using cosine integral formula of a neutrosophic function, we get the following result:

$$\int_0^{\infty}\frac{cos(p_n+q_nI)\lambda x}{\lambda^2+1}d\lambda=\frac{\pi(p_n+q_nI)}{2}e^{-(p_n+q_nI)x}, x\geq 0$$

Solution: Fourier cosine integral formula for a neutrosophic function is

$$f^N(x,I)=\frac{2}{\pi}\int_0^{\infty} cos\lambda(p_n+q_nI)x\left\{\int_0^{\infty} f^N(u,I)cos\lambda(p_n+q_nI)udu\right\}d\lambda \qquad (i)$$

Taking $f^N(x,I) = e^{-(p_n+q_nI)x}$ *so that* $f^N(u,I) = e^{-(p_n+q_nI)u}$

Using these in (i), we get

$$
\begin{aligned}
e^{-(p_n+q_nI)x} &= \frac{2}{\pi}\int_0^\infty cos\lambda(p_n+q_nI)x\left\{\int_0^\infty e^{-(p_n+q_nI)u}cos\lambda(p_n+q_nI)udu\right\}d\lambda \\
&= \frac{2}{\pi}\int_0^\infty cos\lambda(p_n+q_nI)x\left\{\int_0^\infty e^{-v}cos\lambda v\frac{dv}{(p_n+q_nI)}\right\}d\lambda \\
&\quad [Taking\ v=(p_n+q_nI)u\ so\ that\ dv=(p_n+q_nI)du \\
&\quad as\ u\to 0, v\to 0\ and\ u\to\infty, v\to\infty] \\
&= \frac{2}{(p_n+q_nI)\pi}\int_0^\infty cos\lambda(p_n+q_nI)x\left\{\int_0^\infty e^{-v}cos\lambda v dv\right\}d\lambda \\
&= \frac{2}{(p_n+q_nI)\pi}\int_0^\infty cos\lambda(p_n+q_nI)x\left\{\frac{1}{1+\lambda^2}\right\}d\lambda \\
&= \frac{2}{(1+\lambda^2)(p_n+q_nI)\pi}\int_0^\infty cos\lambda(p_n+q_nI)xd\lambda \\
e^{-(p_n+q_nI)x} &= \frac{2}{(p_n+q_nI)\pi}\int_0^\infty \frac{cos\lambda(p_n+q_nI)x}{(1+\lambda^2)}d\lambda \\
Therefore,\ \int_0^\infty \frac{cos\lambda(p_n+q_nI)x}{(1+\lambda^2)}d\lambda &= \frac{\pi(p_n+q_nI)}{2}e^{-(p_n+q_nI)x}, x\geq 0
\end{aligned}
$$

Example 2: Using the Fourier cosine integral formula, we get the following result:

$$
e^{-(p_n+q_nI)x}cos(p_n+q_nI)x = \left(\frac{1}{p_n}-\frac{q_n}{p_n(p_n+q_n)}I\right)\frac{2}{\pi}\int_0^\infty \frac{(s^2+2)coss(p_n+q_nI)x}{s^2+4}ds
$$

Solution: Fourier Cosine Integral formula is given by

$$
f^N(x,I) = \frac{2}{\pi}\int_0^\infty coss(p_n+q_nI)x\left\{\int_0^\infty f(u,I)coss(p_n+q_nI)udu\right\}ds \qquad \text{(i)}
$$

Solution: Taking $f^N(x,I) = e^{-(p_n+q_nI)x}cos(p_n+q_nI)x$ so that $f^N(u,I) = e^{-(p_n+q_nI)u}cos(p_n+q_nI)u$ Using these in (i), we get

$$
\begin{aligned}
&e^{-(p_n+q_nI)x}cos(p_n+q_nI)x \\
&= \frac{2}{\pi}\int_0^\infty coss(p_n+q_nI)x\left\{\int_0^\infty e^{-(p_n+q_nI)u}cos(p_n+q_nI)ucoss(p_n+q_nI)udu\right\}ds \\
&= \frac{1}{\pi}\int_0^\infty coss(p_n+q_nI)x\left\{\int_0^\infty e^{-(p_n+q_nI)u}(2cos(p_n+q_nI)ucoss(p_n+q_nI)u)du\right\}ds \\
&= \frac{1}{\pi}\int_0^\infty coss(p_n+q_nI)x\left[\int_0^\infty e^{-(p_n+q_nI)u}\{cos(p_n+q_nI)(s+1)u+cos(p_n+q_nI)(s-1)u\}du\right]ds \\
&= \frac{1}{\pi}\int_0^\infty coss(p_n+q_nI)x[\int_0^\infty e^{-(p_n+q_nI)u}cos(p_n+q_nI)(s+1)udu \\
&+ \int_0^\infty e^{-(p_n+q_nI)u}cos(p_n+q_nI)(s-1)udu]ds \\
&= \frac{1}{\pi}\int_0^\infty coss(p_n+q_nI)x\left[\frac{(p_n+q_nI)}{(p_n+q_nI)^2+(p_n+q_nI)^2(s+1)^2}+\frac{(p_n+q_nI)}{(p_n+q_nI)^2+(p_n+q_nI)^2(s-1)^2}\right]ds \\
&= \frac{1}{\pi}\int_0^\infty coss(p_n+q_nI)x.\frac{1}{(p_n+q_nI)}\left[\frac{1}{1+(s+1)^2}+\frac{1}{1+(s-1)^2}\right]ds \\
&= \frac{1}{(p_n+q_nI)\pi}\int_0^\infty coss(p_n+q_nI)x\left[\frac{(s^2-2s+2)+(s^2+2s+2)}{(s^2+2s+2)(s^2-2s+2)}\right]ds \\
&= \frac{1}{(p_n+q_nI)\pi}\int_0^\infty \frac{2(s^2+2)}{(s^2+2)^2-(2s)^2}coss(p_n+q_nI)xds \\
&= \frac{2}{(p_n+q_nI)\pi}\int_0^\infty \frac{2(s^2+2)coss(p_n+q_nI)x}{s^4+4}ds \\
&= \left(\frac{1}{p_n}-\frac{q_n}{p_n(p_n+q_n)}I\right)\frac{2}{\pi}\int_0^\infty \frac{2(s^2+2)coss(p_n+q_nI)x}{s^4+4}ds
\end{aligned}
$$

Example 3: Using Fourier sine integral formula, the following result is obtained

$$
\begin{aligned}
e^{-(a_n+b_nI)x}-e^{-(c_n+d_nI)x} =& \frac{2(p_n+q_nI)[(c_n+d_nI)^2-(a_n+b_nI)^2]}{\pi} \\
&\int_0^\infty \frac{sins(p_n+q_nI)xds}{\{(a_n+b_nI)^2+s^2(p_n+q_nI)^2\}\{(c_n+d_nI)^2+s^2(p_n+q_nI)^2\}}.
\end{aligned}
$$

Solution: Fourier sine integral formula is given by

$$
f^N(x,I) = \frac{2}{\pi}\int_0^\infty sins(p_n+q_nI)x\left\{\int_0^\infty f^N(u,I)sins(p_n+q_nI)udu\right\}ds \tag{i}
$$

Taking $f^N(x,I) = e^{-(a_n+b_nI)x}-e^{-(c_n+d_nI)x}$

so that $f^N(u,I) = e^{-(a_n+b_nI)u}-e^{-(c_n+d_nI)u}$

Using these in (i), we get

$$
\begin{aligned}
&e^{-(a_n+b_nI)x} - e^{-(c_n+d_nI)x} \\
&= \frac{2}{\pi}\int_0^\infty sins(p_n+q_nI)x\left\{\int_0^\infty (e^{-(a_n+b_nI)u} - e^{-(c_n+d_nI)u})sins(p_n+q_nI)udu\right\}ds \\
&= \frac{2}{\pi}\int_0^\infty sins(p_n+q_nI)x\left\{\int_0^\infty e^{-(a_n+b_nI)u}sins(p_n+q_nI)udu - \int_0^\infty e^{-(c_n+d_nI)u}sins(p_n+q_nI)udu\right\}ds \\
&= \frac{2}{\pi}\int_0^\infty sins(p_n+q_nI)x\left[\frac{s(p_n+q_nI)}{(a_n+b_nI)^2+s^2(p_n+q_nI)^2} - \frac{s(p_n+q_nI)}{(c_n+d_nI)^2+s^2(p_n+q_nI)^2}\right]ds \\
&= \frac{2}{\pi}\int_0^\infty sins(p_n+q_nI)x.(p_n+q_nI)s\left[\frac{(c_n+d_nI)^2-(a_n+b_nI)^2}{\{(a_n+b_nI)^2+s^2(p_n+q_nI)^2\}\{(c_n+d_nI)^2+s^2(p_n+q_nI)^2\}}\right]ds \\
&= \frac{2(p_n+q_nI)[(c_n+d_nI)^2-(a_n+b_nI)^2]}{\pi}\int_0^\infty \frac{sins(p_n+q_nI)xds}{\{(a_n+b_nI)^2+s^2(p_n+q_nI)^2\}\{(c_n+d_nI)^2+s^2(p_n+q_nI)^2\}}.
\end{aligned}
$$

6. **F.T.s or Complex F.T. of a Neutrosophic Function**

Fourier transform is a mathematical model which helps to transform the signals between two different domains, such as transforming signal from frequency domain to time domain or vice versa. It is an integral transform that converts a function into a form that describes the frequencies present in the original function. The output of the transform is a complex-valued function of frequencies.

Definition: Let $\mathscr{F}(x,I)$ be a function defined on $(-\infty,\infty)$ and be piecewise continuous in each partial interval and absolutely integrable in $(-\infty,\infty)$. Then the F.T. of $\mathscr{F}(x,I)$ is a function of a new variable $'s'$ and it is denoted and defined as

$$\mathscr{F}\{\mathscr{F}(x,I)\} = \overline{\mathscr{F}}(s,I) = f^N(s,I) = \int_{-\infty}^{\infty} e^{is(p_n+q_nI)x}\mathscr{F}(x,I)dx \tag{16}$$

The function $\mathscr{F}(x,I)$ is then known as inverse F.T.s of $\overline{\mathscr{F}}(s,I)$ or $f^N(s,I)$ and is denoted by

$$
\begin{aligned}
\mathscr{F}(x,I) &= \mathscr{F}^{-1}\{\overline{\mathscr{F}}(s,I)\} \\
\mathscr{F}(x,I) &= \mathscr{F}^{-1}\{f^N(s,I)\}
\end{aligned} \tag{17}
$$

6.1. *Inversion Formula for F.T. or Complex F.T. of a Neutrosophic Function:*

If $\overline{\mathscr{F}}(s,I)$ is the F.T. of $\mathscr{F}(x,I)$ and if $\mathscr{F}(x,I)$ satiesfies the Dirichlet's conditions in every finite interval $(-l-I, l+I)$ and further $\mathscr{F}(x,I)$ is absolutely integrable in $(-\infty,\infty)$, then at every point of continuity of $\mathscr{F}(x,I)$,

$$\mathscr{F}(x,I) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\mathscr{F}}(s,I)e^{-is(p_n+q_nI)x}ds$$

Proof: Complex Fourier integral formula is given by

$$\mathscr{F}(x,I) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-is(p_n+q_nI)x}\left\{\int_{-\infty}^{\infty}\mathscr{F}(u,I)e^{is(p_n+q_nI)u}du\right\}ds \tag{18}$$

Complex F.T. of $\mathscr{F}(x,I)$ is given by

$$\begin{aligned}\overline{\mathscr{F}}(s,I) &= \int_{-\infty}^{\infty} e^{is(p_n+q_nI)x}\mathscr{F}(x,I)dx \\ &= \int_{-\infty}^{\infty} \mathscr{F}(u,I)e^{is(p_n+q_nI)u}du \end{aligned} \tag{19}$$

From (18) and (19) we get

$$\mathscr{F}(x,I) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\mathscr{F}}(s,I)e^{-is(p_n+q_nI)x}ds \tag{20}$$

Which is the required inversion formula for complex F.T. of a neutrosophic function.
Thus, we have

$$\begin{aligned}\mathscr{F}\{\mathscr{F}(x,I)\} &= \overline{\mathscr{F}}(s,I) \\ &= f^N(s,I) \\ &= \int_{-\infty}^{\infty} e^{is(p_n+q_nI)x}\mathscr{F}(x,I)dx \end{aligned} \tag{21a}$$

and

$$\begin{aligned}\mathscr{F}(x,I) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-is(p_n+q_nI)x}\overline{\mathscr{F}}(s,I)ds \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-is(p_n+q_nI)x}f^N(s,I)ds \end{aligned} \tag{21b}$$

Theorem 1. If $f^N(s,I)$ is F.T. of $\mathscr{F}(x,I)$, then

$$\begin{aligned}(i)\ \mathscr{F}\{\mathscr{F}(-x,I)\} &= f^N(-s,I) \\ (ii)\ \mathscr{F}\{\overline{\mathscr{F}(-x,I)}\} &= \overline{f^N(s,I)} \\ (iii)\ \mathscr{F}\{\overline{\mathscr{F}(x,I)}\} &= \overline{f^N(-s,I)}\end{aligned}$$

Where bar over a quantity represents its complex conjugate.

Proof: By definition,

$$\mathscr{F}\{\mathscr{F}(x,I)\} = f^N(s,I) = \int_{-\infty}^{\infty} \mathscr{F}(x,I)e^{is(p_n+q_nI)x}dx \tag{22}$$

Now

$$
\begin{aligned}
(i)\ \mathscr{F}\{\mathscr{F}(-x,I)\} &= \int_{-\infty}^{\infty} \mathscr{F}(-x,I)e^{is(p_n+q_nI)x}dx \\
&= \int_{\infty}^{-\infty} \mathscr{F}(y,I)e^{-is(p_n+q_nI)y}(-dy) \\
&\quad \{putting\ y=-x\ \&\ dx=-dy \\
&= \int_{-\infty}^{\infty} \mathscr{F}(y,I)e^{-is(p_n+q_nI)y}(dy) \\
&= \int_{-\infty}^{\infty} \mathscr{F}(x,I)e^{i(-s)(p_n+q_nI)y}(-dy) \\
&= f^N(-s,I)\ \{by(22) \\
(ii)\ \mathscr{F}\{\mathscr{F}(-x,I)\} &= \int_{-\infty}^{\infty} \overline{\{\mathscr{F}(-x,I)\}}e^{is(p_n+q_nI)x}dx \\
&= \int_{-\infty}^{\infty} \overline{\{\mathscr{F}(y,I)\}}e^{-is(p_n+q_nI)y}(-dy) \\
&\quad \{putting\ y=-x\ \&\ dx=-dy \\
&= \int_{-\infty}^{\infty} \overline{\{\mathscr{F}(x,I)\}}e^{-is(p_n+q_nI)x}dx \\
&= \left\{\overline{\int_{-\infty}^{\infty} \mathscr{F}(x,I)e^{is(p_n+q_nI)x}dx}\right\} \\
&= \overline{f^N(s,I)}\ by(22) \\
(iii)\ \mathscr{F}\{\overline{\mathscr{F}(x,I)}\} &= \int_{-\infty}^{\infty} \{\overline{\mathscr{F}(x,I)}\}e^{is(p_n+q_nI)x}dx \\
&= \left\{\int_{-\infty}^{\infty} \mathscr{F}(x,I)e^{i(-s)(p_n+q_nI)x}dx\right\} \\
&= \overline{f^N(-s,I)},\ by(22)
\end{aligned}
$$

7. **F.S.T. or Infinite F.S.T. for a Neutrosophic Function:**

Definition: Let $\mathscr{F}(x,I)$ be a neutrosophic function defined on $(-\infty,\infty)$ and be piecewise continous in each partial interval and absolutely integrable in $(-\infty,\infty)$. Then the infinte F.S.T. of $\mathscr{F}(x,I)$ is a function of a new variable 's' and it is denoted and defined as

$$\mathscr{F}_s\{\mathscr{F}(x,I)\} = \overline{\mathscr{F}_s}(x,I) = f_s^N(s,I) = \int_0^{\infty} \mathscr{F}(x,I)sins(p_n+q_nI)xdx \tag{23}$$

The function $\mathscr{F}(x,I)$ is then called inverse Fourier sine transfom of $\overline{\mathscr{F}_s}(s,I)$ or $f_s^N(s,I)$ and is denoted by

$$\mathscr{F}(x,I) = \mathscr{F}_s{}^{-1}\{\mathscr{F}_s(s,I)\}\ or\ \mathscr{F}(x,I) = \mathscr{F}_s{}^{-1}\{f_s^N(s,I)\} \tag{24}$$

7.1. *Inversion Formula for Infinite F.S.T. of a Neutrosophic Function:*

If $\overline{\mathscr{F}_s}(s,I)$ is the infinite F.S.T. of $\mathscr{F}(x,I)$ and if $\mathscr{F}(x,I)$ satisfies the Dirichlet conditions in every finite interval $(-l-I, l+I)$ and further if $\mathscr{F}(x,I)$ is absolutely integrable in $(-\infty,\infty)$, then at every point of continuity of $\mathscr{F}(x,I)$, then we have

$$\mathscr{F}(x,I) = \frac{2}{\pi}\int_0^\infty \overline{\mathscr{F}_s}(s,I) sins(p_n+q_nI)xds$$

Proof: Fourier sine integral formula is given by

$$\mathscr{F}(x,I) = \frac{2}{\pi}\int_0^\infty sins(p_n+q_nI)x\left\{\int_0^\infty \mathscr{F}(u,I)sins(p_n+q_nI)udu\right\}ds \tag{25}$$

Now, infinite F.S.T. of $\mathscr{F}(x,I)$ is given by

$$\begin{aligned}\overline{\mathscr{F}_s}(s,I) &= \int_0^\infty \mathscr{F}(x,I)sins(p_n+q_nI)xdx\\ &= \int_0^\infty \mathscr{F}(u,I)sins(p_n+q_nI)udu\end{aligned} \tag{26}$$

Thus, from (25) and (26), we get

$$\mathscr{F}(x,I) = \frac{2}{\pi}\int_0^\infty \overline{\mathscr{F}_s}(s,I)sins(p_n+q_nI)xds \tag{27}$$

Which is the required inversion formula for infinite F.S.T..
Thus, we have

$$\begin{aligned}\mathscr{F}_s\{\mathscr{F}(x,I)\} &= \overline{\mathscr{F}_s}(s,I)\\ &= f_s^N(s,I)\\ &= \int_0^\infty \mathscr{F}(x,I)sins(p_n+q_nI)xds \qquad (28a)\\ and\ \mathscr{F}(x,I) &= \mathscr{F}_s{}^{-1}\{\overline{\mathscr{F}_s}(s,I)\}\\ &= \frac{2}{\pi}\int_0^\infty \overline{\mathscr{F}_s}(s,I)sins(p_n+q_nI)xds \qquad (28b)\end{aligned}$$

Equations (28a) and (28b) are also defined as

$$\begin{aligned}\mathscr{F}_s\{\mathscr{F}(x,I)\} &= \overline{\mathscr{F}_s}(s,I)\\ &= f_s^N(s,I)\\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty \mathscr{F}(x,I)sins(p_n+q_nI)xds \qquad (28a^*)\\ and\ \mathscr{F}(x,I) &= \mathscr{F}_s{}^{-1}\{\overline{\mathscr{F}_s}(s,I)\}\\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty \overline{\mathscr{F}_s}(s,I)sins(p_n+q_nI)xds \qquad (28b^*)\end{aligned}$$

8. **F.C.T. or Infinite F.C.T. of a Neutrosophic Function:**

Definition: Let $\mathscr{F}(x,I)$ be a neutrosophic function defined on $(-\infty,\infty)$ and be piecewise continuous in each partial interval and absolutely integrable in $(-\infty,\infty)$. Then the infinite F.C.T. of $\mathscr{F}(x,I)$ is a function of a new variable 's' and it is denoted and defined as

$$\mathscr{F}_c\{\mathscr{F}(x,I)\} = \overline{\mathscr{F}_c}(s,I) = f_c^N(s,I) = \int_0^\infty \mathscr{F}(x,I)coss(p_n+q_nI)xdx \tag{29}$$

The function $\mathscr{F}(x,I)$ is then called inverse F.C.T. of $\overline{\mathscr{F}_c}(s,I)$ or $f_c^N(s,I)$ and is denoted by

$$\begin{aligned}\mathscr{F}(x,I) &= \mathscr{F}_c{}^{-1}\{\overline{\mathscr{F}_c}(s,I)\}\\ or\ \mathscr{F}(x,I) &= \mathscr{F}_c{}^{-1}\{f_c^N(s,I)\}\end{aligned} \tag{30}$$

8.1. *Inversion Formula for Infinite F.C.T. of a Neutrosophic Function:*

If $\overline{\mathscr{F}_c}(s,I)$ is the infinite F.C.T. of $\mathscr{F}(x,I)$ and if $\mathscr{F}(x,I)$ satisfies Dirichlet's conditions in every finite interval $(-l-I, l+I)$ and further if $\mathscr{F}(x,I)$ is absolutely integrable in $(-\infty,\infty)$ then at every point of continuity of $\mathscr{F}(x,I)$, we have

$$\mathscr{F}(x,I) = \frac{2}{\pi}\int_0^\infty \overline{\mathscr{F}_c}(s,I)coss(p_n+q_nI)xds$$

Proof: Fourier cosine integral formula is given by

$$\mathscr{F}(x,I) = \frac{2}{\pi}\int_0^\infty coss(p_n+q_nI)x\left\{\int_0^\infty \mathscr{F}(u,I)coss(p_n+q_nI)udu\right\}ds \tag{31}$$

Now, infinite F.C.T. of $\mathscr{F}(x,I)$ is given by

$$\begin{aligned}\overline{\mathscr{F}_c}(s,I) &= \int_0^\infty \mathscr{F}(x,I)coss(p_n+q_nI)xdx\\ &= \int_0^\infty \mathscr{F}(u,I)coss(p_n+q_nI)udu\end{aligned} \tag{32}$$

Thus, from (31) *and* (32) *we get*

$$\mathscr{F}(x,I) = \frac{2}{\pi}\int_0^\infty \overline{\mathscr{F}_c}(s,I)coss(p_n+q_nI)xds \tag{33}$$

Which is the required inversion formula for infinite F.C.T. of a neutrosophic function.
Thus, we have

$$\mathscr{F}_c\{\mathscr{F}(x,I)\} = \overline{\mathscr{F}_c}(s,I) = f_c(s,I) = \int_0^\infty \mathscr{F}(x,I)coss(p_n+q_nI)xdx \tag{34a}$$

$$and\ \mathscr{F}(x,I) = \mathscr{F}_c{}^{-1}\{\overline{\mathscr{F}_c}(s,I)\} = \frac{2}{\pi}\int_0^\infty \overline{\mathscr{F}_c}(s,I)coss(p_n+q_nI)xds \tag{34b}$$

Equations (34a) and (34b) are also re-written in the following symmetric form as

$$\begin{aligned}\mathscr{F}_c\{\mathscr{F}(x,I)\} &= \overline{\mathscr{F}_c}(s,I)\\ &= f_c^N(s,I)\\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty \mathscr{F}(x,I)coss(p_n+q_nI)xds \qquad (34a^*)\\ and\ \mathscr{F}(x,I) &= \mathscr{F}_c{}^{-1}\{\overline{\mathscr{F}_c}(s,I)\}\\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty \overline{\mathscr{F}_c}(s,I)coss(p_n+q_nI)xds \qquad (34b^*)\end{aligned}$$

9. **Linearity Property of F.T. of a Neutrosophic Function:**

If c_1 and c_2 be constants, then

$$\begin{aligned}&(i)\ \mathscr{F}\{c_1\mathscr{F}_1(x,I)+c_2\mathscr{F}_2(x,I)\} = c_1\mathscr{F}\{\mathscr{F}_1(x,I)\}+c_2\mathscr{F}\{\mathscr{F}_2(x,I)\}\\ &(ii)\ \mathscr{F}_s\{c_1\mathscr{F}_1(x,I)+c_2\mathscr{F}_2(x,I)\} = c_1\mathscr{F}_s\{\mathscr{F}_1(x,I)\}+c_2\mathscr{F}_s\{\mathscr{F}_2(x,I)\}\\ &(iii)\ \mathscr{F}_c\{c_1\mathscr{F}_1(x,I)+c_2\mathscr{F}_2(x,I)\} = c_1\mathscr{F}_c\{\mathscr{F}_1(x,I)\}+c_2\mathscr{F}_c\{\mathscr{F}_2(x,I)\}\end{aligned}$$

Proof: (i) By definition (16), we have

$$\begin{aligned}\mathscr{F}\{c_1\mathscr{F}_1(x,I)+c_2\mathscr{F}_2(x,I)\} &= \int_{-\infty}^{\infty} e^{is(p_n+q_nI)x}\{c_1\mathscr{F}_1(x,I)+c_2\mathscr{F}_2(x,I)\}dx\\ &= c_1\int_{-\infty}^{\infty} e^{is(p_n+q_nI)x}\mathscr{F}_1(x,I)dx\\ &\quad + c_2\int_{-\infty}^{\infty} e^{is(p_n+q_nI)x}\mathscr{F}_2(x,I)dx\\ &= c_1\mathscr{F}\{\mathscr{F}_1(x,I)\}+c_2\mathscr{F}\{\mathscr{F}_2(x,I)\}\end{aligned}$$

(ii) By definition (23), we have

$$\begin{aligned}\mathscr{F}_s\{c_1\mathscr{F}_1(x,I)+c_2\mathscr{F}_2(x,I)\} &= \int_0^\infty \{c_1\mathscr{F}_1(x,I)+c_2\mathscr{F}_2(x,I)\}sins(p_n+q_nI)xdx\\ &= c_1\int_0^\infty \mathscr{F}_1(x,I)sins(p_n+q_nI)xdx\\ &\quad + c_2\int_0^\infty \mathscr{F}_2(x,I)sins(p_n+q_nI)xdx\\ &= c_1\mathscr{F}_s\{\mathscr{F}_1(x,I)\}+c_2\mathscr{F}_s\{\mathscr{F}_2(x,I)\}\end{aligned}$$

(iii) By definition (29), we have

$$\begin{aligned}\mathscr{F}_c\{c_1\mathscr{F}_1(x,I)+c_2\mathscr{F}_2(x,I)\} &= \int_0^\infty \{c_1\mathscr{F}_1(x,I)+c_2\mathscr{F}_2(x,I)\}coss(p_n+q_nI)xdx \\ &= c_1\int_0^\infty \mathscr{F}_1(x,I)coss(p_n+q_nI)xdx \\ &\quad + c_2\int_0^\infty \mathscr{F}_2(x,I)coss(p_n+q_nI)xdx \\ &= c_1\mathscr{F}_c\{\mathscr{F}_1(x,I)\}+c_2\mathscr{F}_c\{\mathscr{F}_2(x,I)\}\end{aligned}$$

10. **Change of Scale Property of a Neutrosophic Function:**

(i) If $f^N(s,I)$ is the F.T. of $\mathscr{F}(x,I)$, then $\frac{1}{|\alpha|}f^N(\frac{s}{\alpha},I)$ is the F.T. of $\mathscr{F}(\alpha x,I)$, where $\alpha\neq 0$.
i.e., If $\mathscr{F}\{\mathscr{F}(x,I)\}=f^N(s,I)$, then $\mathscr{F}\{\mathscr{F}(\alpha x,I)\}=\frac{1}{|\alpha|}f^N(\frac{s}{\alpha},I)$, where $\alpha\neq 0$.
(ii) If $\mathscr{F}_s\{\mathscr{F}(x,I)\}=f_s^N(s,I)$, then $\mathscr{F}_s\{\mathscr{F}(\alpha x,I)\}=\frac{1}{\alpha}f_s^N(\frac{s}{\alpha},I)$
(iii) If $\mathscr{F}_c\{\mathscr{F}(x,I)\}=f_c^N(s,I)$, then $\mathscr{F}_c\{\mathscr{F}(\alpha x,I)\}=\frac{1}{\alpha}f_c^N(\frac{s}{\alpha},I)$

Proof: (i) The following two cases arise:

Case I: Let $\alpha>0$. Then by definition of F.T., we get,

$$\begin{aligned}\mathscr{F}\{\mathscr{F}(\alpha x,I)\} &= \int_{-\infty}^{\infty} e^{is(p_n+q_nI)x}\mathscr{F}(\alpha x,I)dx \\ &= \frac{1}{\alpha}\int_{-\infty}^{\infty} e^{(\frac{s}{\alpha})(p_n+q_nI)t}\mathscr{F}(t,I)dt \\ &\quad [putting\ \alpha x=t\ so\ that\ dx=\frac{1}{\alpha}dt] \\ &= (\frac{1}{\alpha})f^N(\frac{s}{\alpha},I),\ by\ definition\ of\ Fourier\ Transform.\end{aligned}$$

Case II: Let $\alpha<0$. Let $\beta>0$ such that $\alpha=-\beta$. Then by definition, we have

$$\begin{aligned}\mathscr{F}\{\mathscr{F}(\alpha x,I)\} &= \int_{-\infty}^{\infty} e^{is(p_n+q_nI)x}\mathscr{F}(\alpha x,I)dx \\ &= \int_{-\infty}^{\infty} e^{is(p_n+q_nI)t}\mathscr{F}(-\beta x,I)dx \\ &\quad [as\ \alpha=-\beta] \\ &= \int_{\infty}^{-\infty} e^{i(\frac{s}{\alpha})t}\frac{dt}{-\beta},\ [putting\ -\beta x=t\ so\ that\ dx=-\frac{1}{\beta}dt] \\ &= \frac{1}{\beta}\int_{-\infty}^{\infty} e^{i(\frac{s}{\alpha})t}\mathscr{F}(t,I)dt \\ &= \frac{1}{-\alpha}f^N(\frac{s}{\alpha},I),\ by\ definition\ of\ Fourier\ Transform.\end{aligned}$$

Combining the above two cases, we get

$$\mathscr{F}\{\mathscr{F}(\alpha x,I)\}=\frac{1}{|\alpha|}f^N(\frac{s}{\alpha},I)$$

. (ii) By definition (23), we get

$$\int_0^\infty \mathscr{F}(x,I)sins(p_n+q_nI)xdx = f_s^N(s,I) \tag{i}$$

$$\begin{aligned} Now,\ \mathscr{F}_s\{\mathscr{F}(\alpha x,I)\} &= \int_0^\infty \mathscr{F}(\alpha x,I)sins(p_n+q_nI)xdx \\ &= \frac{1}{\alpha}\int_0^\infty \mathscr{F}(t,I)sin\frac{s(p_n+q_nI)t}{\alpha}dt,\ [putting\ \alpha x = t\ so\ that\ dx = \frac{dt}{\alpha}] \\ &= \frac{1}{\alpha}\int_0^\infty \mathscr{F}(x,I)sin\{\frac{s}{\alpha}(p_n+q_nI)x\}dx \\ &= \frac{1}{\alpha}f_s^N(\frac{s}{\alpha},I),\ on\ replacing\ s\ by\ \frac{s}{\alpha}\ in\ (i) \end{aligned}$$

(iii) By definition (29), we get

$$\int_0^\infty \mathscr{F}(x,I)coss(p_n+q_nI)xdx = f_c^N(s,I) \tag{i}$$

$$\begin{aligned} Now,\ \mathscr{F}_c\{\mathscr{F}(\alpha x,I)\} &= \int_0^\infty \mathscr{F}(\alpha x,I)coss(p_n+q_nI)xdx \\ &= \frac{1}{\alpha}\int_0^\infty \mathscr{F}(t,I)cos\frac{s(p_n+q_nI)t}{\alpha}dt,\ [putting\ \alpha x = t\ so\ that\ dx = \frac{dt}{\alpha}] \\ &= \frac{1}{\alpha}\int_0^\infty \mathscr{F}(x,I)cos\{\frac{s}{\alpha}(p_n+q_nI)x\}dx \\ &= \frac{1}{\alpha}f_c^N(\frac{s}{\alpha},I),\ on\ replacing\ s\ by\ \frac{s}{\alpha}\ in\ (i) \end{aligned}$$

Example: If $f_c^N(s,I)$ is F.C.T. of $\mathscr{F}(x,I)$, show that F.C.T. of $\mathscr{F}(\frac{x}{\alpha},I)$ is $\alpha f_s^N(\alpha s,I)$.

Solution: By definition (29), we get

$$\int_0^\infty \mathscr{F}(x,I)coss(p_n+q_nI)xdx = f_c^N(s,I) \tag{i}$$

$$\begin{aligned} Now,\ \mathscr{F}_c\{\mathscr{F}(\frac{x}{\alpha},I)\} &= \int_0^\infty \mathscr{F}(\frac{x}{\alpha},I)coss(p_n+q_nI)xdx \\ &= \alpha\int_0^\infty \mathscr{F}(t,I)cos(\alpha st)(p_n+q_nI)dt,\ [on\ putting\ \frac{x}{\alpha} = t\ so\ that\ dx = \alpha dt] \\ &= \alpha\int_0^\infty \mathscr{F}(x,I)cos\{\alpha s(p_n+q_nI)\}xdx \\ &= \alpha f_c^N(\alpha s,I)\ \ [replacing\ s\ by\ \alpha s\ in\ (i)] \end{aligned}$$

11. **Shifting Property for F.T. of a Neutrosophic Function:**

If $f^N(s,I)$ is the complex F.T. of $\mathscr{F}(x,I)$, then complex F.T. of $\mathscr{F}(x-\alpha,I)$ is $e^{is\alpha(p_n+q_nI)}f^N(s,I)$, i.e., if $\mathscr{F}\{\mathscr{F}(x,I)\} = f^N(s,I)$, then

$$\mathscr{F}\{\mathscr{F}(x-\alpha,I)\} = e^{is\alpha(p_n+q_nI)}f^N(s,I).$$

Proof: By definition (16), we have

$$\int_{-\infty}^{\infty} e^{is(p_n+q_nI)x}\mathscr{F}(x,I)dx = f^N(s,I) \tag{i}$$

Now,

$$\begin{aligned}\mathscr{F}\{\mathscr{F}(x-\alpha,I)\} &= \int_{\infty}^{\infty} e^{is(p_n+q_nI)x}\mathscr{F}(x-\alpha,I)dx \\ &= \int_{\infty}^{\infty} e^{is(p_n+q_nI)(t+\alpha)}\mathscr{F}(t,I)dt,\ [on\ putting\ x-\alpha=t\ so\ that\ dx=dt] \\ &= e^{is\alpha(p_n+q_nI)\alpha}\int_{-\infty}^{\infty} e^{is(p_n+q_nI)t}\mathscr{F}(t,I)dt \\ &= e^{is(p_n+q_nI)\alpha}\int_{-\infty}^{\infty} e^{is(p_n+q_nI)x}\mathscr{F}(x,I)dx \\ &= e^{is(p_n+q_nI)\alpha}f^N(s,I),\ \{using\ (i)\end{aligned}$$

12. **Modulation Theorem for a Neutrosophic Function:**

If $f^N(s,I)$ is the neutrosophic F.T. of a neutrosophic function $f^N(x,I)$, then $\frac{\left[f^N(s+\alpha,I)+f^N(s-\alpha,I)\right]}{2}$ is the F.T. of $\mathscr{F}(x,I)cos\alpha(p_n+q_nI)x$, i.e.,

(i)If $\mathscr{F}\{\mathscr{F}(x,I)\} = f^N(s,I)$, then
$\mathscr{F}_c\{\mathscr{F}(x,I)cos\alpha(p_n+q_nI)x\} = \frac{\left[f^N(s+\alpha,I)+f^N(s-\alpha,I)\right]}{2}$
(ii) If $\mathscr{F}_s\{\mathscr{F}(x,I)\} = f_s^N(s,I)$, then
$\mathscr{F}_s\{\mathscr{F}(x,I)cos\alpha(p_n+q_nI)x\} = \frac{\left[f_s^N(s+\alpha,I)+f_s^N(s-\alpha,I)\right]}{2}$
(iii) If $\mathscr{F}_s\{\mathscr{F}(x,I)\} = f_s^N(s,I)$, then
$\mathscr{F}_c\{\mathscr{F}(x,I)sin\alpha(p_n+q_nI)x\} = \frac{\left[f_s^N(s+\alpha,I)+f_s^N(s-\alpha,I)\right]}{2}$
(iv) If $\mathscr{F}_c\{\mathscr{F}(x,I)\} = f_c^N(s,I)$, then
$\mathscr{F}_s\{\mathscr{F}(x,I)sin\alpha(p_n+q_nI)x\} = \frac{\left[f_c^N(s-\alpha,I)-f_c^N(s+\alpha,I)\right]}{2}$

Proof (i) By definition(16),

$$\int_{-\infty}^{\infty} e^{is(p_n+q_nI)x}\mathscr{F}(x,I)dx = f^N(s,I) \tag{i}$$

$$\begin{aligned}Now,\ \mathscr{F}\{\mathscr{F}(x,I)cos\alpha x\} &= \int_{-\infty}^{\infty} e^{is(p_n+q_nI)x}\mathscr{F}(x,I)cos\alpha(p_n+q_nI)xdx \\ &= \int_{-\infty}^{\infty} e^{is(p_n+q_nI)x}\mathscr{F}(x,I)\frac{e^{i\alpha(p_n+q_nI)x}+e^{-i\alpha(p_n+q_nI)x}}{2}dx \\ &= \frac{1}{2}\int_{-\infty}\infty e^{i(s+\alpha)(p_n+q_nI)x}\mathscr{F}(x,I)dx + \frac{1}{2}\int_{-\infty}\infty e^{i(s-\alpha)(p_n+q_nI)x}\mathscr{F}(x,I)dx \\ &= \frac{1}{2}f^N(s+\alpha,I)+\frac{1}{2}f^N(s-\alpha,I),\ \{using\ (i) \\ &= \frac{\left[f^N(s+\alpha,I)+f^N(s-\alpha,I)\right]}{2}\end{aligned}$$

(ii) By definition(23), we have

$$\int_0^\infty \mathscr{F}(x,I)sins(p_n+q_nI)xdx = f_s^N(s,I) \quad \text{(i)}$$

Now,

$$\begin{aligned}
\mathscr{F}_s\{\mathscr{F}(x,I)cos\alpha(p_n+q_nI)x\} &= \int_{-\infty}^{\infty} \mathscr{F}(x,I)cos\alpha(p_n+q_nI)xsins(p_n+q_nI)xdx, \{By\ defn.\ (i)\\
&= \frac{1}{2}\int_0^\infty \mathscr{F}(x,I)2sins(p_n+q_nI)xcos\alpha(p_n+q_nI)xdx\\
&= \frac{1}{2}\int_0^\infty \mathscr{F}(x,I)[sin(p_n+q_nI)(s+\alpha)x + sin(p_n+q_nI)(s-\alpha)x]dx\\
&= \frac{1}{2}[\int_0^\infty \mathscr{F}(x,I)sin(s+\alpha)(p_n+q_nI)xdx\\
&\quad + \int_0^\infty \mathscr{F}(x,I)sin(s-\alpha)(p_n+q_nI)xdx]\\
&= \frac{1}{2}[f_s^N(s+\alpha,I) + f_s^N(s-\alpha,I)],\ \{using\ (i)
\end{aligned}$$

(iii)

$$\int_0^\infty \mathscr{F}(x,I)sins(p_n+q_nI)xdx = f_s^N(s,I) \quad \text{(i)}$$

Now,

$$\begin{aligned}
\mathscr{F}_c\{\mathscr{F}(x,I)sin\alpha(p_n+q_nI)x\} &= \int_{-\infty}^{\infty} \mathscr{F}(x,I)sin\alpha(p_n+q_nI)xcoss(p_n+q_nI)xdx, \{By\ defn.\ (i)\\
&= \frac{1}{2}\int_0^\infty \mathscr{F}(x,I)2coss(p_n+q_nI)xsin\alpha(p_n+q_nI)xdx\\
&= \frac{1}{2}\int_0^\infty \mathscr{F}(x,I)[sin(p_n+q_nI)(s+\alpha)x - sin(p_n+q_nI)(s-\alpha)x]dx\\
&= \frac{1}{2}[\int_0^\infty \mathscr{F}(x,I)sin(s+\alpha)(p_n+q_nI)xdx\\
&\quad - \int_0^\infty \mathscr{F}(x,I)sin(s-\alpha)(p_n+q_nI)xdx]\\
&= \frac{1}{2}[f_s^N(s+\alpha,I) - f_s^N(s-\alpha,I)],\ \{using\ (i)
\end{aligned}$$

(iv) By definition (28)

$$\int_0^\infty \mathscr{F}(x,I)coss(p_n+q_nI)xdx = f_c^N(s,I) \quad \text{(i)}$$

Now,

$$\begin{aligned}\mathscr{F}_s\{\mathscr{F}(x,I)sin\alpha(p_n+q_nI)x\} &= \int_{-\infty}^{\infty}\mathscr{F}(x,I)sin\alpha(p_n+q_nI)xsins(p_n+q_nI)xdx, \{By\ defn.\ (i)\\ &=\frac{1}{2}\int_0^{\infty}\mathscr{F}(x,I)2sins(p_n+q_nI)xsin\alpha(p_n+q_nI)xdx\\ &=\frac{1}{2}\int_0^{\infty}\mathscr{F}(x,I)[cos(p_n+q_nI)(s-\alpha)x-cos(p_n+q_nI)(s+\alpha)x]dx\\ &=\frac{1}{2}[\int_0^{\infty}\mathscr{F}(x,I)cos(s+\alpha)(p_n+q_nI)xdx\\ &\quad-\int_0^{\infty}\mathscr{F}(x,I)cos(s-\alpha)(p_n+q_nI)xdx]\\ &=\frac{1}{2}[f_c^N(s+\alpha,I)-f_c^N(s-\alpha,I)],\ \{using\ (i)\end{aligned}$$

13. **Solved examples based on Infinite F.T.:**

Example 1. To find the Fourier complex transform of $\mathscr{F}(x,I)$, if

$$\mathscr{F}(x,I)=\begin{cases}e^{iw(p_n+q_nI)x} & \text{if } \alpha<x<\beta\\ 0 & \text{if } x<\alpha \text{ or } x>\beta\end{cases}$$

Solution: By definition (16), we have

$$\begin{aligned}\mathscr{F}\{\mathscr{F}(x,I)\} &= \int_{-\infty}^{\infty}e^{is(p_n+q_nI)x}\mathscr{F}(x,I)dx\\ &= \int_{-\infty}^{\alpha}e^{is(p_n+q_nI)x}\mathscr{F}(x,I)dx+\int_{\alpha}^{\beta}e^{is(p_n+q_nI)x}\mathscr{F}(x,I)dx+\int_{\beta}^{\infty}e^{is(p_n+q_nI)x}\mathscr{F}(x,I)dx\\ &= 0+\int_{\alpha}^{\beta}e^{is(p_n+q_nI)x}.e^{iw(p_n+q_nI)x}dx+0,\ \{\text{using the given values of } \mathscr{F}(x,I).\\ &= \int_{\alpha}^{\beta}e^{i(s+w)(p_n+q_nI)x}dx\\ &= \left[\frac{e^{i(s+w)(p_n+q_nI)x}}{i(s+w)(p_n+q_nI)}\right]_{\alpha}^{\beta}\\ &= i.\frac{e^{i(s+w)(p_n+q_nI)\beta}-e^{i(s+w)(p_n+q_nI)\alpha}}{i^2(s+w)(p_n+q_nI)}\\ &= \frac{i}{(s+w)(p_n+q_nI)}\left[e^{i(s+w)(p_n+q_nI)\alpha}-e^{i(s+w)(p_n+q_nI)\beta}\right],\ \{since\ i^2=1\\ &= (\frac{1}{p_n}-\frac{q_n}{p_n(p_n+q_n)}I)\frac{i}{(s+w)}\left[e^{i(s+w)(p_n+q_nI)\alpha}-e^{i(s+w)(p_n+q_nI)\beta}\right]\end{aligned}$$

Example 2. To find the F.T. of

$$f^N(x,I)=\begin{cases}x+I, & \text{if } |x|\le\alpha\\ 0, & \text{if } |x|>\alpha\end{cases}$$

Solution: By definition (16), we have

$$\begin{aligned}
\mathscr{F}\{f^N(x,I)\} &= \int_{-\infty}^{\infty} f^N(x,I)e^{is(p_n+q_nI)x}dx \\
&= \int_{-\infty}^{-\alpha} f^N(x,I)e^{is(p_n+q_nI)x}dx + \int_{-\alpha}^{\alpha} f^N(x,I)e^{is(p_n+q_nI)x}dx + \int_{\alpha}^{\infty} f^N(x,I)e^{is(p_n+q_nI)x}dx \\
&= \int_{-\alpha}^{\alpha}(x+I)e^{is(p_n+q_nI)x}dx, \ \{\text{using the given values of } f^N(x,I). \\
&= \left[(x+I).\frac{e^{is(p_n+q_nI)x}}{is(p_n+q_nI)}\right]_{-\alpha}^{\alpha} - \int_{-\alpha}^{\alpha}\frac{e^{is(p_n+q_nI)x}}{is(p_n+q_nI)}dx, \ \{\text{using integration by parts.} \\
&= \frac{\alpha+I}{is(p_n+q_nI)}.e^{is(p_n+q_nI)\alpha} - \frac{-\alpha+I}{is(p_n+q_nI)}.e^{is(p_n+q_nI)(-\alpha)} - \left[\frac{.e^{is(p_n+q_nI)x}}{\{is(p_n+q_nI)\}^2}\right]_{-\alpha}^{\alpha} \\
&= \frac{1}{is(p_n+q_nI)}\left[(\alpha+I)e^{is(p_n+q_nI)\alpha} - (I-\alpha)e^{-is(p_n+q_nI)\alpha}\right] \\
&+ \frac{1}{s^2(p_n+q_nI)^2}\left[e^{is(p_n+q_nI)\alpha} - e^{-is(p_n+q_nI)\alpha}\right] \\
&= \frac{2I}{is(p_n+q_nI)}\left[\frac{e^{is(p_n+q_nI)\alpha} - e^{-is(p_n+q_nI)\alpha}}{2i}\right] \\
&+ \frac{2\alpha i}{i^2s(p_n+q_nI)}\left[\frac{e^{is(p_n+q_nI)\alpha} + e^{-is(p_n+q_nI)\alpha}}{2i}\right] \\
&+ \frac{2i}{s^2(p_n+q_nI)}\left[\frac{e^{is(p_n+q_nI)\alpha} - e^{-is(p_n+q_nI)\alpha}}{2i}\right] \\
&= \frac{2I}{s(p_n+q_nI)}sins(p_n+q_nI)\alpha - \frac{2\alpha i}{s(p_n+q_nI)}coss(p_n+q_nI)\alpha \\
&+ \frac{2i}{s^2(p_n+q_nI)^2}sins(p_n+q_nI)\alpha \\
&= (\frac{1}{p_n} - \frac{q_n}{p_n(p_n+q_n)}I).\frac{2}{s}sins(p_n+q_nI)\alpha - \frac{2\alpha i}{s}(\frac{1}{p_n} - \frac{q_n}{p_n(p_n+q_n)}I)coss(p_n+q_nI)\alpha \\
&+ \frac{2i}{s^2}(\frac{1}{p_n} - \frac{q_n}{p_n(p_n+q_n)}I)^2sins(p_n+q_nI)\alpha.
\end{aligned}$$

Example 3 To find the F.T. of $f^N(x,I)$ if

$$f^N(x,I) = \begin{cases} x^2+I, & \text{if } |x| \le \alpha \\ 0, & \text{if } |x| > \alpha \end{cases}$$

Solution: By definition (16), we have

$$\begin{aligned}
\mathscr{F}\{f^N(x,I)\} &= \int_{-\infty}^{\infty} f^N(x,I)e^{is(p_n+q_nI)x}dx \\
&= \int_{-\infty}^{-\alpha} e^{is(a+bIx)}dx + \int_{-\alpha}^{\alpha} f^N(x,I)e^{is(p_n+q_nI)x}dx + \int_{\alpha}^{\infty} f^N(x,I)e^{is(p_n+q_nI)x}dx \\
&= \int_{-\alpha}^{\alpha} (x^2+I)e^{is(p_n+q_nI)x}dx, \ \{\text{using the given values of } f^N(x,I) \\
&= \left[(x^2+I).\frac{e^{is(p_n+q_nI)x}}{is(p_n+q_nI)}\right]_{-\alpha}^{\alpha} - \int_{-\alpha}^{\alpha} 2x.\frac{e^{is(p_n+q_nI)x}}{is(p_n+q_nI)}dx, \ \{\text{using integration by parts.} \\
&= \frac{(\alpha^2+I)}{is(p_n+q_nI)}.e^{is(p_n+q_nI)\alpha} - \frac{(\alpha^2+I)}{is(p_n+q_nI)}.e^{-is(p_n+q_nI)\alpha} \\
&- \frac{2}{is(p_n+q_nI)}.\left[x.\frac{e^{is(p_n+q_nI)x}}{is(p_n+q_nI)}\right]_{-\alpha}^{\alpha} \\
&- \frac{2}{is(p_n+q_nI)}\int_{-\alpha}^{\alpha} 1.\frac{e^{is(p_n+q_nI)x}}{is(p_n+q_nI)}dx \\
&= \frac{(\alpha^2+I)}{is(p_n+q_nI)}.2isins(p_n+q_nI)\alpha - \frac{2}{i^2s^2(p_n+q_nI)^2}\left[\alpha e^{is(p_n+q_nI)\alpha} + \alpha e^{-is(p_n+q_nI)\alpha}\right] \\
&+ \frac{2}{i^2s^2(p_n+q_nI)^2}.\left[\frac{e^{is(p_n+q_nI)x}}{is(p_n+q_nI)}\right]_{-\alpha}^{\alpha} \\
&= \frac{2(\alpha^2+I)}{s(p_n+q_nI)}.sins(p_n+q_nI)\alpha + \frac{4\alpha}{s^2(p_n+q_nI)^2}coss(p_n+q_nI)\alpha \\
&- \frac{4}{s^3(p_n+q_nI)^3}sins(p_n+q_nI)\alpha
\end{aligned}$$

14. Conclusion:

In this article, some basics of Fourier Integrals have been discussed in terms of neutrosophic set. Dirichlet's Conditions, Fourier integral formula and it's five different forms are studied based on neutrosophic set. This article includes the F.T., F.S.T. and F.C.T. of a neutrosophic function and their inversion formulae. In this study, some properties of F.T. are discussed for a neutrosophic function. The purpose of this work is to extend the classical analysis of Fourier integrals and Fourier transform to Neutrosophic form. This study will help to get better results in signal processing, image processing, noise filtering and in some other fields also. Presently, in classical form, there are many applications of Fourier transform in different fields. In future, by using the concept of Neutrosophic in Fourier integrals and Fourier transform it can be axpected to get better results than the present applications of Fourier integrals and Fourier transform.

References

[1] F. Smarandache, Neutrosophy/ Neutrosophic Probability, Set, and Logic, American Research Press, Rehoboth, USA, 1998.

[2] F.Smarandache, Neutrosophy and Neutrosophic Logic, First International Conference on Neutrosophy, Neutrosophic Logic, Set, Probability, and Statistics, University of New Mexico, Gallup, NM 87301, USA 2002.

[3] F. Smarandache, Finite N.C.N.s, by W. B. Vasantha Kandasamy, Zip Publisher, Columbus, Ohio, USA, PP: 1-16, 2011.

[4] F.Smarandache, Introduction to Neutrosophic statistics, Sitech-Education Publisher, PP:34-44, 2014.

[5] F.Smarandache, A Unifying Field in Logics: Neutrosophic Logic, Preface by Charles Le, American Research Press, Rehoboth, 1999, 2000. Second edition of the Proceedings of the First International Conference on Neutrosophy, Neutrosophic Logic, Neutrosophic Set, Neutrosophic Probability and Statistics, University of New Mexico, Gallup, 2001.

[6] F.Smarandache, Proceedings of the First International Conference on Neutrosophy, Neutrosophic Set, Neutrosophic Probability and Statistics, University of New Mexico, 2001.

[7] F. Smarandache, Introduction to Neutrosophic Measure, Neutrosophic Integral, and Neutrosophic Probability, Sitech-Education Publisher, Craiova-Columbus, 2013.

[8] F.Smarandache, Neutrosophic Precalculus and Neutrosophic Calculus, book, 2015.

[9] Madeleine Al- Tahan, Some Results on Single Valued Neutrosophic (Weak) Polygroups, International Journal of Neutrosophic Science, Volume 2 , Issue 1, pp. 38-46 , 2020.

[10] S. A. Edalatpanah, A Direct Model for Triangular Neutrosophic Linear Programming, International Journal of Neutrosophic Science, Volume 1 , Issue 1, pp. 19-28 , 2020.

[11] A. Chakraborty, A New Score Function of Pentagonal Neutrosophic Number and its Application in Networking Problem, International Journal of Neutrosophic Science, Volume 1 , Issue 1, pp.40-51 , 2020.

[12] A. Chakraborty, Application of Pentagonal Neutrosophic Number in Shortest Path Problem, International Journal of Neutrosophic Science, Volume 3 , Issue 1, pp. 21-28 , 2020.

[13] S. Mustafi; T. Latychevskaia, Fourier Transform Holography: A Lensless Imaging Technique, Its Principles and Applications,Multidisciplinary Digital Publishing Institude, 2023.

[14] U. M. Pirzada1 · D. C. Vakaskar2, Fuzzy solution of homogeneous heat equation having solution in Fourier series form. https://doi.org/10.1007/s40324-018-0169.

[15] A.Kharal, a Neutrosophic Multicriteria Decision Making Method, 2011.

[16] A. A. Salama, F.Smarandache, Neutrosophic Crisp Set Theory, Educational Publisher, Columbus, 2015

[17] D. Koundal; S. Gupta; S. Singh, Applications of Neutrosophic Sets in Medical Image Denoising and Segmentation, New Trends in Neutrosophic Theory and Applications, pp. 257-273.

[18] N. Chittora and D. Babel, A brief study on Fourier transform and its applications, International Research Journal of Engineering and Technology, Volume 05, Issue 12, pp. 1127-1131 , 2018.

[19] U. Kadak; F. Basar, On Fourier Series of Fuzzy Valued Functions, The Scientific World of Journal, 2014.

[20] F. Smarandache, New Types of Topologies and Neutrosophic Topologies, Neutrosophic Sysytems with Applications, Vol. 1, 2023, https://doi.org/10.61356/j.nswa.2023.1.

[21] S. N. I. Rosli; M. I. E. Zulkifly, Neutrosophic Bicubic B-spline Surface Interpolation Model for Uncertainty Data, Neutrosophic Systems with Applications, Vol. 10, 2023, https://doi.org/10.61356/j.nswa.2023.69.

Received: June 26, 2024. Accepted: Oct 11, 2024

Neutrosophic Sets and Systems, Vol. 75, 2025
University of New Mexico

Pairwise Approach to Neutrosophic Soft Bi-Topological Spaces: Exploring Neutrosophic Soft Open Sets and Their Real-Life Applications

Maha Mohammed Saeed[1], Samer Al Ghour[2*], Arif Mehmood[3*], Wahid Rehman[4], Fabio Souto de Azevedo[4], R. Seethalakshi[5], Aamir Abbas[3], M. Kaviyarasu[6*]

[1]Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia; (mmmohammed@kau.edu.sa)

[2*]Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid, 22110, Jordan; (algore@just.edu.jo)

[3*]Institute of Numerical Sciences, Department of Mathematics Gomal University Dera Ismail Khan, KPK, Pakistan; (mehdaniyal@gmail.com); (abbasaamir113@gmail.com)

[4]Federal University of Rio Grande do Sul Campus Do Vale, Brazil; (wahidkhattak601@gmail.com); (fabio.azevedo@ufrgs.br);

[5]Department of Mathematics (H&R), Rajalakshmi Institute of Technology Kuthambakkam, Chennai 600124, India; (seethamaths79@gmail.com)

[6*]Department of Mathematics, Vel Tech Rangarajan Dr. Sagunthala R &D Institute of Science and Technology, Avadi, Chennai 600062, Tamil Nadu, India; (kavitamilm@gmail.com)

***Corresponding authors email address**; (algore@just.edu.jo);(mehdaniyal@gmail.com)

Abstract: In this article, eight new definitions are introduced in the context of neutrosophic soft bi-topological spaces with respect to the soft points of the space. Among these new definitions, one important concept known as the soft b^{**} open set is selected for a pairwise discussion of different structures. These structures include separation axioms, subspaces, the connections between separation axioms and closures, results concerning interiors and closures, the product of separation axioms, and their relationships with neutrosophic soft coordinate spaces regarding soft points. Finally, the most significant structure known as entropy is discussed, and based on this structure, an example related to engineering is presented. In this example, to achieve effective risk supervision in the field of engineering, certain risks are classified along with specific parameters, and these risks are evaluated by a team of expert engineers. Mehmood et al. [40] proposed a new approach to generalized neutrosophic soft sets in neutrosophic soft topology concerning the soft points of the space and discussed several structures related to this new approach. The references by Ozturk et al. [32-35] led us to this present work, with [32] serving as a significant source of motivation. In this

particular work, the author defined neutrosophic soft topological space in an exceptionally novel way.

Keywords: Neutrosophic soft set, neutrosophic soft bi-topology, neutrosophic soft open set, neutrosophic soft b^{**} open sets, neutrosophic soft separation axioms and neutrosophic soft entropy.

1. Introduction

Zadeh [1] proposed afuzzy set (FS) to solve ambiguous information. After that, it has been successfully used in different areas [2-7]. Atanassov [8] extended FS to the intuitionistic fuzzy set (IFS) which is very effective to deal with vagueness of the information. The IFS has the ability to express fuzziness and uncertainty of practical situations which each element concludes membership degree, non-membership degree, and hesitation degree and they are presented by exact numbers. The sum of these three functions is number 1. In general, the degrees may be not exact numbers but interval values. Then the theory of IFS was extended to interval-valued IFS [9]. With the development of the theory of FS and its extension, they have been handled many uncertainties in different real-life problems. But, there are many phenomena cannot be dealt by the FS and its extension. For example, when we ask an expert about a complex statement, he or she may not give the exact answer of the problem and say that the possibility that the statement is true is 0.6, that is false is 0.5 and the degree that he or she is not sure is 0.3. This situation cannot be expressed by FS and IFS, and some new set is needed to express the condition. Neutrosophic set (NS) was first proposed by Smarandache [10, 11] from philosophical point of view. A NS is a set that each element has truth membership degree, indeterminacy membership degree and falsity membership degree. As we known, it was difficult to apply the NS in real science and engineering fields. Smarandache [50] made extension of soft set to hyper soft set, and then to plithogenic hyper soft set. The author generalized the soft set to the hyper soft set by transforming the function F into a multi-attribute function and also introduced the hybrids of crisp, fuzzy, intuitionistic fuzzy, neutrosophic, and plithogenic hyper-soft Set. Smarandache [51] generalized the soft set to the hyper-soft set by transforming the function F into a multi-attribute function. and in continuation, introduced the hybrids of crisp, Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Plithogenic Hyper-soft Set Wang et al. [12, 13] proposed single valued neutrosophic set

(SVNS) and interval valued neutrosophic set (IVNS). SVNS and IVNS are the subclasses of simplified neutrosophic set which was proposed by Ye [14] and he presented the operators and relationships of the sets and gave their different properties. Therefore, some new results of SNS appeared [15, 16].

Molodtsov developed soft set theory in a fundamentally different perspective. The application of this theory can be used for meaningfully interpreting real life problems in pure and applied sciences involving imprecise data. Current studies shows that ambiguities in data mining problems can also be solved using soft set theory techniques. The soft set theory could be used to interrogate and extend the idea of probability, fuzzy set, rough set and intuitionistic fuzzy set further. The disadvantage of lack of parameterization tool related to the concepts mentioned above gave a higher realm to soft set theory. In short, unlimited nature of approximate description is the greatest advantage of soft set theory. Molodtsov [17] published the paper titled "Soft set theory: first results", which is considered as the origin of theory of soft sets. Apart from the basic notions of the theory, some of its possible applications and some problems of the future research directions are also discussed in this paper. This theory was further solidified by Maji et al. [18] by defining some fundamentals of the theory such as equality of two soft sets, subset and super set of a soft set etc. As continuation of these ideas, many extensions, hybridizations and extensions were put forward by many authors, some of them are the following: Maji et al. [19] introduced fuzzy soft sets, Wang et al. [20] introduced hesitant fuzzy soft sets and Pei and Miao [21] explore the relationship between fuzzy soft sets and classical information systems. The theory of soft sets is still developing very rapidly both in theoretical as well as application perspectives.

John [22] introduced a soft structure over a set to make a certain discretization of such fundamental mathematical concepts with effectively continuous nature. The benefit is that it provided new tools for the use of the technology of mathematical analysis in real applications catching uncertainty or imperfect data. John [23] studied qualitative properties of certain objects, called topological spaces that are invariant under certain kinds of transformations, called continuous maps. The author also addressed some generalized structures. John [24] have had a quick look into some structures. These are hybridizations including Fuzzy sets, Intuitionistic fuzzy sets, Hesitant fuzzy sets, Rough sets etc.

Al-shami et al. [25] continuously worked on menger spaces and defined a weak type of soft menger spaces, namely, nearly soft menger spaces and gave their complete description using soft s-regular open covers and proved that they coincided with soft menger spaces in the class of soft regular spaces. Also, the authors

studied the role of enriched and soft regular spaces in preserving nearly soft meagerness between soft topological spaces and their parametric topological spaces. Finally, the authors established some properties of nearly soft menger spaces with respect to hereditary and topological properties and product spaces. Al-shami et al. [26] sorted out the weakest conditions that preserve some topologically inspired properties and introduced the concept of an infra soft topological spaces which is a collection of subsets that extend the concept of soft topology by dispensing with the postulate that the collection is closed under arbitrary union. The authors studied the basic concepts of infra soft topological spaces such as infra soft open and infra soft closed sets, infra soft interior and infra soft closure operators, and infra soft limit and infra soft boundary points of a soft set. The authors devised some techniques to generate infra soft topologies and explored the concept of continuity between infra soft topological spaces and determined the conditions under which the continuity is preserved between infra soft topological space and its parametric infra topological spaces. Al-shami et al. [27] discussed weak forms of soft separation axioms and fixed soft points. The authors discussed soft separation axioms and other separation axioms with respect generalized open sets. Al-shami et al. [28] introduced the concept of infra soft connected and infra soft locally connected spaces and discussed the behaviors of infra soft connected and infra soft locally connected spaces under infra soft homeomorphism maps and a finite product of soft spaces. The authors discussed component of a soft point and established its main properties. Al-shami [29] redefined the concept of soft mappings to be convenient for studying the topological concepts and notions in different soft structures and introduced the concepts of open, closed, and homeomorphism mappings in the content of infra soft topology. The authors established main properties and investigated the transmission of these concepts between infra soft topology and its parametric infra topologies and defined a quotient infra soft topology and infra soft quotient mappings and studied their main properties with the aid of illustrative examples.

Al-shami [30] discussed infra soft compact and infra soft Lindelof spaces and described them using a family of infra soft closed sets and displayed their main properties. The author focused on studying the “transmission” of these concepts between infra soft topology and classical infra topology which helped us to discover the behaviors of these concepts in infra soft topology using their counterparts in classical infra topology and vice versa. Among the obtained results, these concepts are closed under infra soft homeomorphisms and finite product of soft spaces. Finally, the author introduced the concept of fixed soft points and reveal main

characterizations, especially those induced from infra soft compact spaces. Asad et al. [31] originated the notion of soft bi-operators. Mehmood et al. [41] studied and discussed new operations of union, intersection, and complement with the help of vague soft sets in a new way. On the basis of these operations, vague soft topology is re-defined. Pairwise vague soft open sets and pairwise vague soft closed sets are also re-defined in vague soft bitopological structures (VSBTS). Moreover, generalized vague soft open sets are introduced in VSBTS concerning soft points of the space. On the basis of generalized vague soft open sets, separation axioms are also introduced. In continuation, these separations axioms are engaged with other important results in VSBTS. Mehmood et al. [42] installed a new concept of vague soft bi-topological space which is based on generalized vague soft open set and is known as vague soft β open set and on the basis of this new concept few structures are re-generated in vague soft bi-topological space with respect to soft points of the spaces. Abobala [48] discussed maximal and minimal ideals of n-refined neutrosophic rings.

Vadivel et al. [52] introduced a new types of δ-open sets and δ-closed sets in neutrosophic topological spaces which are a stronger form. Also, the notion of neutrosophic δ mappings were introduced and also discuss their relationship between near mappings in neutrosophic topological spaces. Moreover, the authors investigated some of their basic properties and examples in neutrosophic topological spaces.

Ozturk et al. [32] pioneers of new operations on neutrosophic soft sets. These operations are union and intersections and discussed basic results. Ozturk et al. [33] unveiled the concept of neutrosophic soft mapping, neutrosophic soft open mapping and neutrosophic soft homeomorphism on the basis of operation defined in [32]. Some results are secured with best understandable examples. Gunduz et al. [34] introduced separation axioms in neutrosophic soft topological spaces with respect to soft points. Ozturk, Taha Yasin [35] introduced new concepts in neutrosophic soft topological spaces. These concepts are boundary, dense set and neutrosophic soft basis. In addition, the concept of soft subspace on neutrosophic soft topological spaces. Some interesting results are addressed with respect to soft points. Some complicated results are secured with best examples. Acikgoz and Esenbel [53] introduced the concepts of neutrosophic soft δ−interior, neutrosophic soft quasi coincidence, neutrosophic soft q-neighborhood, neutrosophic regular open soft set, neutrosophic soft δ−closure, neutrosophic soft θ−closure, neutrosophic semi open soft set and established that the set of all neutrosophic soft δ−open sets is also a neutrosophic soft topology, which is called the neutrosophic soft δ−topology and the

authors obtained equivalent forms of neutrosophic soft δ−continuity. Moreover, the notions of neutrosophic soft δ−compactness and neutrosophic soft locally δ−compactness are defined and their basic properties under neutrosophic soft δ−continuous mappings are investigated.

Mehmood et al. [36] introduced a new approach to neutrosophic soft bi topological spaces relative to soft points of the spaces**.** Mehmood et al. [43] introduced neutrosophic soft $**_b$ open set with the help of neutrosophic soft α open set and neutrosophic soft β open set and with the help of this new definition some neutrosophic soft separation axioms and neutrosophic soft other separation axioms are addressed and in continuation, soft countability and its engagements with other neutrosophic soft results are also addressed.

Deli and Broumi [37] installed concept of a relation on neutrosophic soft sets which permit to compose two neutrosophic soft sets. It is devised to derive advantageous information through the composition of two neutrosophic soft sets. Then, the authors scanned symmetric, transitive and reflexive neutrosophic soft relations and many related concepts such as equivalent neutrosophic soft set relation, partition of neutrosophic soft sets, equivalence classes, quotient neutrosophic soft sets, neutrosophic soft composition are given and their propositions are discussed. Finally, a decision making technique on neutrosophic soft sets is presented. Bera and Mahapatra [38] installed the new structure to the world of mathematics. This structure is known as neutrosophic soft topology and in addition the authors discussed all the fundamentals. Maji [39] introduced concept neutrosophic soft set. Some definitions and operations are introduced and established some properties. Mehmood et al. [40] launched a new way to attempt generalized neutrosophic soft sets in neutrosophic soft topology concerning neutrosophic soft points of the space and discussed few structures concerning these new approached. Hayat et al. [57] developed a framework based on soft sets, TOPSIS and Shannon entropy to identify customer preferences. TOPSIS provides integrated evaluations to obtain the best concept for two customers, acceptable for both, satisfactory for both, and vise versa. Hayat et al. A simple graph's underlying subgraphs (regular subgraphs, irregular subgraphs, cycles, and trees) were used by Hayat et al. [58] to characterize type 2 soft graphs. The authors then presented regular type 2 soft graphs, irregular type 2 soft graphs, and type 2 soft trees. Moreover, type 2 soft cycles, type 2 soft cut-nodes, and type 2 soft bridges were introduced by the authors. In order to illustrate these new ideas, several operations on type 2 soft trees are shown, along with various instances. Following references Ozturk et al. [32-35] leaded me to this present work.

In section 1, the basic definitions are given which are necessary for the up-coming sections. These basic definitions are neutrosophic soft set, complement of neutrosophic soft set, equality of neutrosophic soft sets, union and intersection of neutrosophic soft sets, absolute neutrosophic soft set, neutrosophic soft topology and neutrosophic soft closer. In section 2, pair-wise neutrosophic soft open sets in neutrosophic soft bi-topological space and for the supportment of this structure an example is given. Intersection of two neutrosophic soft topological space, the non-validity of union of two neutrosophic soft topological spaces is discussed with an example. In this section, new types of open and close sets have been introduced in neutrosophic soft bi topological spaces. Eight new neutrosophic soft open sets are introduced. Among these, neutrosophic soft b^{**} is chosen to generate the results pairwisly in neutrosophic soft bi-topological space with respect to soft points. These results are different neutrosophic soft separations axioms and their interconnection among themselves.

In this section 3, neutrosophic soft limit point, neutrosophic soft interior point, few results on the basis of these definitions are discussed. These results are discussed terms of neutrosophic soft interior and neutrosophic soft closures with respect to soft points. In this section 4, some spaces are discussed in terms of product with respect to soft points of the spaces. Product of neutrosophic soft T_0 spaces, neutrosophic soft T_1 spaces, neutrosophic soft T_2 spaces and their engagement with neutrosophic soft coordinate spaces with respect to soft points are studied under soft b^{**} open sets in neutrosophic soft bi-topological space. Finally the most important structure known as entropy is discussed and on the basis of this structures an example is given which is related to engineering which goes like this, in order to obtain an effective risk supervision in the field of engineering, certain risks are classified along with some parameters and these risks are evaluated by the team of experts (engineers). In the final section some concluding comments are summarized and future planning is given.

2. Preliminaries

In this section the basic definitions are given which are necessary for the up-coming sections. These basic definitions are neutrosophic soft set, complement of neutrosophic soft set, equality of neutrosophic soft sets, union and intersection of neutrosophic soft sets, absolute neutrosophic soft set, neutrosophic soft topology and neutrosophic soft closer.

Definition 1.1[18] Let $\mathcal{X}$ be a universal set and $\Re$ be a set of parameters. Let $P(\mathcal{X})$ denote the set of all neutrosophic sets of $\mathcal{X}$ then a neutrosophic soft set $(\tilde{F},\Re)$ o v er $\mathcal{X}$ is a set defined by a set of valued function representing a mapping $\tilde{F}:\Re \to P(\mathcal{X})$ where $\tilde{F}$

is called approximate function of the neutrosophic soft set $(\tilde{F},\Re)$. In other words the neutrosophic soft set is a parameterized family of some elements of the set $P(\mathcal{X})$ and therefore it can be written as a set of ordered pairs,$(\tilde{F},\Re)=\left\{\left(\left(e,\langle x,\mathbb{T}_{\tilde{f}(e)}(x),\ \mathbb{I}_{\tilde{f}(e)}(x),\mathbb{F}_{\tilde{f}(e)}(x)\rangle:x\in\mathcal{X}\right):e\in\Re\right)\right\}$,

Where, $\mathbb{T}_{\tilde{f}(e)}(x),\ \mathbb{I}_{\tilde{f}(e)}(x),\mathbb{F}_{\tilde{f}(e)}(x)\in[0,1]$, respectively called the truth-membership, indeterminacy-membership, falsity-membership function of $\tilde{F}(e)$.Since supremum of each $\mathbb{T},\mathbb{I},\mathbb{F}$ is 1 so the inequality $0\leq\mathbb{T}_{\tilde{f}(e)}(x)+\mathbb{I}_{\tilde{f}(e)}(x)+\mathbb{F}_{\tilde{f}(e)}(x)\leq 3$ is obvious.

Definition 1.2[19] Let$(\tilde{F},\Re)$be a neutrosophic soft set over universal set $\mathcal{X}$ and the complement of $(\tilde{F},\Re)$ is denoted by $(\tilde{F},\Re)^c$ and is defined by:

$(\tilde{F},\Re)^c=\left\{\left(\left(e,\langle x,\mathbb{F}_{\tilde{f}(e)}(x),1-\mathbb{I}_{\tilde{f}(e)}(x),\mathbb{T}_{\tilde{f}(e)}(x)\rangle:x\in\mathcal{X}\right):e\in\Re\right)\right\}$ and It is clear that $\left((\tilde{F},\Re)^c\right)^c=(\tilde{F},\Re)$.

Definition 1.3[20] Let $(\tilde{F},\Re)$ and $(\tilde{G},\Re)$ two neutrosophic soft sets over universal set $\mathcal{X}$.$(\tilde{F},\Re)$ is said to be neutrosophic soft sub set of $(\tilde{G},\Re)$ if $\mathbb{T}_{\tilde{F}(e)}(x)\preccurlyeq\mathbb{T}_{\tilde{G}(e)}(x),\mathbb{I}_{\tilde{F}(e)}(x)\preccurlyeq\mathbb{I}_{\tilde{G}(e)}(x),\mathbb{F}_{\tilde{F}(e)}(x)\succcurlyeq\mathbb{F}_{\tilde{G}(e)}(x),\forall\ e\in\Re$ and $\forall\ x\in\mathcal{X}$. It is denoted by $(\tilde{F},\Re)\subseteq(\tilde{G},\Re)$.$(\tilde{F},\Re)$is is said to be neutrosophic soft equal to $(\tilde{G},\Re)$ if $(\tilde{F},\Re)$is neutrosophic soft sub set of $(\tilde{G},\Re)$ and $(\tilde{G},\Re)$ is neutrosophic soft sub set if $(\tilde{F},\Re)$and is denoted by $(\tilde{F},\Re)=(\tilde{G},\Re)$.

Definition 1.4[20] let $(\tilde{F},\Re)$and$(\tilde{G},\Re)$ be two neutrosophic soft sub sets over universal set $\mathcal{X}$ such that$\left((\tilde{F},\Re)\neq(\tilde{G},\Re)\right.$ then their union is denoted by $(\tilde{F},\Re)\cup(\tilde{G},\Re)=(\tilde{H},\Re)$ and is defined by $(\tilde{H},\Re)=\{\left(\left(e,[\![x,\mathbb{T}_{\tilde{H}(e)}(x),\ \mathbb{I}_{\tilde{H}(e)}(x),\mathbb{F}_{\tilde{H}(e)}(x):x\in\mathcal{X}]\!]\right):e\in\Re\right)\}$

Where,

$$\mathbb{T}_{\tilde{H}(e)}(x) = max\left[\mathbb{T}_{\tilde{F}(e)}(x), \mathbb{T}_{\tilde{G}(e)}(x)\right]$$

$$\mathbb{I}_{\tilde{H}(e)}(x) = max\left[\mathbb{I}_{\tilde{F}(e)}(x), \mathbb{I}_{\tilde{G}(e)}(x)\right]$$

$$\mathbb{F}_{\tilde{H}(e)}(x) = min\left[\mathbb{F}_{\tilde{F}(e)}(x), \mathbb{F}_{\tilde{G}(e)}(x)\right]$$

Definition 1.5 [13] Let $(\tilde{F}, \Re)$ and $\tilde{G}, \Re)$ be two neutrosophic soft sub sets over the universal set $\mathcal{X}$ such that $(\tilde{F}, \Re) \neq (\tilde{G}, \Re)$ then their intersection is denoted by $(\tilde{F}, \Re) \cap (\tilde{G}, \Re) = (\tilde{H}, \Re)$ and is defined as follows $(\tilde{H}, \Re) = \{\left((e, [\![x, \mathbb{T}_{\tilde{H}(e)}(x), \mathbb{I}_{\tilde{H}(e)}(x), \mathbb{F}_{\tilde{H}(e)}(x) : x \in \mathcal{X}]\!]) : e \in \Re\right\}$ where

$$\mathbb{T}_{\tilde{H}(e)}(x) = min\left[\mathbb{T}_{\tilde{F}(e)}(x), \mathbb{T}_{\tilde{G}(e)}(x)\right]$$

$$\mathbb{I}_{\tilde{H}(e)}(x) = min\left[\mathbb{I}_{\tilde{F}(e)}(x), \mathbb{I}_{\tilde{G}(e)}(x)\right]$$

$$\mathbb{F}_{\tilde{H}(e)}(x) = max\left[\mathbb{F}_{\tilde{F}(e)}(x), \mathbb{F}_{\tilde{G}(e)}(x)\right]$$

Definition 1.6 [16] Neutrosophic soft set $(\tilde{F}, \Re)$ over the universal set $\mathcal{X}$ is said to be a null neutrosophic soft set

If $\mathbb{T}_{\tilde{F}(e)}(x) = 0, \mathbb{I}_{\tilde{F}(e)}(x) = 0, \mathbb{F}_{\tilde{F}(e)}(x) = 1; \forall\, e \in \Re$ and $\forall\, x \in \mathcal{X}$.

It is denoted by $0_{(\mathcal{X},\Re)}$.

Definition 1.7 [21] Neutrosophic soft set $(\tilde{F}, \Re)$ over universal set $\mathcal{X}$ is said to be an absolute neutrosophical softness if $\mathbb{T}_{\tilde{F}(e)}(x) = 1, \mathbb{I}_{\tilde{F}(e)}(x) = 1, \mathbb{F}_{\tilde{F}(e)}(x) = 0; \forall\, e \in \Re$ and $\forall\, x \in \mathcal{X}$. It is signified as $1_{(\mathcal{X},\Re)}$ clearly, ${0_{(\mathcal{X},\Re)}}^c = 1_{(\mathcal{X},\Re)}$ and ${1_{(\mathcal{X},\Re)}}^c = 0_{(\mathcal{X},\Re)}$.

Definition 1.8[13] Let $NSS\ (\mathcal{X})$ be the family of all neutrosophic soft sets over universal set $\mathcal{X}$ and $\tau \subset \text{NSS}\ (\mathcal{X})$ then τ is said to be neutrosophic soft topology on $\mathcal{X}$ if:

(1): $0_{(\mathcal{X},\Re)}, 1_{(\mathcal{X},\Re)}. \in \tau,$

(2): The union of any number of neutrosophic soft sets in τ belongs to τ,

(3): The intersection of a finite number of neutrosophic soft sets in in τ belongs to τ then ($\mathcal{X}$, τ,$\Re$) is said to be a neutrosophic soft topological space over $\mathcal{X}$. Each member of τ is said to be a neutrosophic soft open set.

Definition 1.9 [13] *Let* ($\mathcal{X}$, τ , $\Re$) be a neutrosophic soft topological space over $\mathcal{X}$ and $(\tilde{F},\Re)$ be a neutrosophic soft set over $\mathcal{X}$ then $(\tilde{F},\Re)$ is supposed to be a neutrosophic soft close set if and only if its complement is a neutrosophic set open set.

2. Neutrosophic Pairwise Approach

In this section, pair-wise neutrosophic soft open sets in neutrosophic soft bi-topological space and for the supportment of this structure an example is given. Intersection of two neutrosophic soft topological space, the non-validity of union of two neutrosophic soft topological spaces is discussed with an example. In this section, new types of open and close sets have been introduced in neutrosophic soft bi topological spaces. Eight new neutrosophic soft open sets are introduced. Among these, neutrosophic soft b^{**} is chosen to generate the results pairwisly in neutrosophic soft bi-topological space with respect to soft points. These results are different neutrosophic soft separations axioms and their interconnection among themselves.

Definition 2.1 If$(\mathcal{X},\tau_1,\partial)$ and $(\mathcal{X},\tau_2,\partial)$ are two neutrosophic soft topological space (NSTS) then $(\mathcal{X},\tau_1,\tau_2,\partial)$ is called neutrosophic soft bi topological space. A neutrosophic soft sub-set $\langle\tilde{\eta},\partial\rangle$ is said to be neutrosophic soft pairwise open in $(\mathcal{X},\tau_1,\tau_2,\partial)$ if there exists a neutrosophic soft open set $\langle\widetilde{\eta_1},\partial\rangle$ in τ_1 neutrosophic soft open set $\langle\widetilde{\eta_2},\partial\rangle$ in τ_2 such that $\langle\tilde{\eta},\partial\rangle = \langle\widetilde{\eta_1},\partial\rangle \,\tilde{\cup}\, \langle\widetilde{\eta_2},\partial\rangle$.

Example 2.2 Let $\mathcal{X} = \{x_1, x_2, x_3\}$, $\partial = \{s_1, s_2\}$ and $\tau_1 = \{0_{(\mathcal{X},\partial)}, 1_{(\mathcal{X},\partial)}, (\mathcal{P}_1,\partial), (\mathcal{P}_2,\partial)\}$,

$\tau_2 = \{0_{(\mathcal{X},\partial)}, 1_{(\mathcal{X},\partial)}, (\mathbb{J}_1,\partial), (\mathbb{J}_2,\partial)\}$, where$(\mathcal{P}_1,\partial), (\mathcal{P}_2,\partial), (\mathbb{J}_1,\partial)$, $(\mathbb{J}_2,\partial)$ being neutrosophic soft sub sets are as following:

$$f_{(\mathcal{P}_1,\partial)}(s_1) = \begin{pmatrix} < x_1, 02\times10^{-1}, 03\times10^{-1}, 08\times10^{-1} >, \\ < x_2, 02\times10^{-4}, 03\times10^{-4}, 08\times10^{-4} >, \\ < x_3, 02\times10^{-1}, 04\times10^{-1}, 03\times10^{-1} > \end{pmatrix}$$

$$f_{(\mathcal{P}_1,\partial)}(s_2)=\begin{pmatrix}<x_1,03\times10^{-1},02\times10^{-1},06\times10^{-1}>,\\<x_2,01\times10^{-1},05\times10^{-1},05\times10^{-1}>,\\<x_3,04\times10^{-1},03\times10^{-1},04\times10^{-1}>\end{pmatrix}$$

$$f_{(\mathcal{P}_2,\partial)}(s_1)=\begin{pmatrix}<x_1,04\times10^{-1},03\times10^{-1},06\times10^{-1}>,\\<x_2,04\times10^{-1},05\times10^{-1},03\times10^{-1}>,\\<x_3,03\times10^{-1},05\times10^{-1},02\times10^{-1}>\end{pmatrix}$$

$$f_{(\mathcal{P}_2,\partial)}(s_2)=\begin{pmatrix}<x_1,03\times10^{-1},04\times10^{-1},05\times10^{-1}>,\\<x_2,02\times10^{-1},06\times10^{-1},04\times10^{-1}>,\\<x_3,04\times10^{-1},06\times10^{-1},03\times10^{-1}>\end{pmatrix}$$

$$f_{(\mathbb{J}_1,\partial)}(s_1)=\begin{pmatrix}<x_1,05\times10^{-1},04\times10^{-1},04\times10^{-1}>,\\<x_2,06\times10^{-1},06\times10^{-1},02\times10^{-1}>,\\<x_3,04\times10^{-1},06\times10^{-1},01\times10^{-1}>\end{pmatrix}$$

$$f_{(\mathbb{J}_1,\partial)}(s_2)=\begin{pmatrix}<x_1,04\times10^{-1},06\times10^{-1},03\times10^{-1}>,\\<x_2,03\times10^{-1},07\times10^{-1},03\times10^{-1}>,\\<x_3,05\times10^{-1},07\times10^{-1},01\times10^{-1}>\end{pmatrix}$$

$$f_{(\mathbb{J}_2,\partial)}(s_1)=\begin{pmatrix}<x_1,01\times10^{-1},02\times10^{-1},07\times10^{-1}>,\\<x_2,03\times10^{-1},03\times10^{-1},03\times10^{-1}>,\\<x_3,01\times10^{-1},02\times10^{-1},02\times10^{-1}>\end{pmatrix}$$

$$f_{(\mathbb{J}_2,\partial)}(s_2)=\begin{pmatrix}<x_1,01\times10^{-1},02\times10^{-1},07\times10^{-1}>,\\<x_2,03\times10^{-1},03\times10^{-1},03\times10^{-1}>,\\<x_3,01\times10^{-1},02\times10^{-1},02\times10^{-1}>\end{pmatrix}.$$

Therefore, τ_1 and τ_2 are neutrosophic soft topological spaces on $\mathcal{X}$ and so $(\mathcal{X},\tau_1,\tau_2,\partial)$ is a neutrosophic soft bi topological spaces.

Theorem 2.3 Let $(\mathcal{X},\tau_1,\tau_2,\partial)$ be a neutrosophic soft topological space then $\tau_1 \tilde{\cap} \tau_2$ is a neutrosophic soft topological space on $\mathcal{X}$.

Proof. For this we have to verify all the three conditions of neutrosophic soft bi topological space. Conditions (1) and (3) are obvious, for condition (2) , let $\{(\mathcal{P}_i,\partial); i\in I\}\in \tau_1 \tilde{\cap} \tau_2$ then $(\mathcal{P}_i,\partial)\in\tau_1$, $(\mathcal{P}_i,\partial)\in\tau_2$ as τ_1 , τ_2 are neutrosophic soft topological spaces on $\mathcal{X}$, then $\cup_i(\mathcal{P}_i,\partial)\in\tau_1, \cup_i(\mathcal{P}_i,\partial)\in\tau_2$. Therefore $\cup_i(\mathcal{P}_i,\partial)\in\tau_1 \tilde{\cap} \tau_2$.

Remark 2.4 Let $(\mathcal{X},\tau_1,\tau_2,\partial)$ be a neutrosophic soft topological space then $\tau_1 \tilde{\cup} \tau_2$need not be a neutrosophic soft topological space on $\mathcal{X}$.

Example 2.5 Let $\mathcal{X} = \{x_1, x_2, x_3\}$, $\partial = \{s_1, s_2\}$ and $\tau_1 = \{0_{(\mathcal{X},\partial)}, 1_{(\mathcal{X},\partial)}, (\mathcal{P}_1, \partial), (\mathcal{P}_2, \partial), (\mathcal{P}_3, \partial)\}$, $\tau_2 = \{0_{(\mathcal{X},\partial)}, 1_{(\mathcal{X},\partial)}, (\mathbb{J}_1, \partial), (\mathbb{J}_2, \partial)\}$, where $(\mathcal{P}_1, \partial), (\mathcal{P}_2, \partial), (\mathcal{P}_3, \partial), (\mathbb{J}_1, \partial)$, $(\mathbb{J}_2, \partial)$and being neutrosophic soft sub sets are as following:

$$f_{(\mathcal{P}_1,\partial)}(s_1) = \begin{pmatrix} < x_1, 02 \times 10^{-1}, 03 \times 10^{-1}, 08 \times 10^{-1} >, \\ < x_2, 04 \times 10^{-1}, 04 \times 10^{-1}, 04 \times 10^{-1} >, \\ < x_3, 02 \times 10^{-1}, 04 \times 10^{-1}, 03 \times 10^{-1} > \end{pmatrix}$$

$$f_{(\mathcal{P}_1,\partial)}(s_2) = \begin{pmatrix} < x_1, 03 \times 10^{-1}, 02 \times 10^{-1}, 06 \times 10^{-1} >, \\ < x_2, 01 \times 10^{-1}, 05 \times 10^{-1}, 05 \times 10^{-1} >, \\ < x_3, 04 \times 10^{-1}, 03 \times 10^{-1}, 05 \times 10^{-1} > \end{pmatrix}$$

$$f_{(\mathcal{P}_2,\partial)}(s_1) = \begin{pmatrix} < x_1, 04 \times 10^{-1}, 03 \times 10^{-1}, 06 \times 10^{-1} >, \\ < x_2, 04 \times 10^{-1}, 05 \times 10^{-1}, 03 \times 10^{-1} >, \\ < x_3, 03 \times 10^{-1}, 05 \times 10^{-1}, 02 \times 10^{-1} > \end{pmatrix}$$

$$f_{(\mathcal{P}_2,\partial)}(s_2) = \begin{pmatrix} < x_1, 03 \times 10^{-1}, 04 \times 10^{-1}, 05 \times 10^{-1} >, \\ < x_2, 02 \times 10^{-1}, 06 \times 10^{-1}, 04 \times 10^{-1} >, \\ < x_3, 04 \times 10^{-1}, 06 \times 10^{-1}, 03 \times 10^{-1} > \end{pmatrix}$$

$$f_{(\mathcal{P}_3,\partial)}(s_1) = \begin{pmatrix} < x_1, 05 \times 10^{-1}, 04 \times 10^{-1}, 04 \times 10^{-1} >, \\ < x_2, 06 \times 10^{-1}, 06 \times 10^{-1}, 02 \times 10^{-1} >, \\ < x_3, 04 \times 10^{-1}, 06 \times 10^{-1}, 00 \times 10^{-1} > \end{pmatrix}$$

$$f_{(\mathcal{P}_3,\partial)}(s_2) = \begin{pmatrix} < x_1, 04 \times 10^{-1}, 06 \times 10^{-1}, 03 \times 10^{-1} >, \\ < x_2, 03 \times 10^{-1}, 07 \times 10^{-1}, 03 \times 10^{-1} >, \\ < x_3, 05 \times 10^{-1}, 07 \times 10^{-1}, 01 \times 10^{-1} > \end{pmatrix}$$

$$f_{(\mathbb{J}_1,\partial)}(s_1) = \begin{pmatrix} < x_1, 05 \times 10^{-1}, 04 \times 10^{-1}, 04 \times 10^{-1} >, \\ < x_2, 06 \times 10^{-1}, 06 \times 10^{-1}, 02 \times 10^{-1} >, \\ < x_3, 04 \times 10^{-1}, 06 \times 10^{-1}, 01 \times 10^{-1} > \end{pmatrix}$$

$$f_{(\mathbb{J}_1,\partial)}(s_2) = \begin{pmatrix} < x_1, 04 \times 10^{-1}, 06 \times 10^{-1}, 03 \times 10^{-1} >, \\ < x_2, 03 \times 10^{-1}, 07 \times 10^{-1}, 03 \times 10^{-1} >, \\ < x_3, 05 \times 10^{-1}, 07 \times 10^{-1}, 01 \times 10^{-1} > \end{pmatrix}$$

$$f_{(\mathbb{J}_2,\partial)}(s_1) = \begin{pmatrix} < x_1, 01 \times 10^{-1}, 02 \times 10^{-1}, 07 \times 10^{-1} >, \\ < x_2, 03 \times 10^{-1}, 03 \times 10^{-1}, 03 \times 10^{-1} >, \\ < x_3, 01 \times 10^{-1}, 02 \times 10^{-1}, 02 \times 10^{-1} > \end{pmatrix}$$

$$f_{(\mathbb{J}_2,\partial)}(s_2) = \begin{pmatrix} < x_1, 01 \times 10^{-1}, 02 \times 10^{-1}, 07 \times 10^{-1} >, \\ < x_2, 03 \times 10^{-1}, 03 \times 10^{-1}, 03 \times 10^{-1} >, \\ < x_3, 01 \times 10^{-1}, 02 \times 10^{-1}, 02 \times 10^{-1} > \end{pmatrix}.$$

Here $\tau_1 \tilde{\cup} \tau_2 = \{0_{(\mathcal{X},\partial)}, 1_{(\mathcal{X},\partial)}, (\mathcal{P}_1, \partial), (\mathcal{P}_2, \partial), (\mathcal{P}_3, \partial), (\mathbb{J}_1, \partial), (\mathbb{J}_2, \partial)\}$ is not a neutrosophic soft topological space on $\mathcal{X}$.

Definition 2.6 Let $(\mathcal{X}, \tau_1, \tau_2, \partial)$ be a neutrosophic soft topological space. Then a neutrosophic soft set

$$(\mathbb{J}, \partial) = \{(s, \{< x, T_{\mathbb{J}(s)}(x), I_{\mathbb{J}(s)}(x), F_{\mathbb{J}(s)}(x) >\} : x \in \mathcal{X}, s \in \partial\}$$

is called as a pairwise neutrosophic soft open set if there exist a neutrosophic soft open $(\mathbb{J}_1, \partial)$ in τ_1, neutrosophic soft open $(\mathbb{J}_2, \partial)$ in τ_2 such that for all $x \in \mathcal{X}$ such that $(\mathbb{J}, \partial) = (\mathbb{J}_1, \partial) \tilde{\cup} (\mathbb{J}_2, \partial) =$

$$\left\{\left(s, \begin{Bmatrix} < x, \max\{T_{\mathbb{J}(s)}(x), T_{\mathbb{J}(s)}(x)\}, \\ \max\{I_{\mathbb{J}(s)}(x), I_{\mathbb{J}(s)}(x)\}, \min\{F_{\mathbb{J}(s)}(x), F_{\mathbb{J}(s)}(x)\} > \end{Bmatrix}\right) : s \in \partial\right\}$$

Definition 2.7 Let $(\mathcal{X}, \tau_1, \tau_2, \partial)$ be a neutrosophic soft topological space. Then a neutrosophic soft set

$$(\mathbb{J}, \partial) = \{(s, \{< x, T_{\mathbb{J}(s)}(x), I_{\mathbb{J}(s)}(x), F_{\mathbb{J}(s)}(x) >\}) : x \in \mathcal{X}, s \in \partial\}$$

is called as a pairwise neutrosophic soft closed set if $(\mathbb{J}, \partial)^c$ is a pairwise neutrosophic soft open . It is clear that $(\mathbb{J}, \partial)$is a pairwise neutrosophic soft closed set if there exist a pairwise neutrosophic soft closed $(\mathbb{J}_1, \partial)$ in τ_1, pairwise neutrosophic soft closed $(\mathbb{J}_2, \partial)$ in τ_2 such that for all $x \in \mathcal{X}$ $(\mathbb{J}, \partial) = (\mathbb{J}_1, \partial) \tilde{\cap} (\mathbb{J}_2, \partial) =$

$$\left\{\left(s, \begin{Bmatrix} < x, \min\{I_{\mathbb{J}(s)}(x), T_{\mathbb{J}(s)}(x)\}, \min\{I_{\mathbb{J}(s)}(x), I_{\mathbb{J}(s)}(x)\}, \\ \max\{F_{\mathbb{J}(s)}(x), F_{\mathbb{J}(s)}(x)\} > \end{Bmatrix}\right) : s \in \partial\right\}$$

The set of all pairwise neutrosophic soft closed in $(\mathcal{X}, \tau_1, \tau_2, \partial)$ is denoted by $PNSC(\mathcal{X}, \tau_1, \tau_2, \partial)$.

Definition 2.8 Let $(\mathcal{X}, \tau_1, \theta)$ and $(\mathcal{X}, \tau_2, \theta)$ be two neutrosophic soft topological spaces over $\mathcal{X}$ then $(\mathcal{X}, \tau_2 \vee \tau_2, \theta)$ is the smallest neutrosophic soft topology on $\mathcal{X}$ that contains $(\mathcal{X}, \tau_2 \cup \tau_2, \theta)$

Definition 2.9 Let $(\mathcal{X}, \tau_1, \tau_2, \theta)$ be a neutrosophic soft topological space over $\mathcal{X}$ and $(\tilde{f}, \theta)$ be a neutrosophic soft set over $\mathcal{X}$ then $(\tilde{f}, \theta)$is said

(1): neutrosophic soft semi-open if $(\tilde{f}, \theta) \subseteq NScl\left(NSint(\tilde{f}, \theta)\right)$. (2): neutrosophic soft pre-open if$(\tilde{f}, \theta) \subseteq NSint\left(NScl(\tilde{f}, \theta)\right)$.

(3): neutrosophic soft α-open if $(\tilde{f},\theta) \subseteq NSint\left(NScl\left(NSint\,(\tilde{f},\theta)\right)\right)$. (4): neutrosophic soft β-open if $(\tilde{f},\theta) \subseteq NScl\left(NSint\left(NScl\,(\tilde{f},\theta)\right)\right)$.

(5): neutrosophic soft b-open if $(\tilde{f},\theta) \subseteq NScl\left(NSint(\tilde{f},\theta)\right) \cup NSint\left(NScl(\tilde{f},\theta)\right)$. (6): neutrosophic soft $*_b$-open if $(\tilde{f},\theta) \subseteq NScl\left(NSint(\tilde{f},\theta)\right) \cap NSint\left(NScl(\tilde{f},\theta)\right)$. (7): neutrosophic soft b^{**}-open if $(\tilde{f},\theta) \subseteq NSint\left(NScl\left(NSint(\tilde{f},\theta)\right)\right) \cup NScl\left(NSint\left(NScl(\tilde{f},\theta)\right)\right)$. (8): neutrosophic soft $**_b$-open if $(\tilde{f},\theta) \subseteq NSint\left(NScl\left(NSint(\tilde{f},\theta)\right)\right) \cap NScl\left(NSint\left(NScl(\tilde{f},\theta)\right)\right)$.

Definition 2.10 A neutrosophic soft bitopological space $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ is called pairwise neutrosophic soft T_0 space if $x_{1^{(s_1,t_1,u_1)}}^{e_1}, x_{2^{(s_2,t_2,u_2)}}^{e_2}$ be two neutrosophic soft points such that $x_{1^{(s_1,t_1,u_1)}}^{e_1} \widetilde{\cap} x_{2^{(s_2,t_2,u_2)}}^{e_2} \cong 0_{(\mathcal{X},\theta)}$.

If there exists neutrosophic soft b^{**} open sets $(\tilde{f},\theta) \,\tilde{\in}\, \tau_1$ and $(\tilde{g},\theta) \,\tilde{\in}\, \tau_2$ such that $x_{1^{(s_1,t_1,u_1)}}^{e_1} \,\tilde{\in}\, (\tilde{f},\theta), x_{1^{(s_1,t_1,u_1)}}^{e_1} \widetilde{\cap} (\tilde{g},\theta) \cong 0_{(\mathcal{X},\theta)}$ or $x_{2^{(s_2,t_2,u_2)}}^{e_2} \,\tilde{\in}\, (\tilde{g},\theta),\ x_{2^{(s_2,t_2,u_2)}}^{e_2} \widetilde{\cap} (\tilde{f},\theta) \cong 0_{(\mathcal{X},\theta)}$.

Definition 2.11 A neutrosophic soft bitopological space $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ is called pair-wise neutrosophic soft T_1 space if $x_{1^{(s_1,t_1,u_1)}}^{e_1}, x_{2^{(s_1,t_1,u_1)}}^{e_2}$ be two neutrosophic soft points such that $x_{1^{(s_1,t_1,u_1)}}^{e_1} \widetilde{\cap} x_{2^{(s_2,t_2,u_2)}}^{e_2} \cong 0_{(\mathcal{X},\theta)}$.

If there exists neutrosophic b^{**} open sets $(\tilde{f},\theta) \,\tilde{\in}\, \tau_1$, $(\tilde{g},\theta) \,\tilde{\in}\, \tau_2$ such that $x_{1^{(s_1,t_1,u_1)}}^{e_1} \,\tilde{\in}\, (\tilde{f},\theta), x_{2^{(s_1,t_1,u_1)}}^{e_2} \widetilde{\cap} (\tilde{g},\theta) \cong 0_{(\mathcal{X},\theta)}$ and $x_{2^{(s_2,t_2,u_2)}}^{e_2} \,\tilde{\in}\, (\tilde{g},\theta),\ x_{2^{(s_2,t_2,u_2)}}^{e_2} \widetilde{\cap} (\tilde{f},\theta) \cong 0_{(\mathcal{X},\theta)}$.

Definition 2.12 A neutrosophic soft bitopological space $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ is called pair-wise neutrosophic soft T_2 space if $x_{1_{(s_1,t_1,u_1)}}^{e_1}, x_{2_{(s_1,t_1,u_1)}}^{e_2}$ be two neutrosophic soft points such that $x_{1_{(s_1,t_1,u_1)}}^{e_1} \widetilde{\cap} x_{2_{(s_2,t_2,u_2)}}^{e_2} \cong 0_{(\mathcal{X},\theta)}$.

If there exists neutrosophic soft b^{**} open sets $(\tilde{f},\theta) \widetilde{\in} \tau_1$, $(\tilde{g},\theta) \widetilde{\in} \tau_2$ such that $x_{1_{(s_1,t_1,u_1)}}^{e_1} \widetilde{\in} (\tilde{f},\theta)$ and $x_{2_{(s_2,t_2,u_2)}}^{e_2} \widetilde{\in} (\tilde{g},\theta)$, $(\tilde{f},\theta)\widetilde{\cap}(\tilde{g},\theta) \cong 0_{(\mathcal{X},\theta)}$.

Definition 2.13 In neutrosophic soft bitopological space $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$

τ_1 said to be neutrosophic soft T_0 space with respect to τ_2 if for each pair of neutrosophic soft points $x_{1_{(s_1,t_1,u_1)}}^{e_1}, x_{2_{(s_2,t_2,u_2)}}^{e_2}$, $x_{1_{(s_1,t_1,u_1)}}^{e_1} \widetilde{\cap} x_{2_{(s_2,t_2,u_2)}}^{e_2} \cong 0_{(\mathcal{X},\theta)}$ there exists τ_1 neutrosophic soft b^{**}open set $(\tilde{f},\theta)$ and τ_2 neutrosophic soft b^{**}open set $(\tilde{g},\theta)$ such that $x_{1_{(s_1,t_1,u_1)}}^{e_1} \widetilde{\in} (\tilde{f},\theta)$ and $x_{2_{(s_2,t_2,u_2)}}^{e_2} \widetilde{\cap}(\tilde{g},\theta) \cong 0_{(\mathcal{X},\theta)}$ or $x_{2_{(s_2,t_2,u_2)}}^{e_2} \widetilde{\in} (\tilde{g},\theta)$ and $x_{1_{(s_1,t_1,u_1)}}^{e_1} \widetilde{\cap}(\tilde{g},\theta) \cong 0_{(\mathcal{X},\theta)}$, Similarly, τ_2 said to be neutrosophic soft T_0 space with respect to τ_1 if for each pair of neutrosophic soft points $x_{1_{(s_1,t_1,u_1)}}^{e_1}, x_{2_{(s_2,t_2,u_2)}}^{e_2}$, $x_{1_{(s_1,t_1,u_1)}}^{e_1} \widetilde{\cap} x_{2_{(s_2,t_2,u_2)}}^{e_2} \cong 0_{(\mathcal{X},\theta)}$ there exists τ_2 neutrosophic soft b^{**}open set $(\tilde{f},\theta)$ and τ_1 neutrosophic soft b^{**} open set $(\tilde{g},\theta)$ such that $x_{1_{(s_1,t_1,u_1)}}^{e_1} \widetilde{\in} (\tilde{f},\theta)$ and $x_{2_{(s_2,t_2,u_2)}}^{e_2} \widetilde{\cap}(\tilde{g},\theta) \cong 0_{(\mathcal{X},\theta)}$ or $x_{2_{(s_2,t_2,u_2)}}^{e_2} \widetilde{\in} (\tilde{g},\theta)$ and $x_{1_{(s_1,t_1,u_1)}}^{e_1} \widetilde{\cap}(\tilde{g},\theta) \cong 0_{(\mathcal{X},\theta)}$.

Definition 2.14 In neutrosophic soft bitopological space $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$

τ_1 said to be neutrosophic soft T_1 space with respect to τ_2 if for each pair of neutrosophic soft points $x_{1_{(s_1,t_1,u_1)}}^{e_1}, x_{2_{(s_2,t_2,u_2)}}^{e_2}$, $x_{1_{(s_1,t_1,u_1)}}^{e_1} \widetilde{\cap} x_{2_{(s_2,t_2,u_2)}}^{e_2} \cong 0_{(\mathcal{X},\theta)}$ there exists τ_1 neutrosophic soft b^{**}*open set* $(\tilde{f},\theta)$ and τ_2 neutrosophic soft b^{**}open set $(\tilde{g},\theta)$ such that $x_{1_{(s_1,t_1,u_1)}}^{e_1} \widetilde{\in} (\tilde{f},\theta)$ and $x_{2_{(s_2,t_2,u_2)}}^{e_2} \widetilde{\cap}(\tilde{g},\theta) \cong 0_{(\mathcal{X},\theta)}$ and

$x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} (\tilde{g},\theta), x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\cap}(\tilde{g},\theta) \cong 0_{(\tilde{x},\theta)}$, Similarly, τ_2 said to be neutrosophic soft T_1 space with respect to τ_1 if for each pair of neutrosophic soft points $x_{1_{(s_1,t_1,u_1)}}^{e_1}, x_{2_{(s_2,t_2,u_2)}}^{e_2}$, $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\cap} x_{2_{(s_2,t_2,u_2)}}^{e_2} \cong 0_{(\tilde{x},\theta)}$ there exists τ_2 neutrosophic soft b^{**} o pe n set $(\tilde{f},\theta)$ an d τ_1 neutrosophic soft b^{**} open set $(\tilde{g},\theta)$ such that $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} (\tilde{f},\theta)$ and $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\cap}(\tilde{g},\theta) \cong 0_{(\tilde{x},\theta)}$ and $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} (\tilde{g},\theta)$, $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\cap}(\tilde{g},\theta) \cong 0_{(\tilde{x},\theta)}$.

Definition 2.15 In neutrosophic soft bitopological space $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$

τ_1 said to be neutrosophic soft T_2 space with respect to τ_2 if for each pair of neutrosophic soft points $x_{1_{(s_1,t_1,u_1)}}^{e_1}, x_{2_{(s_2,t_2,u_2)}}^{e_2}$, $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\cap} x_{2_{(s_2,t_2,u_2)}}^{e_2} \cong 0_{(\tilde{x},\theta)}$ there exists τ_1 neutrosophic soft b^{**}open set $(\tilde{f},\theta)$ and τ_2 neutrosophic soft b^{**}open set $(\tilde{g},\theta)$ such that $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} (\tilde{f},\theta)$ and $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} (\tilde{g},\theta)$ $(\tilde{f},\theta) \tilde{\cap}(\tilde{g},\theta) \cong 0_{(\tilde{x},\theta)}$, Similarly, τ_2 said to be neutrosophic soft T_2 space with respect to τ_1 if for each pair of neutrosophic soft points $x_{1_{(s_1,t_1,u_1)}}^{e_1}, x_{2_{(s_2,t_2,u_2)}}^{e_2}$, $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\cap} x_{2_{(s_2,t_2,u_2)}}^{e_2} \cong 0_{(\tilde{x},\theta)}$ there exists τ_2 neutrosophic $soft\ b^{**}$open set $(\tilde{f},\theta)$ and τ_1 neutrosophic soft b^{**}open set $(\tilde{g},\theta)$ such that $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} (\tilde{f},\theta)$ and $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} (\tilde{g},\theta)$ $(\tilde{f},\theta) \tilde{\cap}(\tilde{g},\theta) \cong 0_{(\tilde{x},\theta)}$.

Theorem 2.16 Let $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ be neutrosophic soft bitopological space. If $\langle \mathcal{X}, \tau_1, \theta \rangle$ and $\langle \mathcal{X}, \tau_2, \theta \rangle$ are neutrosophic soft T_0 spaces. Then $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ is a pair wise neutrosophic soft T_0 space.

Proof. Let $x_{1_{(s_1,t_1,u_1)}}^{e_1}, x_{2_{(s_2,t_2,u_2)}}^{e_2}$ be neutrosophic soft points such that $x_{1_{(s_1,t_1,u_1)}}^{e_1} \widetilde{\cap} x_{2_{(s_2,t_2,u_2)}}^{e_2} \cong 0_{(\mathcal{X},\theta)}$ and suppose $\langle \mathcal{X}, \tau_1, \theta \rangle$ is neutrosophic soft T_0 space with respect to $\langle \mathcal{X}, \tau_2, \theta \rangle$ space then there exists τ_1 neutrosophic soft b^{**}open set $(\tilde{f}, \theta)$ and τ_2 neutrosophic soft b^{**} open set $(\tilde{g}, \theta)$ such that $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} (\tilde{f}, \theta)$, $x_{2_{(s_2,t_2,u_2)}}^{e_2} \widetilde{\cap} (\tilde{f}, \theta) \cong 0_{(\mathcal{X},\theta)}$ or $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} (\tilde{g}, \theta), x_{1_{(s_1,t_1,u_1)}}^{e_1} \widetilde{\cap} (\tilde{f}, \theta) \cong 0_{(\mathcal{X},\theta)}$. Similarly, Let $x_{1_{(s_1,t_1,u_1)}}^{e_1}, x_{2_{(s_2,t_2,u_2)}}^{e_2}$ be neutrosophic soft points such that $x_{1_{(s_1,t_1,u_1)}}^{e_1} \widetilde{\cap} x_{2_{(s_2,t_2,u_2)}}^{e_2} \cong 0_{(\mathcal{X},\theta)}$ and suppose $\langle \mathcal{X}, \tau_2, \theta \rangle$ is neutrosophic soft T_0 space with respect to $\langle \mathcal{X}, \tau_1, \theta \rangle$ space then there exists τ_2 neutrosophic soft b^{**}open set $(\tilde{f}, \theta)$ and τ_1 neutrosophic soft open set $(\tilde{g}, \theta)$ such that $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} (\tilde{f}, \theta)$, $x_{2_{(s_2,t_2,u_2)}}^{e_2} \widetilde{\cap} (\tilde{f}, \theta) \cong 0_{(\mathcal{X},\theta)}$ or $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} (\tilde{g}, \theta), x_{1_{(s_1,t_1,u_1)}}^{e_1} \widetilde{\cap} (\tilde{f}, \theta) \cong 0_{(\mathcal{X},\theta)}$. Hence, $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ is a pair wise neutrosophic soft T_0 space.

Theorem 2.17 Let $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ be neutrosophic soft bitopological space. If $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ is a pair wise neutrosophic soft T_0 space then $\langle \mathcal{X}, \tau_1 v \tau_2, \theta \rangle$ is a pair wise neutrosophic soft T_0 space.

Proof. Let $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ be neutrosophic soft bitopological space. If $x_{1_{(s_1,t_1,u_1)}}^{e_1}, x_{2_{(s_2,t_2,u_2)}}^{e_2}$ be neutrosophic soft points such that $x_{1_{(s_1,t_1,u_1)}}^{e_1} \widetilde{\cap} x_{2_{(s_2,t_2,u_2)}}^{e_2} \cong 0_{(\mathcal{X},\theta)}$. Since $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ is a pair wise neutrosophic soft T_0 space. So $\langle \mathcal{X}, \tau_1, \theta \rangle$ is neutrosophic soft T_0 space with respect to $\langle \mathcal{X}, \tau_2, \theta \rangle$ space then there exists τ_1 neutrosophic soft b^{**}open set $(\tilde{f}, \theta)$ and τ_2 neutrosophic soft b^{**} open set $(\tilde{g}, \theta)$ such that $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} (\tilde{f}, \theta)$, $x_{2_{(s_2,t_2,u_2)}}^{e_2} \widetilde{\cap} (\tilde{f}, \theta) \cong 0_{(\mathcal{X},\theta)}$ or $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} (\tilde{g}, \theta), x_{1_{(s_1,t_1,u_1)}}^{e_1} \widetilde{\cap} (\tilde{f}, \theta) \cong 0_{(\mathcal{X},\theta)}$. Similarly,

Let $x_{1_{(s_1,t_1,u_1)}}^{e_1}, x_{2_{(s_2,t_2,u_2)}}^{e_2}$ be neutrosophic soft points such that $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\cap} x_{2_{(s_2,t_2,u_2)}}^{e_2} \cong 0_{(\mathcal{X},\theta)}$ and suppose $\langle \mathcal{X}, \tau_2, \theta \rangle$ is neutrosophic soft T_0 space with respect to $\langle \mathcal{X}, \tau_1, \theta \rangle$ space then there exists τ_2 neutrosophic soft b^{**}open set $(\tilde{f}, \theta)$ and τ_1 neutrosophic soft b^{**} open set $(\tilde{g}, \theta)$ such that $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} (\tilde{f}, \theta)$, $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\cap} (\tilde{f}, \theta) \cong 0_{(\mathcal{X},\theta)}$ or $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} (\tilde{g}, \theta), x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\cap} (\tilde{f}, \theta) \cong 0_{(\mathcal{X},\theta)}$. In either, case $(\tilde{f}, \theta), (\tilde{g}, \theta) \tilde{\in} \langle \mathcal{X}, \tau_1 v \tau_2, \theta \rangle$ and hence $\langle \mathcal{X}, \tau_1 v \tau_2, \theta \rangle$ is a pair wise neutrosophic soft T_0 space.

Theorem 2.18 Let $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ be neutrosophic soft bitopological space over $\mathcal{X}$ and Y be a non-empty neutrosophic soft subset of $\mathcal{X}$. If $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ is pair wise neutrosophic soft T_0 space. Then $\langle Y, \tau_{1Y}, \tau_{2Y}, \theta \rangle$ is pair wise neutrosophic soft T_0 space

Proof. Let $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ be neutrosophic soft bitopological space over $\mathcal{X}$ If $x_{1_{(s_1,t_1,u_1)}}^{e_1}, x_{2_{(s_2,t_2,u_2)}}^{e_2}$ be neutrosophic soft points such that $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\cap} x_{2_{(s_2,t_2,u_2)}}^{e_2} \cong 0_{(\mathcal{X},\theta)}$. If $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ is pair wise neutrosophic soft T_0 space, then there exist τ_1 neutrosophic soft b^{**} open set $(\tilde{f}, \theta)$ and τ_2 neutrosophic soft b^{**} open set $(\tilde{g}, \theta)$ Such that $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} (\tilde{f}, \theta)$, $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\cap} (\tilde{f}, \theta) \cong 0_{(\mathcal{X},\theta)}$ or $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} (\tilde{g}, \theta), x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\cap} (\tilde{f}, \theta) \cong 0_{(\mathcal{X},\theta)}$. Now, $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} Y$ and $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} (\tilde{f}, \theta)$. Hence, where $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} Y \tilde{\cap} (\tilde{f}, \theta) = (Y_{\tilde{f}}, \theta)$ where $(\tilde{f}, \theta) \tilde{\in} \tau_1$.Consider $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\cap} (\tilde{f}, \theta) \cong 0_{(\mathcal{X},\theta)}$,this means that $\alpha \in \theta$ for some $\alpha \in \theta$. $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\notin} Y \tilde{\cap} (\tilde{f}, \theta) = (Y_{\tilde{f}}, \theta)$. There fore τ_{1Y} is neutrosophic soft T_0 space with respect τ_{2Y}. Similarly, can proved thatτ_{2Y} is neutrosophic soft T_0 space with respect to τ_{1Y}, that

is $x_{2}^{e_2}_{(s_2,t_2,u_2)} \tilde{\in} (\tilde{g},\theta)$ and $x_{1}^{e_1}_{(s_1,t_1,u_1)} \tilde{\cap} (\tilde{g},\theta) \cong 0_{(\tilde{x},\theta)}$ then $x_{2}^{e_2}_{(s_2,t_2,u_2)} \tilde{\in} (Y_{\tilde{g}},\theta)$ and $x_{1}^{e_1}_{(s_1,t_1,u_1)} \tilde{\cap} (Y_{\tilde{g}},\theta) \cong 0_{(\tilde{x},\theta)}$. Thus $\langle Y, \tau_{1Y}, \tau_{2Y}, \theta\rangle$ is pair wise neutrosophic soft T_0 space.

Theorem 2.19 Let $\langle \mathcal{X}, \tau_1, \tau_2, \theta\rangle$ be neutrosophic soft bitopological space. Then $\langle \mathcal{X}, \tau_1, \tau_2, \theta\rangle$ is a pair wise neutrosophic soft T_1 space if and only if $\langle \mathcal{X}, \tau_1, \theta\rangle$ and $\langle \mathcal{X}, \tau_2, \theta\rangle$ are neutrosophic soft T_1 space.

Proof. Suppose $\langle \mathcal{X}, \tau_1, \theta\rangle$ and $\langle \mathcal{X}, \tau_2, \theta\rangle$ are neutrosophic soft T_0 spaces. If $x_{1}^{e_1}_{(s_1,t_1,u_1)}, x_{2}^{e_2}_{(s_2,t_2,u_2)}$ be neutrosophic soft points such that $x_{1}^{e_1}_{(s_1,t_1,u_1)} \tilde{\cap} x_{2}^{e_2}_{(s_2,t_2,u_2)} \cong 0_{(\tilde{x},\theta)}$ then

1). τ_1 is a neutrosophic soft T_1 space with respect to τ_2. So there exists τ_1 neutrosophic soft b^{**}open set $(\tilde{f},\theta)$ and τ_2 neutrosophic soft b^{**}open set $(\tilde{g},\theta)$ such that $x_{1}^{e_1}_{(s_1,t_1,u_1)} \tilde{\in} (\tilde{f},\theta)$, $x_{2}^{e_2}_{(s_2,t_2,u_2)} \tilde{\cap} (\tilde{f},\theta) \cong 0_{(\tilde{x},\theta)}$ and $x_{2}^{e_2}_{(s_2,t_2,u_2)} \tilde{\in} (\tilde{g},\theta), x_{1}^{e_1}_{(s_1,t_1,u_1)} \tilde{\cap} (\tilde{f},\theta) \cong 0_{(\tilde{x},\theta)}$

2). τ_2 is a neutrosophic soft T_1 space with respect to τ_1. So there exists τ_2 neutrosophic soft b^{**}open set $(\tilde{f},\theta)$ and τ_1 neutrosophic soft b^{**}open set $(\tilde{g},\theta)$ such that $x_{1}^{e_1}_{(s_1,t_1,u_1)} \tilde{\in} (\tilde{f},\theta)$, $x_{2}^{e_2}_{(s_2,t_2,u_2)} \tilde{\cap} (\tilde{f},\theta) \cong 0_{(\tilde{x},\theta)}$ and $x_{2}^{e_2}_{(s_2,t_2,u_2)} \tilde{\in} (\tilde{g},\theta), x_{1}^{e_1}_{(s_1,t_1,u_1)} \tilde{\cap} (\tilde{f},\theta) \cong 0_{(\tilde{x},\theta)}$ .All possibilities give the same result. So $\langle \mathcal{X}, \tau_1, \tau_2, \theta\rangle$ is a pair wise neutrosophic soft T_1 space.

Conversely, we suppose that $\langle \mathcal{X}, \tau_1, \tau_2, \theta\rangle$ is a pair wise neutrosophic soft T_1 space. Then

1). There exists some neutrosophic soft b^{**}open set $(\tilde{f},\theta) \in \tau_1$ with respect to neutrosophic soft b^{**} open set $(\tilde{g},\theta) \in \tau_2$ such $x_{1}^{e_1}_{(s_1,t_1,u_1)} \tilde{\in} (\tilde{f},\theta)$, $x_{2}^{e_2}_{(s_2,t_2,u_2)} \tilde{\cap} (\tilde{f},\theta) \cong 0_{(\tilde{x},\theta)}$ and $x_{2}^{e_2}_{(s_2,t_2,u_2)} \tilde{\in} (\tilde{g},\theta), x_{1}^{e_1}_{(s_1,t_1,u_1)} \tilde{\cap} (\tilde{g},\theta) \cong 0_{(\tilde{x},\theta)}$.

2). there exists some neutrosophic soft b^{**}open set $(\tilde{f},\theta) \in \tau_2$with respect to neutrosophic soft b^{**}open set$(\tilde{g},\theta) \in \tau_1$such that $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} (\tilde{f},\theta)$, $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\cap}(\tilde{f},\theta) \cong 0_{(\tilde{X},\theta)}$ and $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} (\tilde{g},\theta), x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\cap}(\tilde{f},\theta) \cong 0_{(\tilde{X},\theta)}$.

Thus$\langle \mathcal{X},\tau_1,\theta\rangle$ and $\langle \mathcal{X},\tau_2,\theta\rangle$ are neutrosophic soft T_1 spaces.

Theorem 2.20 Let $\langle \mathcal{X},\tau_1,\tau_2,\theta\rangle$ be neutrosophic soft bitopological space. if $\langle \mathcal{X},\tau_1,\tau_2,\theta\rangle$ is pair wise neutrosophic soft T_1space then $\langle \mathcal{X},\tau_1 v\tau_2,\theta\rangle$ is also neutrosophic soft T_1 space.

Proof. If $x_{1_{(s_1,t_1,u_1)}}^{e_1}, x_{2_{(s_2,t_2,u_2)}}^{e_2}$ be neutrosophic soft points such that $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\cap} x_{2_{(s_2,t_2,u_2)}}^{e_2} \cong 0_{(\tilde{X},\theta)}$ then exists n neutrosophic soft b^{**}open set $(\tilde{f},\theta) \in \tau_1$with respect to neutrosophic soft b^{**}open set $(\tilde{g},\theta) \in \tau_2$such that

$x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} (\tilde{f},\theta)$, $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\cap}(\tilde{f},\theta) \cong 0_{(\tilde{X},\theta)}$ and $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} (\tilde{g},\theta), x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\cap}(\tilde{f},\theta) \cong 0_{(\tilde{X},\theta)}$.

Similarly, exists neutrosophic soft b^{**}open set $(\tilde{f},\theta) \in \tau_2$with respect to neutrosophic soft b^{**} open set $(\tilde{g},\theta) \in \tau_1$such that $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} (\tilde{f},\theta)$, $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\cap}(\tilde{f},\theta) \cong 0_{(\tilde{X},\theta)}$ and $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} (\tilde{g},\theta), x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\cap}(\tilde{f},\theta) \cong 0_{(\tilde{X},\theta)}$.

$\Longrightarrow (\tilde{f},\theta),(\tilde{g},\theta) \in \tau_1 v\tau_2$. So $\langle \mathcal{X},\tau_1 v\tau_2,\theta\rangle$ is neutrosophic soft T_1 space.

Theorem 2.21 Let $\langle \mathcal{X},\tau_1,\tau_2,\theta\rangle$ be neutrosophic soft bitopological space and Y be non-empty sub set of $\mathcal{X}$.If $\langle \mathcal{X},\tau_1,\tau_2,\theta\rangle$ is pair wise neutrosophic soft T_1 space then $\langle Y,\tau_{1Y},\tau_{2Y},\theta\rangle$ is pair wise neutrosophic soft T_1 space.

Proof. Let $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ be neutrosophic soft bitopological space and $x_{1_{(s_1,t_1,u_1)}}^{e_1}, x_{2_{(s_2,t_2,u_2)}}^{e_2} \in Y, x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\cap} x_{2_{(s_2,t_2,u_2)}}^{e_2} \cong 0_{(\mathcal{X},\theta)}$. If $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ is pair wise soft neutrosophic soft T_1 then there exists τ_1 neutrosophic soft b^{**} open set $(\tilde{f}, \theta)$ and τ_2 neutrosophic soft b^{**} open set $(\tilde{g}, \theta)$ su ch that $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} (\tilde{f}, \theta)$, $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\cap} (\tilde{f}, \theta) \cong 0_{(\mathcal{X},\theta)}$ and $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} (\tilde{g}, \theta), x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\cap} (\tilde{f}, \theta) \cong 0_{(\mathcal{X},\theta)}$.

Obviously, $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} Y \tilde{\cap} (\tilde{f}, \theta) \cong (Y_{\tilde{f}}, \theta)$ .Then $x_{2_{(s_2,t_2,u_2)}}^{e_2} \notin Y \tilde{\cap} \tilde{f}(\alpha)$ for some $\alpha \in \theta$ .This means that $\alpha \in \theta$ then $x_{2_{(s_2,t_2,u_2)}}^{e_2} \notin Y \tilde{\cap} \tilde{f}(\alpha)$ for some $\alpha \in \theta$. Therefore, $_{(s_2,t_2,u_2)}^{e_2} \notin Y \tilde{\cap} (\tilde{f}, \theta) \cong (Y_{\tilde{f}}, \theta)$.

Obviously, $x_{2_{(s_2,t_2,u_2)}}^{e_2} \in Y \tilde{\cap} (\tilde{g}, \theta) = (Y_{\tilde{g}}, \theta)$ where $(\tilde{g}, \theta) \tilde{\in} \tau_2$ .Consider $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\cap} (\tilde{g}, \theta) \cong 0_{(\mathcal{X},\theta)}$ and this means that $\alpha \in \theta$ then $x_{1_{(s_1,t_1,u_1)}}^{e_1} \notin Y \tilde{\cap} g(\alpha)$ for some $\alpha \in \theta$ therefore $x_{1_{(s_1,t_1,u_1)}}^{e_1} \notin Y \cap (\tilde{g}, \theta) = (Y_{\tilde{g}}, \theta)$ hence τ_{1Y} is neutrosophic soft T_1 space with respect $((Y_{\tilde{g}}, \theta)$ to τ_{2Y}. Similarly, it can be proved that τ_{2Y} is neutrosophic soft T_1 space with respect to τ_{1Y} , that is $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} (\tilde{g}, \theta)$ and $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\cap} (\tilde{g}, \theta) \cong 0_{(\mathcal{X},\theta)}$.Then $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} (Y_{\tilde{g}}, \theta)$ and $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\cap} (Y_{\tilde{g}}, \theta) \cong 0_{(\mathcal{X},\theta)}$. Thus $\langle Y, \tau_{1Y}, \tau_{2Y}, \theta \rangle$ is pair wise neutrosophic soft T_1 space.

Theorem 2.22 Every pair wise neutrosophic soft T_1 space is pair wise neutrosophic soft T_0 space.

Proof. Let $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ be neutrosophic soft open bitopological space and if $x_{1_{(s_1,t_1,u_1)}}^{e_1}, x_{2_{(s_2,t_2,u_2)}}^{e_2}$ be neutrosophic soft points such that

$x_{1}^{e_1}_{(s_1,t_1,u_1)} \widetilde{\cap} x_{2}^{e_2}_{(s_2,t_2,u_2)} \cong 0_{(\mathcal{X},\theta)}$. If $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ is pair wise neutrosophic soft T_1 space and that is, $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ is pair wise neutrosophic soft T_1space with respect to τ_2 and τ_2 is neutrosophic soft T_1 space with respect to τ_1 and if τ_1 is neutrosophic soft T_1 space with respect to τ_2 then there exists a τ_1 neutrosophic soft b^{**}open set $(\tilde{f}, \theta)$ and a τ_2 neutrosophic soft b^{**} open set $(\tilde{g}, \theta)$ such that $x_{1}^{e_1}_{(s_1,t_1,u_1)} \tilde{\in} (\tilde{f}, \theta)$, $x_{2}^{e_2}_{(s_2,t_2,u_2)} \widetilde{\cap} (\tilde{f}, \theta) \cong 0_{(\mathcal{X},\theta)}$ and $x_{2}^{e_2}_{(s_2,t_2,u_2)} \tilde{\in} (\tilde{g}, \theta), x_{1}^{e_1}_{(s_1,t_1,u_1)} \widetilde{\cap} (\tilde{g}, \theta) \cong 0_{(\mathcal{X},\theta)}$. Obviously $x_{1}^{e_1}_{(s_1,t_1,u_1)} \tilde{\in} (\tilde{f}, \theta)$, $x_{2}^{e_2}_{(s_2,t_2,u_2)} \widetilde{\cap} (\tilde{f}, \theta) \cong 0_{(\mathcal{X},\theta)}$ or $x_{2}^{e_2}_{(s_2,t_2,u_2)} \tilde{\in} (\tilde{g}, \theta), x_{1}^{e_1}_{(s_1,t_1,u_1)} \widetilde{\cap} (\tilde{g}, \theta) \cong 0_{(\mathcal{X},\theta)}$. Therefore τ_1 is neutrosophic soft T_0 space with respect to τ_2. Similarly, if τ_2 is a neutrosophic soft T_1 space with respect to τ_1 then, there exists τ_2 neutrosophic soft b^{**}open set $(\tilde{f}, \theta)$ and τ_1 neutrosophic soft open set $(\tilde{g}, \theta)$ such that $x_{1}^{e_1}_{(s_1,t_1,u_1)} \tilde{\in} (\tilde{f}, \theta), x_{2}^{e_2}_{(s_2,t_2,u_2)} \widetilde{\cap} (\tilde{f}, \theta) \cong 0_{(\mathcal{X},\theta)}$ and $x_{2}^{e_2}_{(s_2,t_2,u_2)} \tilde{\in} (\tilde{g}, \theta), x_{1}^{e_1}_{(s_1,t_1,u_1)} \widetilde{\cap} (\tilde{g}, \theta) \cong 0_{(\mathcal{X},\theta)}$. Obviously $x_{1}^{e_1}_{(s_1,t_1,u_1)} \tilde{\in} (\tilde{f}, \theta)$, $x_{2}^{e_2}_{(s_2,t_2,u_2)} \widetilde{\cap} (\tilde{f}, \theta) \cong 0_{(\mathcal{X},\theta)}$ or $x_{2}^{e_2}_{(s_2,t_2,u_2)} \tilde{\in} (\tilde{g}, \theta), x_{1}^{e_1}_{(s_1,t_1,u_1)} \widetilde{\cap} (\tilde{g}, \theta) \cong 0_{(\mathcal{X},\theta)}$ .Therefore τ_2 neutrosophic soft T_0 space with respect to τ_1. Thus $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ is a pair wise neutrosophic soft T_0 space.

Theorem 2.23 Every pair wise neutrosophic soft T_2 space is pair wise neutrosophic soft T_1 space

Proof. Let $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ be a neutrosophic soft bitopological space and if $x_{1}^{e_1}_{(s_1,t_1,u_1)}, x_{2}^{e_2}_{(s_2,t_2,u_2)}$ be neutrosophic soft points such that

$x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\cap} x_{2_{(s_2,t_2,u_2)}}^{e_2} \cong 0_{(\mathcal{X},\theta)}$. If $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ is pair wise neutrosophic soft T_2space and that is $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ is pair wise neutrosophic soft T_2 space. If τ_1 is neutrosophic soft T_2 space with respect τ_2 then there exists a τ_1 neutrosophic soft open set $(\tilde{f}, \theta)$ and τ_2 neutrosophic soft b^{**} open set $(\tilde{g}, \theta)$ such that $(\tilde{f}, \theta) \tilde{\in} \tau_1$, $(\tilde{g}, \theta) \tilde{\in} \tau_2$ such that $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} (\tilde{f}, \theta)$ and $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} (\tilde{g}, \theta)$, $(\tilde{f}, \theta) \tilde{\cap} (\tilde{g}, \theta) \cong 0_{(\mathcal{X},\theta)}$.

Obviously $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} (\tilde{f}, \theta)$ and $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\cap} (\tilde{g}, \theta) \cong 0_{(\mathcal{X},\theta)}$ and $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} (\tilde{g}, \theta)$, $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\cap} (\tilde{g}, \theta) \cong 0_{(\mathcal{X},\theta)}$.

Therefore τ_1 is neutrosophic soft T_1 space with respect to τ_2. Similarly, if τ_2 is neutrosophic soft T_2 space with respect to τ_1 then there exists a τ_2 n neutrosophic soft b^{**} open set $(\tilde{f}, \theta)$ set and τ_1 neutrosophic soft b^{**} open set $(\tilde{g}, \theta)$ such that $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} (\tilde{f}, \theta)$ and $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} (\tilde{g}, \theta)$, $(\tilde{f}, \theta) \tilde{\cap} (\tilde{g}, \theta) \cong 0_{(\mathcal{X},\theta)}$.

Obviously $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} (\tilde{f}, \theta)$ and $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\cap} (\tilde{g}, \theta) \cong 0_{(\mathcal{X},\theta)}$ and $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} (\tilde{g}, \theta)$, $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\cap} (\tilde{g}, \theta) \cong 0_{(\mathcal{X},\theta)}$.

Therefore τ_2 is neutrosophic soft T_1space with respect to τ_1 thus $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ is pair wise neutrosophic soft T_1space.

Theorem 2.24 Let $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ be neutrosophic soft bitopological space. If $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ is

Pair wise neutrosophic soft T_2space. Then $\langle \mathcal{X}, \tau_1 \nu \tau_2, \theta \rangle$ also a neutrosophic soft T_2 space.

Proof. Since $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ be neutrosophic soft bitopological space. If $x_{1_{(s_1,t_1,u_1)}}^{e_1}, x_{2_{(s_2,t_2,u_2)}}^{e_2}$ be neutrosophic soft points such that $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\cap} x_{2_{(s_2,t_2,u_2)}}^{e_2} \cong 0_{(\mathcal{X},\theta)}$.

τ_1 is neutrosophic soft T_2 space with respect to τ_2 so for each pair of neutrosophic soft points $x_{1(s_1,t_1,u_1)}^{e_1}, x_{2(s_2,t_2,u_2)}^{e_2}$, $x_{1(s_1,t_1,u_1)}^{e_1} \widetilde{\cap} x_{2(s_2,t_2,u_2)}^{e_2} \cong 0_{(\mathcal{X},\theta)}$ there exists τ_1 neutrosophic soft open set $(\tilde{f},\theta)$ and τ_2 neutrosophic soft b^{**}open set $(\tilde{g},\theta)$ such that $x_{1(s_1,t_1,u_1)}^{e_1} \tilde{\in} (\tilde{f},\theta)$ and $x_{2(s_2,t_2,u_2)}^{e_2} \tilde{\in} (\tilde{g},\theta)$ $(\tilde{f},\theta) \widetilde{\cap} (\tilde{g},\theta) \cong 0_{(\mathcal{X},\theta)}$, Similarly, τ_2 is neutrosophic soft T_2 space with respect to τ_1 so for each pair of neutrosophic soft points $x_{1(s_1,t_1,u_1)}^{e_1}, x_{2(s_2,t_2,u_2)}^{e_2}$, $x_{1(s_1,t_1,u_1)}^{e_1} \widetilde{\cap} x_{2(s_2,t_2,u_2)}^{e_2} \cong 0_{(\mathcal{X},\theta)}$ there exists τ_2 neutrosophic soft b^{**} open set $(\tilde{f},\theta)$ and τ_1 neutrosophic soft b^{**} open set $(\tilde{g},\theta)$ such that $x_{1(s_1,t_1,u_1)}^{e_1} \tilde{\in} (\tilde{f},\theta)$ and $x_{2(s_2,t_2,u_2)}^{e_2} \tilde{\in} (\tilde{g},\theta)$ $(\tilde{f},\theta) \widetilde{\cap} (\tilde{g},\theta) \cong 0_{(\mathcal{X},\theta)}$. In each case $(\tilde{f},\theta), (\tilde{g},\theta) \tilde{\in} \tau_1 \nu \tau_2$ and hence $\langle \mathcal{X}, \tau_1 \nu \tau_2, \theta \rangle$ is neutrosophic soft T_2 space.

Theorem 2.25 Let $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ be neutrosophic soft bitopological space and Y be a non-empty sub set of $\mathcal{X}$. If $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ is pair-wise neutrosophic soft T_2 space then$\langle Y, \tau_{1Y}, \tau_{2Y}, \theta \rangle$ is pair wise neutrosophic soft T_2 space

Proof. Let $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ be neutrosophic soft bitopological space.Let $x_{1(s_1,t_1,u_1)}^{e_1}, x_{2(s_2,t_2,u_2)}^{e_2}$be neutrosophic soft points then $x_{1(s_1,t_1,u_1)}^{e_1} \widetilde{\cap} x_{2(s_2,t_2,u_2)}^{e_2} \cong 0_{(\mathcal{X},\theta)}$. If $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ is pairwise neutrosophic soft T_2 space, Then there exists τ_1 neutrosophic soft b^{**}open set $(\tilde{f},\theta)$ and τ_2 n neutrosophic soft b^{**}open set $(\tilde{g},\theta)$ such that $x_{1(s_1,t_1,u_1)}^{e_1} \tilde{\in} (\tilde{f},\theta)$ and $x_{2(s_2,t_2,u_2)}^{e_2} \tilde{\in} (\tilde{g},\theta)$ $(\tilde{f},\theta) \widetilde{\cap} (\tilde{g},\theta) \cong 0_{(\mathcal{X},\theta)}$. So for each $\alpha \in \theta$, $x_{1(s_1,t_1,u_1)}^{e_1} \tilde{\in} \tilde{f}(\alpha), x_{2(s_2,t_2,u_2)}^{e_2} \tilde{\in} \tilde{g}(\alpha),$ $\tilde{f}(\alpha) \cap \tilde{g}(\alpha) \cong 0_{(\mathcal{X},\theta)}$ and this implies

$x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} Y\tilde{\cap}\tilde{f}(\alpha), x_{2_{(s_2,t_2,u_2)}}^{e_2} \in Y\tilde{\cap}\tilde{g}(\alpha)$, $\tilde{f}(\alpha)\tilde{\cap}\tilde{g}(\alpha) \cong 0_{(\tilde{X},\theta)}$. Hence

$x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} \langle Y_{\tilde{f}}, \theta\rangle, x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} \langle Y_{\tilde{g}}, \theta\rangle$

$\langle Y_{\tilde{f}}, \theta\rangle \cap \langle Y_{\tilde{g}}, \theta\rangle \cong 0_{(\tilde{X},\theta)}$. Where $\langle Y_{\tilde{f}}, \theta\rangle$ is neutrosophic soft b^{**} open set in τ_{1Y}, $\langle Y_{\tilde{g}}, \theta\rangle$ is neutrosophic soft b^{**}open set in τ_{2Y} therefore $\tau_{1Y}$is neutrosophic soft T_2 space with respect to τ_{2Y}. Similarly, it can be proved that τ_{2Y} is neutrosophic soft T_2 space with respect to τ_{1Y}. Thus $\langle Y, \tau_{1Y}, \tau_{2Y}, \theta\rangle$ is pair wise neutrosophic soft T_2 space.

Theorem 2.26 Let $\langle \mathcal{X}, \tau_1, \tau_2, \theta\rangle$ be neutrosophic soft bitopological space then the following are equivalent

1). $\langle \mathcal{X}, \tau_1, \tau_2, \theta\rangle$ be a pair wise neutrosophic soft T_2 space.

2). Let $x_{1_{(s_1,t_1,u_1)}}^{e_1}, x_{2_{(s_2,t_2,u_2)}}^{e_2}$ be neutrosophic soft points then $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\cap} x_{2_{(s_2,t_2,u_2)}}^{e_2} \cong 0_{(\tilde{X},\theta)}$ then there is a neutrosophic soft b^{**} open set $(\tilde{f}, \theta) \tilde{\in} \tau_1$ and $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} \mathcal{X} - (\overline{(\tilde{f}, \theta)})^{\tau_1}$

Proof. 1) implies 2) suppose $\langle \mathcal{X}, \tau_1, \tau_2, \theta\rangle$ is a pair wise neutrosophic soft T_2 space and for $x_{1_{(s_1,t_1,u_1)}}^{e_1}, x_{2_{(s_2,t_2,u_2)}}^{e_2}$ be neutrosophic soft points then $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\cap} x_{2_{(s_2,t_2,u_2)}}^{e_2} \cong 0_{(\tilde{X},\theta)}$, $\langle \mathcal{X}, \tau_1, \tau_2, \theta\rangle$ pair wise neutrosophic soft T_2 space implies there exists neutrosophic soft b^{**}open set $(\tilde{f}, \theta) \tilde{\in} \tau_1$ and $(\tilde{g}, \theta) \tilde{\in} \tau_2$ such that $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} (\tilde{f}, \theta) \tilde{\in} \tau_1$ and $(\tilde{g}, \theta) \tilde{\in} \tau_2$ such that $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} (\tilde{f}, \theta)$ and $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} (\tilde{g}, \theta)$ $(\tilde{f}, \theta) \tilde{\cap} (\tilde{g}, \theta) \cong 0_{(\tilde{X},\theta)}$. Similarly for the second case so that $(\tilde{f}, \theta) \subseteq ((\tilde{g}, \theta))^c$ also $(\overline{(\tilde{f}, \theta)})^{\tau_1}$ is the smallest n neutrosophic soft b^{**}close set in τ_2 that contains $(\tilde{f}, \theta)$, $(\tilde{g}, \theta)$ and is a neutrosophic soft b^{**} close set $\tilde{\in} \tau_2$.

So $(\overline{(\tilde{f},\theta)})^{\tau_2} \tilde{\subseteq} ((\tilde{g},\theta))^c$, $(\overline{(\tilde{g},\theta)}) \tilde{\subseteq} ((\overline{(\tilde{f},\theta)})^{\tau_2})^c$ .Thus $x_{2^{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} (\tilde{g},\theta) \tilde{\subseteq} (\overline{(\tilde{f},\theta)})^{\tau_2})^c$ or $x_{2^{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} \mathcal{X} - (\overline{(\tilde{f},\theta)})^{\tau_2}$. Let $x_{1^{(s_1,t_1,u_1)}}^{e_1}, x_{2^{(s_2,t_2,u_2)}}^{e_2}$ be neutrosophic soft points, $x_{1^{(s_1,t_1,u_1)}}^{e_1} \tilde{\cap} x_{2^{(s_2,t_2,u_2)}}^{e_2} \cong 0_{(\mathcal{X},\theta)}$ then there exists a neutrosophic soft b^{**} open set $(\tilde{f},\theta) \tilde{\in} \tau_1$ such that $x_{1^{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} (\tilde{f},\theta)$ and $x_{2^{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} \mathcal{X} - (\overline{(\tilde{f},\theta)})^{\tau_2}$ is $(\overline{(\tilde{f},\theta)})^{\tau_2}$ neutrosophic soft b^{**} close set in τ_2 so $(\tilde{g},\theta) \cong \mathcal{X} - (\overline{\tilde{f},\theta})^{\tau_2} \in \tau_2$. Now $x_{1^{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} (\tilde{f},\theta)$ and $x_{2^{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} (\tilde{g},\theta)$and $(\tilde{f},\theta) \cap (\tilde{g},\theta) \cong (\tilde{f},\theta) \tilde{\cap} (\overline{\mathcal{X}} - (\overline{\tilde{f},\theta})^{\tau_2} \tilde{\subseteq} (\tilde{f},\theta) \tilde{\cap} (\overline{\mathcal{X}} - (\tilde{f},\theta)$.

Since $(\tilde{f},\theta) \tilde{\subseteq} (\overline{(\tilde{f},\theta)})^{\tau_2}$ and hence $(\tilde{f},\theta) \tilde{\cap} (\tilde{g},\theta) \tilde{\subseteq} (\tilde{f},\theta) \tilde{\cap} (\overline{\mathcal{X}} - (\tilde{f},\theta) \cong 0_{(\mathcal{X},\theta)}$ and thus $(\tilde{f},\theta) \tilde{\cap} (\tilde{g},\theta) \cong 0_{(\mathcal{X},\theta)}$.

3. Results in Terms of Interior and Closure

In this section, neutrosophic soft limit point, neutrosophic soft interior point, few results on the basis of these definitions are discussed. These results are discussed terms of neutrosophic soft interior and neutrosophic soft closures with respect to soft points.

Definition 3.1 Let $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ be neutrosophic soft bitopological space and $(\tilde{f},\theta)$ neutrosophic soft open set. A neutrosophic soft point $x_{1^{(s_1,t_1,u_1)}}^{e_1}$ is called a neutrosophic soft limit point of $(\tilde{f},\theta)$ if every neutrosophic soft b^{**} open set$(\tilde{g},\theta)$ where $(\tilde{g},\theta) \cong (\widetilde{g_1},\theta) \tilde{\cup} (\widetilde{g_2},\theta)$ such that $(\widetilde{g_1},\theta) \tilde{\in} \tau_1$ and $(\widetilde{g_2},\theta) \tilde{\in} \tau_1$ containing $x_{1^{(s_1,t_1,u_1)}}^{e_1}$contains at least one neutrosophic soft b^{**} open set $(\tilde{g},\theta)$ other than $x_{1^{(s_1,t_1,u_1)}}^{e_1}$ that is$\left[(\tilde{g},\theta) - \left\{ x_{1^{(s_1,t_1,u_1)}}^{e_1} \right\}\right] \tilde{\cap} (\tilde{f},\theta) \neq 0_{(\mathcal{X},\theta)}$.The set of all neutrosophic soft limit point

of $(\tilde{f},\theta)$ is called the derived neutrosophic soft set and is denoted by $(\tilde{f},\theta)^{d}$. If there exists neutrosophic soft b^{**} open set $(\tilde{g},\theta)$ such that $\left((\tilde{g},\theta)-x_{1_{(s_1,t_1,u_1)}}^{e_1}\right)\tilde{\cap}(\tilde{f},\theta)\cong 0_{(x,\theta)}$.Then $x_{1_{(s_1,t_1,u_1)}}^{e_1}$ is unable to be the neutrosophic soft limit point of $(\tilde{f},\theta)$.

Definition 3.2 Let $\langle X,\tau_1,\tau_2,\theta\rangle$ be neutrosophic soft bitopological space and $(\tilde{f},\theta)$ neutrosophic soft b^{**} open set. A soft point $x_{1_{(s_1,t_1,u_1)}}^{e_1}\tilde{\in}(\tilde{f},\theta)$ is called neutrosophic soft interior point of $(\tilde{f},\theta)$ if we can search out neutrosophic soft b^{**} open set $(\tilde{g},\theta)$, where $(\tilde{g},\theta)\cong(\widetilde{g_1},\theta)\tilde{\cup}(\widetilde{g_2},\theta)$ such that $(\widetilde{g_1},\theta)\tilde{\in}\tau_1$ and $(\widetilde{g_2},\theta)\tilde{\in}\tau_1$ containing $x_{1_{(s_1,t_1,u_1)}}^{e_1}$ such that $x_{1_{(s_1,t_1,u_1)}}^{e_1}\tilde{\in}(\tilde{g},\theta)\tilde{\subseteq}(\tilde{f},\theta)$.

Theorem 3.3 Let $\langle X,\tau_1,\tau_2,\theta\rangle$ be a neutrosophic soft bitopological space. If $(\tilde{f},\theta),(\tilde{g},\theta)$ be neutrosophic soft subsets then

(1) $(\tilde{f},\theta)\tilde{\subseteq}(\tilde{g},\theta)\Rightarrow(\tilde{f},\theta)^{d}\tilde{\subseteq}(\tilde{g},\theta)^{d}$

(2) $[(\tilde{f},\theta)\tilde{\cup}(\tilde{g},\theta)]^{d}\cong(\tilde{f},\theta)^{d}\tilde{\cup}(\tilde{g},\theta)^{d}$

(3) $[(\tilde{f},\theta)\tilde{\cap}((\tilde{g},\theta)]^{d}\tilde{\subseteq}(\tilde{f},\theta)^{d}\tilde{\cap}(\tilde{g},\theta)^{d}$

Proof. Let $x_{(s,t,u)}^{e}\tilde{\in}(\tilde{f},\theta)^{d}$ so for every neutrosophic soft b^{**} open set $(\mathfrak{h},\theta)$ where $(\mathfrak{h},\theta)\cong(\mathfrak{h}_1,\theta)\tilde{\cup}(\mathfrak{h}_2,\theta)$ such that $(\mathfrak{h}_1,\theta)\tilde{\in}\tau_1$ and $(\mathfrak{h}_2,\theta)\tilde{\in}\tau_2$ contains $x_{(s,t,u)}^{e}$, we have $\left((\mathfrak{h},\theta)-x_{(s,t,u)}^{e}\right)\tilde{\cap}(\tilde{f},\theta)\cong 0_{(x,\theta)}$.

But $(\tilde{f},\theta)\tilde{\subseteq}(\tilde{g},\theta)$ and so $\left((\mathfrak{h},\theta)-x_{(s,t,u)}^{e}\right)\tilde{\cap}(\tilde{g},\theta)\cong 0_{(x,\theta)}$ implies $x_{(s,t,u)}^{e}\tilde{\in}(\tilde{g},\theta)^{d}$.

Hence $(\tilde{f},\theta)^{d}\tilde{\subseteq}(\tilde{g},\theta)^{d}$ Rest are trivial.

Theorem 3.4 Let $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ be neutrosophic soft bitopological space and $(\tilde{f}, \theta)$ be neutrosophic soft subset, then $(\tilde{f}, \theta) \tilde{\cup} (\tilde{f}, \theta)^d$ is neutrosophic soft b^{**} close set.

Proof. We will prove $\left((\tilde{f}, \theta) \tilde{\cup} (\tilde{f}, \theta)^d\right)^c$ is neutrosophic soft b^{**} open set. Let $x^e_{(s,t,u)} \tilde{\in} \left((\tilde{f}, \theta) \tilde{\cup} (\tilde{f}, \theta)^d\right)^c$ implies $x^e_{(s,t,u)} \tilde{\notin} (\tilde{f}, \theta) \tilde{\cup} (\tilde{f}, \theta)^d$ implies $x^e_{(s,t,u)} \tilde{\cap} (\tilde{f}, \theta) \neq 0_{(\mathcal{X},\theta)}$, $x^e_{(s,t,u)} \tilde{\cap} (\tilde{f}, \theta)^d \neq 0_{(\mathcal{X},\theta)}$ and now $x^e_{(s,t,u)} \tilde{\cap} (\tilde{f}, \theta) \neq 0_{(\mathcal{X},\theta)}$ so we can find neutrosophic soft b^{**} open set $(\tilde{g}, \theta)$ where $(\tilde{g}, \theta) \tilde{\cong} (\tilde{g_1}, \theta) \tilde{\cup} (\tilde{g_2}, \theta)$ such that $(\tilde{g_1}, \theta) \tilde{\in} \tau_1$ and $(\tilde{g_2}, \theta) \tilde{\in} \tau_1$ such that $\left((\tilde{g}, \theta) - x^e_{(s,t,u)}\right) \tilde{\cap} (\tilde{f}, \theta) \neq 0_{(\mathcal{X},\theta)}$ implies $(\tilde{g}, \theta) \tilde{\cap} (\tilde{f}, \theta) \tilde{\cong} 0_{(\mathcal{X},\theta)}$…..(1). we claim that $(\tilde{g}, \theta) \tilde{\cap} (\tilde{f}, \theta)^d \tilde{\cong} 0_{(\mathcal{X},\theta)}$ because if $(\tilde{g}, \theta) \tilde{\cap} (\tilde{f}, \theta)^d \tilde{\neq} 0_{(\mathcal{X},\theta)}$ then there exists $y^{e'}_{(s',t',u')} \tilde{\in} (\tilde{g}, \theta) \tilde{\cap} (\tilde{f}, \theta)^d$ implies $y^{e'}_{(s',t',u')} \tilde{\in} (\tilde{g}, \theta)$ and $y^{e'}_{(s',t',u')} \tilde{\in} (\tilde{f}, \theta)^d$ implies $y^{e'}_{(s',t',u')}$ is neutrosophic soft limit point of $(\tilde{f}, \theta)$ so every neutrosophic sof t b^{**} open set $(\tilde{g}, \theta)$contains $y^{e'}_{(s',t',u')}$intersects $(\tilde{f}, \theta)$in some neutrosophic soft point other than $y^{e'}_{(s',t',u')}$ that is $\left((\tilde{g}, \theta) - y^{e'}_{(s',t',u')}\right) \tilde{\cap} (\tilde{f}, \theta) \tilde{\cong} 0_{(\mathcal{X},\theta)}$implies $(\tilde{g}, \theta) \tilde{\cap} (\tilde{f}, \theta) \neq 0_{(\mathcal{X},\theta)}$ but it contradicts results (1) $thus (\tilde{g}, \theta) \tilde{\cap} (\tilde{f}, \theta) \tilde{\cup} (\tilde{g}, \theta) \tilde{\cap} (\tilde{f}, \theta)^d \tilde{\cong} 0_{(\mathcal{X},\theta)}$ implies $(\tilde{g}, \theta) \tilde{\subseteq} \left((\tilde{f}, \theta) \tilde{\cup} (\tilde{f}, \theta)^d\right)^c$ or $x^e_{(s,t,u)} \tilde{\in} (\tilde{g}, \theta) \tilde{\subseteq} \left((\tilde{f}, \theta) \tilde{\cup} (\tilde{f}, \theta)^d\right)^c$ implies $x^e_{(s,t,u)}$ is neutrosophic soft interior point of $\left((\tilde{f}, \theta) \tilde{\cup} (\tilde{f}, \theta)^d\right)$. Thus $\left((\tilde{f}, \theta) \tilde{\cup} (\tilde{f}, \theta)^d\right)^c$ is neutrosophic soft b^{**} open and consequently $(\tilde{f}, \theta) \tilde{\cup} (\tilde{f}, \theta)^d$ is neutrosophic soft b^{**} close.

Theorem 3.5 Let $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ be neutrosophic soft bitopological space and $(\tilde{f}, \theta)$ be neutrosophic soft subset, then $(\tilde{f}, \theta)$ is neutrosophic soft b^{**} close set if $(\tilde{f}, \theta)^d \cong (\tilde{f}, \theta)$

Proof. We show $(\tilde{f}, \theta)^d \cong (\tilde{f}, \theta)$. Let $x^e_{(s,t,u)} \tilde{\in} (\tilde{f}, \theta)^d$ implies $x^e_{(s,t,u)}$ is neutrosophic soft limit point of $(\tilde{f}, \theta)$ so for $\tilde{\in} \tau_1 \; and \; (\tilde{g}_2, \theta) \tilde{\in} \tau_2$ contains $x^e_{(s,t,u)}$, we have $\left((\tilde{g}, \theta) - x^e_{(s,t,u)}\right) \tilde{\cap} (\tilde{f}, \theta) \tilde{\neq} 0_{(\mathcal{X}, \theta)}$ ... (1). We are to show that $x^e_{(s,t,u)} \tilde{\in} (\tilde{f}, \theta)$ suppose on contrary that $x^e_{(s,t,u)} \tilde{\notin} (\tilde{f}, \theta)$ Then $x^e_{(s,t,u)} \tilde{\in} (\tilde{f}, \theta)^c$. Then $(\tilde{f}, \theta)^c$ is neutrosophic $soft \; b^{**}$ open because $(\tilde{f}, \theta)$ is neutrosophic soft b^{**} close. Since result (1) is true for all neutrosophic soft b^{**} close sets $(\tilde{g}, \theta)$ containing $x^e_{(s,t,u)}$ so replace $(\tilde{g}, \theta)$ by $(\tilde{f}, \theta)^c$ in (1), we get $((\tilde{f}, \theta)^c - x^e_{(s,t,u)}) \tilde{\cap} (\tilde{f}, \theta) \tilde{\neq} 0_{(\mathcal{X}, \theta)}$. Obviously $(\tilde{f}, \theta)^c - x^e_{(s,t,u)} \tilde{\subseteq} (\tilde{f}, \theta)^c$. This implies $(\tilde{f}, \theta)^c \tilde{\cap} (\tilde{f}, \theta) \tilde{\neq} 0_{(\mathcal{X}, \theta)}$. So we are forced to accept that $x^e_{(s,t,u)} \tilde{\in} (\tilde{f}, \theta)$ that is, $x^e_{(s,t,u)} \in (\tilde{f}, \theta)^d$ implies $x^e_{(s,t,u)} \tilde{\in} (\tilde{f}, \theta)$ therefore $(\tilde{f}, \theta)^d \cong (\tilde{f}, \theta)$ and conversely let $((\tilde{f}, \theta)^d \cong (\tilde{f}, \theta)$. We are to prove that neutrosophic soft close. $(\tilde{f}, \theta)$ such that $(\tilde{f}, \theta) \tilde{\neq} (\tilde{f}_1, \theta) \tilde{\cup} (\tilde{f}_2, \theta)$ such that $(\tilde{f}_1, \theta) \tilde{\in} \tau_1$ and $(\tilde{f}_2, \theta) \tilde{\in} \tau_2$ such that $(\tilde{f}_1, \theta)$ and $(\tilde{f}_2, \theta)$ are neutrosophic soft b^{**} close in their respective structures. Let $x^e_{(s,t,u)} \tilde{\in} (\tilde{f}, \theta)^c$ implies $x^e_{(s,t,u)} \tilde{\in} (\tilde{f}, \theta)$ but $(\tilde{f}, \theta)^d \cong (\tilde{f}, \theta)$ so $x^e_{(s,t,u)} \tilde{\notin} (\tilde{f}, \theta)^d$ implies $x^e_{(s,t,u)}$ is not neutrosophic soft limit point of $(\tilde{f}, \theta)$ we can search out neutrosophic soft $b^{**} open \; set$ $(\tilde{f}_3, \theta)$ where $(\tilde{f}_3, \theta) \tilde{\neq} (\tilde{f}_4, \theta) \tilde{\cup} (\tilde{f}_5, \theta)$ such that

$\left(\tilde{f}_4,\theta\right) \tilde{\in} \tau_1$ and $\left(\tilde{f}_5,\theta\right) \tilde{\in} \tau_2$ such that $\tilde{\subseteq} (\tilde{f},\theta)^c$ or $x^{\varepsilon}_{(s,t,u)} \tilde{\in} \left(\tilde{f}_3,\theta\right) \tilde{\subseteq} (\tilde{f},\theta)^c$ implies $x^{\varepsilon}_{(s,t,u)}$ is neutrosophic soft interior point of $(\tilde{f},\theta)^c$ implies $(\tilde{f},\theta)^c$ is neutrosophic soft b^{**} open and consequently $(\tilde{f},\theta)$ is neutrosophic soft b^{**} close.

Theorem 3.6 Let $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ be neutrosophic soft bitopological space if $(\tilde{f},\theta)$ be neutrosophic soft sub set then $\overline{\left((\tilde{f},\theta)\right)} \tilde{=} (\tilde{f},\theta) \tilde{\cup} (\tilde{f},\theta)^d$

Proof. Since $\overline{(\tilde{f},\theta)}$ is the smallest neutrosophic soft b^{**} close set containing $(\tilde{f},\theta)$ and so $(\tilde{f},\theta) \tilde{\subseteq} \overline{(\tilde{f},\theta)}$ implies $\left((\tilde{f},\theta)\right)^d \tilde{\subseteq} \overline{(\tilde{f},\theta)}^d$. Also $\overline{(\tilde{f},\theta)}^d \tilde{\subseteq} \overline{(\tilde{f},\theta)}$.Therefore $(\tilde{f},\theta)^d \tilde{\subseteq} \overline{(\tilde{f},\theta)}^d \tilde{\subseteq} \overline{(\tilde{f},\theta)} \Rightarrow (\tilde{f},\theta)^d \tilde{\subseteq} \overline{(\tilde{f},\theta)}$ also $(\tilde{f},\theta) \tilde{\subseteq} \overline{(\tilde{f},\theta)} \Rightarrow$ $(\tilde{f},\theta) \tilde{\cup} (\tilde{f},\theta)^d \tilde{\subseteq} \overline{(\tilde{f},\theta)}$... (1)

Again since $(\tilde{f},\theta) \tilde{\cup} (\tilde{f},\theta)^d$ is a neutrosophic soft b^{**}close set containing $(\tilde{f},\theta)$. *But* $\overline{(\tilde{f},\theta)}$ is the smallest neutrosophic soft close set containing $(\tilde{f},\theta)$, so $(\tilde{f},\theta) \tilde{\subseteq} \overline{(\tilde{f},\theta)} \tilde{\subseteq} (\tilde{f},\theta) \tilde{\cup} (\tilde{f},\theta)^d$ that is $\overline{(\tilde{f},\theta)} \tilde{\subseteq} (\tilde{f},\theta) \tilde{\cup} (\tilde{f},\theta)^d$ … (2). *Hence* $\overline{(\tilde{f},\theta)} \tilde{=} (\tilde{f},\theta) \tilde{\cup} (\tilde{f},\theta)^d$

Theorem 3.7 *Let* $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ be neutrosophic soft bitopological space. If $(\tilde{f},\theta)$ be neutrosophic soft set then $(\tilde{f},\theta)$ is neutrosophic soft b^{**} close set if and only if $\overline{(\tilde{f},\theta)} \tilde{=} (\tilde{f},\theta)$.

Proof. Let $(\tilde{f},\theta)$ be neutrosophicsoft set then $(\tilde{f},\theta)$ is neutrosophic soft b^{**}close set so that $(\tilde{f},\theta) \cong (\tilde{f}_1,\theta) \tilde{\cup} (\tilde{f}_2,\theta)$ such that $(\tilde{f}_1,\theta) \in \tau_1$ and $(\tilde{f}_2,\theta) \in \tau_1$ and are both neutrosophic soft b^{**}close sets in their respective domain. Then $(\tilde{f}_1,\theta)^d \cong (\tilde{f}_1,\theta)$this implies $(\tilde{f}_1,\theta) \tilde{\cup} (\tilde{f}_1,\theta)^d \cong (\tilde{f}_1,\theta) \Rightarrow \overline{(\tilde{f}_1,\theta)} \cong (\tilde{f},\theta)$ and conversely let $\overline{(\tilde{f},\theta)} \cong (\tilde{f},\theta)$this implies that $(\tilde{f},\theta) \tilde{\cup} (\tilde{f},\theta)^d \cong (\tilde{f},\theta) \Rightarrow (\tilde{f},\theta)^d \cong (\tilde{f},\theta) \Rightarrow (\tilde{f},\theta)$is neutrosophic soft b^{**}close.

Theorem 3.8 Let $\langle X,\tau_1,\tau_2,\theta\rangle$ be neutrosophic soft bitopological space. If $(\tilde{f},\theta)$ and $(\tilde{g},\theta)$ are neutrosophic soft subsets such that $(\tilde{f},\theta) \tilde{\subseteq} (\tilde{g},\theta)$ then i) $\overline{(\tilde{f},\theta)} \tilde{\subseteq} \overline{(\tilde{g},\theta)}$ ii) $\overline{(\tilde{f},\theta)} \tilde{\cup} (\tilde{g},\theta) \cong \overline{(\tilde{f},\theta)} \tilde{\cup} \overline{(\tilde{g},\theta)}$ iii) $\overline{(\tilde{f},\theta)} \tilde{\cup} (\tilde{g},\theta) \tilde{\subseteq} (\tilde{f},\theta) \tilde{\cap} \overline{(\tilde{g},\theta)}$

Proof. i) If $(\tilde{f},\theta) \tilde{\subseteq} (\tilde{g},\theta) \Rightarrow (\tilde{f},\theta) \tilde{\cup} (\tilde{f},\theta)^d \tilde{\subseteq} (\tilde{g},\theta) \tilde{\cup} (\tilde{g},\theta)^d \Rightarrow \overline{(\tilde{f},\theta)} \tilde{\subseteq} \overline{(\tilde{g},\theta)}$ ii) since $(\tilde{f},\theta) \tilde{\subseteq} (\tilde{f},\theta) \tilde{\cup} (\tilde{g},\theta) \Rightarrow \overline{(\tilde{f},\theta)} \tilde{\subseteq} \overline{(\tilde{f},\theta) \tilde{\cup} (\tilde{g},\theta)}$… (1)

Also since $(\tilde{g},\theta) \tilde{\subseteq} (\tilde{f},\theta) \tilde{\cup} (\tilde{g},\theta) \Rightarrow \overline{(\tilde{g},\theta)} \tilde{\subseteq} \overline{(\tilde{f},\theta) \tilde{\cup} (\tilde{g},\theta)}$… (2). Maxing results (1) and (2) we have $\overline{(\tilde{f},\theta)} \tilde{\cup} \overline{(\tilde{g},\theta)} \tilde{\subseteq} \overline{(\tilde{f},\theta) \tilde{\cup} (\tilde{g},\theta)}$…(3) to get the required result we proceeds as since $(\tilde{f},\theta) \tilde{\subseteq} \overline{(\tilde{f},\theta)}$ and $(\tilde{g},\theta) \tilde{\subseteq} \overline{\overline{(\tilde{g},\theta)}} \Rightarrow (\tilde{f},\theta) \tilde{\cup} (\tilde{g},\theta) \tilde{\subseteq} \overline{(\tilde{f},\theta)} \tilde{\cup} \overline{(\tilde{g},\theta)}$ since $\overline{(\tilde{f},\theta)}$ *and* $\overline{(\tilde{g},\theta)}$ are neutrosophic soft b^{**}close sets so is $\overline{(\tilde{f},\theta)} \tilde{\cup} \overline{(\tilde{g},\theta)}$ and it contains $(\tilde{f},\theta) \tilde{\cup} (\tilde{g},\theta)$ but $\overline{(\tilde{f},\theta) \tilde{\cup} (\tilde{g},\theta)}$ is the smallest neutrosophic soft b^{**} close sub set containing $(\tilde{f},\theta) \tilde{\cup} (\tilde{g},\theta)$. So $(\tilde{f},\theta) \tilde{\cup} (\tilde{g},\theta) \tilde{\subseteq} \overline{(\tilde{f},\theta) \tilde{\cup} (\tilde{g},\theta)} \tilde{\subseteq} \overline{(\tilde{f},\theta)} \tilde{\cup} \overline{(\tilde{g},\theta)}$ $\Rightarrow \overline{(\tilde{f},\theta) \tilde{\cup} (\tilde{g},\theta)} \tilde{\subseteq} \overline{(\tilde{f},\theta)} \tilde{\cup} \overline{(\tilde{g},\theta)}$ …(4) maxing results (3) and (4) we have $\overline{(\tilde{f},\theta) \tilde{\cup} (\tilde{g},\theta)} \cong \overline{(\tilde{f},\theta)} \tilde{\cup} \overline{(\tilde{g},\theta)}$

iii) Since $(\tilde{f},\theta) \tilde{\subseteq} \overline{(\tilde{f},\theta)}$ ` *and* $(\tilde{g},\theta) \tilde{\subseteq} \overline{(\tilde{f},\theta)} \Rightarrow (\tilde{f},\theta)\,\tilde{\cap}(\tilde{g},\theta) \tilde{\subseteq} \overline{(\tilde{f},\theta)}\,\tilde{\cup}\,\overline{(\tilde{g},\theta)} \tilde{\subseteq} \overline{(\tilde{f},\theta)}\,\tilde{\cup}\,\overline{(\tilde{g},\theta)}$ that is $\overline{(\tilde{f},\theta)}\,\tilde{\cup}\,\overline{(\tilde{g},\theta)} \tilde{\subseteq} (\tilde{f},\theta)\tilde{\cap}\overline{(\tilde{g},\theta)}$.

4. Product of Few Mixed Structures

In this section some spaces are discussed in terms of product with respect to soft points of the spaces. Product of neutrosophic soft T_0 spaces, neutrosophic soft T_1 spaces, neutrosophic soft T_2 spaces and their engagement with neutrosophic soft coordinate spaces with respect to soft points are studied under soft b^{**} open sets in neutrosophic soft bi-topological space. Finally, the most important structure known as entropy is discussed and on the basis of this structures an example is given which is related to engineering which goes like this, in order to obtain an effective risk supervision in the field of engineering, certain risks are classified along with some parameters and these risks are evaluated by the team of experts (engineers).

Theorem 4.1. Let $\langle \mathcal{X}, \tau_1, \tau_2, \theta\rangle$ be neutrosophic soft bi-topological space such that it is neutrosophic soft Hausdorff space if and only if for every two neutrosophic soft points $x_1^{e_1}_{(s_1,t_1,u_1)}, x_2^{e_2}_{(s_2,t_2,u_2)}$, $x_1^{e_1}_{(s_1,t_1,u_1)}\,\tilde{\cap} x_2^{e_2}_{(s_2,t_2,u_2)} \tilde{=} 0_{(\mathcal{X},\theta)}$ there exists neutrosophic soft b^{**}close sets $(\tilde{l_1},\theta)$ and $(\tilde{l_2},\theta)$ that $x_1^{e_1}_{(s_1,t_1,u_1)} \tilde{\in} (\tilde{l_1},\theta), x_2^{e_2}_{(s_2,t_2,u_2)}\,\tilde{\cap}(\tilde{l_1},\theta) \tilde{=} 0_{(\mathcal{X},\theta)}$ and $x_2^{e_2}_{(s_2,t_2,u_2)} \tilde{\in} (\tilde{l_2},\theta), x_1^{e_1}_{(s_1,t_1,u_1)}\,\tilde{\cap}(\tilde{l_2},\theta) \tilde{=} 0_{(\mathcal{X},\theta)}$ and $(\tilde{l_1},\theta)\,\tilde{\cup}\,(\tilde{l_2},\theta) \tilde{=} 0_{(\mathcal{X},\theta)}$.

Proof. Let $\langle \mathcal{X}, \tau_1, \tau_2, \theta\rangle$ be neutrosophic soft bi-topological space, then for neutrosophic soft points $x_1^{e_1}_{(s_1,t_1,u_1)}, x_2^{e_2}_{(s_2,t_2,u_2)}$, $x_1^{e_1}_{(s_1,t_1,u_1)}\,\tilde{\cap} x_2^{e_2}_{(s_2,t_2,u_2)} \tilde{=} 0_{(\mathcal{X},\theta)}$ we can find two neutrosophic soft b^{**}open sets such that $x_1^{e_1}_{(s_1,t_1,u_1)} \tilde{\in} (\tilde{g},\theta)$ and $x_2^{e_2}_{(s_2,t_2,u_2)} \tilde{\in} (\tilde{h},\theta)$ such

that $(\tilde{g},\theta)\,\tilde{\cap}(\tilde{h},\theta) \cong 0_{(\tilde{X},\theta)}$. Since $(\tilde{g},\theta)\,\tilde{\cap}(\tilde{h},\theta) \cong 0_{(\tilde{X},\theta)}$ .So $x_{1}^{e_1}_{(s_1,t_1,u_1)} \tilde{\in} (\tilde{g},\theta)$ and $x_{2}^{e_2}_{(s_2,t_2,u_2)} \tilde{\in} (\tilde{h},\theta) \Longrightarrow$ $x_{1}^{e_1}_{(s_1,t_1,u_1)} \tilde{\in} (\tilde{g},\theta), x_{2}^{e_2}_{(s_2,t_2,u_2)}\,\tilde{\cap}(\tilde{g},\theta) \cong 0_{(\tilde{X},\theta)}$ and $x_{2}^{e_2}_{(s_2,t_2,u_2)} \tilde{\in} (\tilde{h},\theta)$, $x_{1}^{e_1}_{(s_1,t_1,u_1)}\,\tilde{\cap}(\tilde{h},\theta) \cong 0_{(\tilde{X},\theta)}$ this implies $x_{1}^{e_1}_{(s_1,t_1,u_1)} \tilde{\notin} (\tilde{g},\theta)^c, x_{2}^{e_2}_{(s_2,t_2,u_2)} \tilde{\in} (\tilde{g},\theta)^c$ and $x_{2}^{e_2}_{(s_2,t_2,u_2)} \tilde{\notin} (\tilde{h},\theta)^c, x_{1}^{e_1}_{(s_1,t_1,u_1)} \tilde{\in} (\tilde{h},\theta)^c$ or $x_{1}^{e_1}_{(s_1,t_1,u_1)} \tilde{\in} (\tilde{h},\theta)^c$, $x_{2}^{e_2}_{(s_2,t_2,u_2)} \tilde{\notin} (\tilde{h},\theta)^c$ and $x_{2}^{e_2}_{(s_2,t_2,u_2)} \tilde{\in} (\tilde{g},\theta)^c, x_{1}^{e_1}_{(s_1,t_1,u_1)} \tilde{\notin} (\tilde{g},\theta)^c$.

Let $(\tilde{l_1},\theta) \cong (\tilde{h},\theta)^c$ and $(\tilde{l_2},\theta) = (\tilde{g},\theta)^c$ .Then $(\tilde{l_2},\theta)$ and $(\tilde{l_1},\theta)$ are neutrosophic soft b^{**} close sets and $x_{1}^{e_1}_{(s_1,t_1,u_1)} \tilde{\in} (\tilde{l_1},\theta), x_{2}^{e_2}_{(s_2,t_2,u_2)}\,\tilde{\cap}(\tilde{l_1},\theta) \cong 0_{(\tilde{X},\theta)}$ and $x_{2}^{e_2}_{(s_2,t_2,u_2)} \tilde{\in} (\tilde{l_2},\theta), x_{1}^{e_1}_{(s_1,t_1,u_1)}\,\tilde{\cap}(\tilde{l_2},\theta) \cong 0_{(\tilde{X},\theta)}$. Also $(\tilde{l_1},\theta)\,\tilde{\cup}\,(\tilde{l_1},\theta) \cong (\tilde{g},\theta)^c\,\tilde{\cup}\,(\tilde{h},\theta)^c \cong \left((\tilde{g},\theta)\tilde{\cap}(\tilde{h},\theta)\right)^c \cong \left(0_{(\tilde{X},\theta)}\right)^c \cong \tilde{X}$. Conversely, let us suppose that for every two neutrosophic soft points $x_{1}^{e_1}_{(s_1,t_1,u_1)}$ and $x_{2}^{e_2}_{(s_2,t_2,u_2)}$ such that $x_{1}^{e_1}_{(s_1,t_1,u_1)}\,\tilde{\cap}x_{2}^{e_2}_{(s_2,t_2,u_2)} \cong 0_{(\tilde{X},\theta)}$, there exists neutrosophic soft b^{**} close sets $(\tilde{l_1},\theta)$ and $(\tilde{l_2},\theta)$ so $x_{1}^{e_1}_{(s_1,t_1,u_1)} \tilde{\in} (\tilde{l_1},\theta)$, $x_{2}^{e_2}_{(s_2,t_2,u_2)} \tilde{\notin} (\tilde{l_1},\theta)$ and $x_{2}^{e_2}_{(s_2,t_2,u_2)} \tilde{\in} (\tilde{l_2},\theta), x_{1}^{e_1}_{(s_1,t_1,u_1)} \tilde{\notin} (\tilde{l_2},\theta)$ and $(\tilde{l_1},\theta)\,\tilde{\cup}\,(\tilde{l_2},\theta) \cong \tilde{X}$. We prove that $\langle X, \tau_1, \tau_2, \theta\rangle$ be a neutrosophic soft Hausdorff space. We proceed as fallow to catch the required space. $(\tilde{g},\theta) \cong (\tilde{l_1},\theta)^c$ and $(\tilde{h},\theta) = (\tilde{l_2},\theta)^c$ then $(\tilde{g},\theta)$ and $(\tilde{h},\theta)$ are neutrosophic soft b^{**} open sets and $x_{1}^{e_1}_{(s_1,t_1,u_1)} \tilde{\in} (\tilde{l_1},\theta), x_{2}^{e_2}_{(s_2,t_2,u_2)} \tilde{\notin} (\tilde{l_1},\theta)$ and $x_{2}^{e_2}_{(s_2,t_2,u_2)} \tilde{\in} (\tilde{l_2},\theta), x_{1}^{e_1}_{(s_1,t_1,u_1)} \tilde{\notin} (\tilde{l_2},\theta)$ implies that $x_{2}^{e_2}_{(s_2,t_2,u_2)}\,\tilde{\cap}(\tilde{l_1},\theta) \cong 0_{(\tilde{X},\theta)}$ and

$x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\cap}(\tilde{l}_2,\theta) \cong 0_{(\tilde{x},\theta)} \Rightarrow x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} (\tilde{l}_1,\theta)^c$ and $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} (\tilde{l}_2,\theta)^c$ $\Rightarrow x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} (\tilde{h},\theta)$ and $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} (\tilde{g},\theta)$. Moreover $(\tilde{g},\theta)\,\tilde{\cap}(\tilde{h},\theta) \cong (\tilde{l}_2,\theta)^c \tilde{\cap}(\tilde{l}_1,\theta)^c = \left((\tilde{l}_2,\theta)\tilde{\cap}(\tilde{l}_1,\theta)\right)^c \cong \mathcal{X} \cong 0_{(\tilde{x},\theta)}$.This proves that for every two soft neutrosophic soft points $x_{1_{(s_1,t_1,u_1)}}^{e_1}$ and $x_{2_{(s_2,t_2,u_2)}}^{e_2}$ such that $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\cap} x_{2_{(s_2,t_2,u_2)}}^{e_2} \cong 0_{(\tilde{x},\theta)}$ there exists neutrosophic soft b^{**} open sets $(\tilde{g},\theta)$ and $(\tilde{h},\theta)$such hat $x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} (\tilde{g},\theta)$and $x_{2_{(s_2,t_2,u_2)}}^{e_2} \tilde{\in} (\tilde{h},\theta)$such that$(\tilde{g},\theta)\,\tilde{\cap}(\tilde{h},\theta) \cong 0_{(\tilde{x},\theta)}$

Theorem 4.2 If $\langle \mathcal{X}, \tau_1, \tau_2, \theta\rangle$ and $\langle Y, \mathcal{T}_1, \mathcal{T}_2, \theta\rangle$ be two neutrosophic soft Hausdorff space then their product is also neutrosophic soft Hausdorff space.

Proof. Let $(\tilde{\lambda}_1,\theta)$ and $(\tilde{\lambda}_2,\theta)$ be two neutrosophic soft points such that $(\tilde{\lambda}_1,\theta) \cong (x_{1_{(s_1,t_1,u_1)}}^{e_1}, x_{2_{(s_2,t_2,u_2)}}^{e_2})$ and $(\tilde{\lambda}_2,\theta) \cong (y_{1_{(s_1,t_1,u_1)}}^{e_1}, y_{2_{(s_2,t_2,u_2)}}^{e_2})$ and $(\tilde{\lambda}_1,\theta)\tilde{\cap}(\tilde{\lambda}_2,\theta) \cong 0_{(\tilde{x},\theta)}$. Suppose $(x_{1_{(s_1,t_1,u_1)}}^{e_1}, y_{1_{(s_1,t_1,u_1)}}^{e_1}) \tilde{\in} \widetilde{\mathcal{X}_\theta}$ and$(x_{1_{(s_1,t_1,u_1)}}^{e_1}, y_{1_{(s_1,t_1,u_1)}}^{e_1}) \tilde{\in} \widetilde{Y_\theta}$ if $x_{1_{(s_1,t_1,u_1)}}^{e_1} \neq y_{1_{(s_1,t_1,u_1)}}^{e_1}$ are two distinct points of $\widetilde{\mathcal{X}_\theta}$ but $\widetilde{\mathcal{X}_\theta}$ is given to be neutrosophic soft Hausdorff space so there exists two distinct neutrosophic soft b^{**}open sets $(\widetilde{g_1},\theta)$ and $(\widetilde{h_1},\theta)$such that $(x_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} (\widetilde{g_1},\theta)$ and $(y_{1_{(s_1,t_1,u_1)}}^{e_1} \tilde{\in} (\widetilde{h_1},\theta)$. Now $(\widetilde{g_1},\theta) \times Y$ and $(\widetilde{h_1},\theta) \times Y$ are two distinct neutrosophic soft b^{**} open sets in $\mathcal{X} \times Y$ containing $(\tilde{\lambda}_1,\theta)$ and $(\tilde{\lambda}_2,\theta)$ respectively that is $(\tilde{\lambda}_1,\theta) \tilde{\in} (\widetilde{g_1},\theta) \times Y$and $(\tilde{\lambda}_1,\theta) \tilde{\in} (\widetilde{h_1},\theta) \times Y$ and $(\widetilde{g_1},\theta) \times Y)\tilde{\cap}((\widetilde{h_1},\theta) \times Y) \cong 0_{(\tilde{x},\theta)}$

Thus $\mathcal{X} \times Y$ is a neutrosophic soft Hausdorff space. Now if $x_{2(s_2,t_2,u_2)}^{e_2} \tilde{\neq} y_{2(s_2,t_2,u_2)}^{e_2}, x_{2(s_2,t_2,u_2)}^{e_2} \tilde{\cap} y_{2(s_2,t_2,u_2)}^{e_2} \cong 0_{(\mathcal{X},\theta)}$ are two neutrosophic soft points of $\mathcal{X}$ but Y is given to neutrosophic soft Hausdorff space so we can find two distinct neutrosophic soft b^{**} open sets $(\widetilde{g_2},\theta)$ and $(\widetilde{h_2},\theta)$ such that $x_{2(s_2,t_2,u_2)}^{e_2} \tilde{\in} (\widetilde{g_1},\theta)$ and $y_{2(s_2,t_2,u_2)}^{e_2} \tilde{\in} (\widetilde{h_2},\theta)$. Now $\mathcal{X} \times (\widetilde{g_2},\theta)$ and $\mathcal{X} \times (\widetilde{h_2},\theta)$ are two distinct neutrosophic soft b^{**} open sets in $\mathcal{X} \times Y$ containing $(\widetilde{\lambda_1},\theta)$ and $(\widetilde{\lambda_2},\theta)$ respectively that is $(\widetilde{\lambda_1},\theta) \in \mathcal{X} \times (\widetilde{g_2},\theta)$ and $(\widetilde{\lambda_2},\theta) \in \mathcal{X} \times (\widetilde{h_2},\theta)$ such that $\mathcal{X} \times (\widetilde{g_2},\theta)) \tilde{\cap} (\mathcal{X} \times (\widetilde{h_2},\theta) \cong 0_{(\mathcal{X},\theta)}$. Hence the product of any finite number of neutrosophic soft Hausdorff spaces is a neutrosophic soft Hausdorff space.

Theorem 4.3 The product of any number of neutrosophic soft Hausdorff spaces is a neutrosophic soft Hausdorff space.

Proof. Let us suppose that $\mathcal{X} \cong \prod_{i \tilde{\in} I} \mathcal{X}_i$ provided $\mathcal{X}_i \forall\, i \in I$ is a neutrosophic soft Hausdorff space at its own right. We need to show that $\mathcal{X}$ is neutrosophic soft Hausdorff space. For this let $x_{1(s_1,t_1,u_1)}^{e_1}$ and $x_{2(s_2,t_2,u_2)}^{e_2}$ such that $x_{1(s_1,t_1,u_1)}^{e_1} \tilde{\cap} x_{2(s_2,t_2,u_2)}^{e_2} \cong 0_{(\mathcal{X},\theta)}$ be two neutrosophic soft points of $\prod_{i \tilde{\in} I} \mathcal{X}_i$ such that they are disjoint. This implies $x_{(s,t,u)}^{e} \cong (x_{(s,t,u)_i}^{e})_{i \in I}$ and $y_{(s',t',u')}^{e'} \cong (y_{(s',t',u')_i}^{e'})_{i \in I}$ where $x_{(s,t,u)_i}^{e}$ and $y_{(s',t',u')_i}^{e'} \tilde{\in} \widetilde{\mathcal{X}_{\theta_i}} \forall\, i \in I$. But $x_{(s,t,u)}^{e}$ and $y_{(s',t',u')}^{e'}$ are distinct, so $(x_{(s,t,u)_i}^{e})_{i \in I}$ and $(y_{(s',t',u')_i}^{e'})_{i \in I}$ are distinct which leads to $x_{(s,t,u)_i}^{e}$ and $y_{(s',t',u')_i}^{e'}$ are distinct for some $i \in I$.

Since the neutrosophic soft projection $\pi_i : \prod_{i \tilde{\in} I} \widetilde{\mathcal{X}_{\theta_i}} \to \widetilde{\mathcal{X}_{\theta_i}}$ defined by

$\pi_1\left(x^{e}_{(s,t,u)}\right) \cong x^{e}_{(s,t,u)_i}$ for all $x^{e}_{(s,t,u)} \cong \left(x^{e}_{(s,t,u)_i}\right)_{i\in I}$. Since $x^{e}_{(s,t,u)_i}$ and $y^{e'}_{(s',t',u')} \tilde{\in} \widetilde{\mathcal{X}_{\theta_i}}$ such that $x^{e}_{(s,t,u)_i}$ and $y^{e'}_{(s',t',u')_i}$ are distinct and more over $\widetilde{\mathcal{X}_{\theta_i}}$ is neutrosophic soft Hausdorff space. So there exist neutrosophic soft b^{**} open sets $(\widetilde{g_i},\theta)$ and $(\widetilde{h_i},\theta)$ in $\mathcal{X}_i$ so that $x_1{}^{e_1}_{(s_1,t_1,u_1)} \tilde{\in} (\widetilde{g_{i1}},\theta)$, $y_1{}^{e'_1}_{(s'_1,t'_1,u'_1)} \tilde{\in} (\widetilde{h_{i1}},\theta)$, $(\widetilde{g_{i1}},\theta)\tilde{\cap}(\widetilde{h_{i1}},\theta) \cong 0_{(\mathcal{X},\theta)}$, $x_2{}^{e_2}_{(s_2,t_2,u_2)} \tilde{\in} (\widetilde{g_{i2}},\theta), y_2{}^{e'_2}_{(s'_2,t'_2,u'_2)} \tilde{\in} (\widetilde{h_{i2}},\theta), (\widetilde{g_{i2}},\theta)\tilde{\cap}(\widetilde{h_{i2}},\theta) \cong 0_{(\mathcal{X},\theta)}$, $x_3{}^{e_3}_{(s_3,t_3,u_3)} \tilde{\in} (\widetilde{g_{i3}},\theta)$, $y_3{}^{e'_3}_{(s'_3,t'_3,u'_3)} \tilde{\in} (\widetilde{h_{i3}},\theta), (\widetilde{g_{i3}},\theta)\tilde{\cap}(\widetilde{h_{i3}},\theta) \cong 0_{(\mathcal{X},\theta)}$ and in general $x^{e}_{(s,t,u)_i} \tilde{\in} (\widetilde{g_i},\theta)$, $y^{e'}_{(s',t',u')_i} \tilde{\in} (\widetilde{h_i},\theta)$ and $(\widetilde{g_i},\theta)\tilde{\cap}(\widetilde{h_i},\theta) \cong 0_{(\mathcal{X},\theta)} \Longrightarrow \pi_i\left(x^{e}_{(s,t,u)}\right) \tilde{\in} (\widetilde{g_i},\theta)$, $\pi_i\left(y^{e'}_{(s',t',u')}\right) \tilde{\in} (\widetilde{h_i},\theta) \Longrightarrow x^{e}_{(s,t,u)} \in {\pi_i}^{-1}(\widetilde{g_i},\theta)$ and $\left(y^{e'}_{(s',t',u')}\right) \tilde{\in} {\pi_i}^{-1}(h_i, \theta)$ and let $(\tilde{g},\theta) \cong {\pi_i}^{-1}(\widetilde{g_i},\theta)$ and $(\tilde{h},\theta) \cong {\pi_i}^{-1}(\widetilde{h_i},\theta)$. Then obviously, $x^{e}_{(s,t,u)} \tilde{\in} (\widetilde{g_i},\theta)$ and $y^{e'}_{(s',t',u')} \tilde{\in} (\tilde{h},\theta)$ since, ${\pi_i}^{-1}(\widetilde{g_i},\theta)$ and ${\pi_i}^{-1}(\widetilde{h_i},\theta)$ are neutrosophic soft open in $\prod_{i\tilde{\in}I}\mathcal{X}_i$, so do $(\tilde{g},\theta)$ and $(\tilde{h},\theta)$ it is enough to show that $(\tilde{g},\theta)\tilde{\cap}(\tilde{h},\theta) \cong 0_{(\mathcal{X},\theta)}$. Consider $(\tilde{g},\theta)\tilde{\cap}(\tilde{h},\theta) \cong {\pi_i}^{-1}(\widetilde{g_i},\theta)$ and $\tilde{\cap}{\pi_i}^{-1}(\widetilde{h_i},\theta) \cong {\pi_i}^{-1}((g_i,\theta)\tilde{\cap}(\widetilde{h_i},\theta)) \cong {\pi_i}^{-1}(0_{(\mathcal{X},\theta)}) \cong 0_{(\mathcal{X},\theta)}$.

Theorem 4.4 Let $\langle \pi, \tilde{\tau}_1, \varpi\rangle$and $\langle \Omega, \tilde{\Im}_1, \varpi\rangle$ be two neutrosophic soft topological spaces on the crisp set $\pi$and Ω respectively. $\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \Omega, \tilde{\Im}_1, \varpi\rangle$ be the neutrosophic soft product space, then the neutrosophic soft product space $\times\left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \Omega, \tilde{\Im}_1, \varpi\rangle\right)^{j:j\in\lambda}\right]$ is

neutrosophic soft b_1^{**} space iff each neutrosophic soft coordinate space $\left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \mathfrak{L}, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^j\right]$ is neutrosophic soft b_1^{**}space.

Proof. Suppose each neutrosophic soft coordinate space $\left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \mathfrak{L}, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^j\right]$ is neutrosophic soft $b^{**}{}_1$ space and let $\left[\left(\mathfrak{t}_1{}^m{}_{(p_1,p_2,p_3)}, \varpi\right)^{j:j\in\lambda}\right]$ be an element of $\times\left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \mathfrak{L}, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{j:j\in\lambda}\right]$then, $\left(\mathfrak{t}_1{}^m{}_{(p_1,p_2,p_3)}, \varpi\right)^j \tilde{\in} \left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \mathfrak{L}, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^j\right]$ for each $j \tilde{\in} \Delta$. Since $\left[\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \mathfrak{L}, \tilde{\mathfrak{F}}_1, \varpi\rangle\right]^j$ is neutrosophic soft b_1^{**} space, it follows that $\left[\left(\mathfrak{t}_1{}^m{}_{(p_1,p_2,p_3)}, \varpi\right)^j\right]$ is $\tilde{\tau}_1\tilde{\mathfrak{F}}_1$ neutrosophic soft b^{**}closed for each $j \tilde{\in} \Delta$. Now, each neutrosophic soft projection mapping $\mathfrak{H}^j$ being neutrosophic soft continuous, it follows that $(\mathfrak{H}^j)^{-1}\left[\left(\mathfrak{t}_1{}^m{}_{(p_1,p_2,p_3)}, \varpi\right)^j\right]$ is soft $\tilde{\tau}_1\tilde{\mathfrak{F}}_1$ neutrosophic soft b^{**} closed is $\times\left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \mathfrak{L}, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{j:j\in\lambda}\right]$ for every $j \tilde{\in} \Delta$. Consequently,$\bigcap_j(\mathfrak{H}^\partial)^{-1}\left[\left(\mathfrak{t}_1{}^m{}_{(p_1,p_2,p_3)}, \varpi\right)^j\right] = \left(\mathfrak{t}_1{}^m{}_{(p_1,p_2,p_3)}, \varpi\right)^j$ so, every singleton neutrosophic soft subs$p$ace of $\times\left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \mathfrak{L}, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{j:j\in\lambda}\right]$ is neutrosophic soft $\tilde{\tau}_1\tilde{\mathfrak{F}}_1$ neutrosophic soft b^{**}closed . Hence, $\times\left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \mathfrak{L}, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{j:j\in\lambda}\right]$ is neutrosophic soft $**_{b_1}$ space and conversely, let $\times\left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \mathfrak{L}, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{j:j\in\lambda}\right]$ is neutrosophic soft $*_{b_1}$ space and let $\left(\times\left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \mathfrak{L}, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{j:j\in\lambda}\right]\right)^\beta$ be an arbitrary neutrosophic soft coordinate space of $\times\left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \mathfrak{L}, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{j:j\in\lambda}\right]$. Let $\left[\left[\left(\mathfrak{t}_1{}^m{}_{(p_1,p_2,p_3)}, \varpi\right)^\partial\right]\right]^\beta$ and $\left[\left(\mathfrak{t}_2{}^{m'}{}_{(p_1',p_2',p_3')}, \varpi\right)^j\right]^\beta$ be any two neutrosophic soft distinct points of $\times\left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \mathfrak{L}, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{j:j\in\lambda}\right]^\beta$. Choose $\left(\mathfrak{t}_1{}^m{}_{(p_1,p_2,p_3)}, \varpi\right)$ and $\left(\mathfrak{t}_2{}^{m'}{}_{(p_1',p_2',p_3')}, \varpi\right)$ in $\times\left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \mathfrak{L}, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{j:j\in\lambda}\right]$ whose β - th coordinate is $\left[\left[\left(\mathfrak{t}_1{}^m{}_{(p_1,p_2,p_3)}, \varpi\right)^j\right]\right]^\beta$ and $\left[\left[\left(\mathfrak{t}_2{}^{m'}{}_{(p_1',p_2',p_3')}, \varpi\right)^j\right]\right]$ respect i vely. Since $\left[\left[\left(\mathfrak{t}_1{}^m{}_{(p_1,p_2,p_3)}, \varpi\right)^j\right]\right]^\beta \neq \left[\left[\left(\mathfrak{t}_2{}^{m'}{}_{(p_1',p_2',p_3')}, \varpi\right)^j\right]\right]^\beta$ we have $\left(\mathfrak{t}_1{}^m{}_{(p_1,p_2,p_3)}, \varpi\right) \succ \left(\mathfrak{t}_2{}^{m'}{}_{(p_1',p_2')}, \varpi\right)$ or $\left(\mathfrak{t}_1{}^m{}_{(p_1,p_2,p_3)}, \varpi\right) \prec \left(\mathfrak{t}_2{}^{m'}{}_{(p_1',p_2',p_3')}, \varpi\right)$ or $\left(\mathfrak{t}_1{}^m{}_{(p_1,p_2,p_3)}, \varpi\right) \succ\succ \left(\mathfrak{t}_2{}^{m'}{}_{(p_1',p_2',p_3')}, \varpi\right)$ or $\left(\mathfrak{t}_1{}^m{}_{(p_1,p_2,p_3)}, \varpi\right) \preccurlyeq \left(\mathfrak{t}_2{}^{m'}{}_{(p_1',p_2',p_3')}, \varpi\right)$ But $\times\left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \mathfrak{L}, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{j:j\in\lambda}\right]$ being soft neutrosophic soft $b^{**}{}_1$space , corresponding to the neutrosophic soft distinct points $\left(\mathfrak{t}_1{}^m{}_{(p_1,p_2,p_3)}, \varpi\right)$ and $\left(\mathfrak{t}_2{}^{m'}{}_{(p_1',p_2',p_3')}, \varpi\right)$ of

$\left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \Omega, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{j:j\in\lambda}\right]$ there exists neutrosophic soft b^{**} open sets $\mathfrak{f}_{1\varpi}$ and $\mathfrak{f}_{2\varpi}$ in $\times\left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \Omega, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{j:j\in\lambda}\right]$ such that $\left({\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)}, \varpi\right) \in \mathfrak{f}_{1\varpi}$ but $\left({\mathfrak{t}_2}^{m'}{}_{(\mathfrak{p}_1',\mathfrak{p}_2',\mathfrak{p}_3')}, \varpi\right) \tilde{\notin} \mathfrak{f}_{1\varpi}$ and $\left({\mathfrak{t}_2}^{m'}{}_{(\mathfrak{p}_1',\mathfrak{p}_2',\mathfrak{p}_3')}, \varpi\right) \in \mathfrak{f}_{2\varpi}$ but $\left({\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)}, \varpi\right) \tilde{\notin} \mathfrak{f}_{2\varpi}$. So, there exists basic neutrosophic soft b^{**} open sets $\mathfrak{f}_{3\varpi} = \times\{\mathfrak{f}_{1\varpi}{}^{\alpha} : \alpha \in \Delta\}$ and $\mathfrak{f}_{4\varpi} = \times\{(\mathfrak{f}_{2\varpi})^{\alpha} : \alpha \in \Delta\}$ such that $\left({\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)}, \varpi\right) \tilde{\in} \mathfrak{f}_{3\varpi} \tilde{\notin} \mathfrak{f}_{1\varpi}$ and $\left({\mathfrak{t}_2}^{m'}{}_{(\mathfrak{p}_1',\mathfrak{p}_2',\mathfrak{p}_3')}, \varpi\right) \tilde{\in} \mathfrak{f}_{4\varpi} \tilde{\notin} (\mathfrak{f}_{2\varpi})$. Clearly, $\left({\mathfrak{t}_2}^{m'}{}_{(\mathfrak{p}_1',\mathfrak{p}_2',\mathfrak{p}_3')}, \varpi\right) \notin \mathfrak{f}_{3\varpi}$ and $\left({\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)}, \varpi\right) \tilde{\notin} \mathfrak{f}_{4\varpi}$. Thus, $\mathfrak{f}_{1\varpi}{}^{\beta}$ is neutrosophic soft b^{**} open set containing $\left({\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)}, \varpi\right)^{\beta}$ but not $\left({\mathfrak{t}_2}^{m'}{}_{(\mathfrak{p}_1',\mathfrak{p}_2',\mathfrak{p}_3')}, \varpi\right)^{\beta}$; and, $(\mathfrak{f}_{2\varpi})^{\beta}$ is neutrosophic soft b^{**}open set containing $\left({\mathfrak{t}_2}^{m'}{}_{(\mathfrak{p}_1',\mathfrak{p}_2',\mathfrak{p}_3')}, \varpi\right)^{\beta}$ but not $\left({\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)}, \varpi\right)^{\beta}$.This shows that $\left(\times\left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \Omega, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{j:j\in\lambda}\right]\right)^{\beta}$ is neutrosophic soft $\mathfrak{b}_1{}^{**}$space.

Theorem 4.5 Let $\langle \pi, \tilde{\tau}_1, \varpi\rangle$ and $\langle \Omega, \tilde{\mathfrak{F}}_1, \varpi\rangle$be two neutrosophic soft topological spaces such that they are neutrosophic soft $\mathfrak{b}_2{}^{**}$space on the crisp set π and Ωrespectively. $\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \Omega, \tilde{\mathfrak{F}}_1, \varpi\rangle$ be the neutrosophic soft product space, then the product space $\times\left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \Omega, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{j:j\in\lambda}\right]$ is neutrosophic soft $\mathfrak{b}_2{}^{**}$ space if and only if each neutrosophic soft coordinate space $\left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \Omega, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{j}\right]$ is neutrosophic soft $\mathfrak{b}_2{}^{**}$space.

Proof. Suppose each neutrosophic soft coordinate space $\times\left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \Omega, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{j}\right]$ is neutrosophic soft $\mathfrak{b}_2{}^{**}$ space and let $\langle {\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)}, \varpi^{j:j\in\lambda}\rangle$ and $\langle\left({\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)}, \varpi\right)^{j:j\in\lambda}\rangle$ be two neutrosophic soft space points of $\times\left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \Omega, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{j:j\in\lambda}\right]$ such that $\langle\left({\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)}, \varpi\right)^{j:j\in\lambda}\rangle \succ \langle\left({\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)}, \varpi\right)^{j:j\in\lambda}\rangle$ Then, $\left({\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)}, \varpi\right)^{\beta} \succ \left({\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)}, \varpi\right)^{\beta}$ for some for each $\beta \tilde{\in} \Delta$ where $\left({\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)}, \varpi\right)^{\beta}, \left({\mathfrak{t}_2}^{m'}{}_{(\mathfrak{p}_1',\mathfrak{p}_2')}, \varpi\right)^{\beta} \tilde{\in} \left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \Omega, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{\beta}\right]$. Now, $\left[\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \Omega, \tilde{\mathfrak{F}}_1, \varpi\rangle\right]^{\beta}$ is neutrosophic soft $\mathfrak{b}_2{}^{**}$ space and $\left({\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)}, \varpi\right)^{\beta} \succ \left({\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)}, \varpi\right)^{\beta}$ are points of $\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \Omega, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{\beta}$, So there exists a $\tilde{\tau}_1\tilde{\mathfrak{F}}_1$ neutrosophic soft b^{**} open sets $(\mathfrak{f}_{1\varpi})^{\beta}$ and $(\mathfrak{f}_{2\varpi})^{\beta}$ such that $\left({\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)}, \varpi\right)^{\beta} \in (\mathfrak{f}_{1\varpi})^{\beta}$ and $\left({\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)}, \varpi\right)^{\beta} \tilde{\in} (\mathfrak{f}_{2\varpi})^{\beta}$ and $(\mathfrak{f}_{1\varpi})^{\beta}\widetilde{\cap}(\mathfrak{f}_{2\varpi})^{\beta} = \widetilde{0_{(\pi,\varpi)}}$. Since $\mathfrak{H}^{\beta}\left[\left(\left({\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)}, \varpi\right)^{\beta}\right)\right] = \left({\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)}, \varpi\right)^{\beta} \in (\mathfrak{f}_{1\varpi})^{\beta}$ and

$\mathfrak{H}^{\beta}\left(\left({\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)},\varpi\right)^{\beta}\right)=\left({\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)},\varpi\right)^{\beta}\ \widetilde{\in}\ (\mathfrak{f}_{2\varpi})^{\beta}$ each neutrosophic soft projection mapping π^{β} being neutrosophic soft continuous, it follows that $\left({\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)},\varpi\right)\in(\mathfrak{H}^{\beta})^{-1}((\mathfrak{f}_{1\varpi})^{\beta})$ and $\left({\mathfrak{t}_2}^{m/}{}_{(\mathfrak{p}_1/,\mathfrak{p}_2/,\mathfrak{p}_3/)},\varpi\right)\ \widetilde{\in}\ (\mathfrak{H}^{\beta})^{-1}((\mathfrak{f}_{2\varpi})^{\beta})$

and $(\mathfrak{H}^{\beta})^{-1}\left[(\mathfrak{f}_{1\varpi})^{\beta}\widetilde{\cap}(\mathfrak{f}_{2\varpi})^{\beta}\right]=(\mathfrak{H}^{\beta})^{-1}\left[\widetilde{0_{(\pi,\varpi)}}\right]=\widetilde{0_{(\pi,\varpi)}}$. Moreover by neutrosophic soft continuity of $\mathfrak{H}^{\beta}$, $(\mathfrak{H}^{\beta})^{-1}\left((\mathfrak{f}_{1\varpi})^{\beta}\right),(\mathfrak{H}^{\beta})^{-1}\left((\mathfrak{f}_{2\varpi})^{\beta}\right)$ are a $\tilde{\tau}_1\tilde{\mathfrak{F}}_1$ neutrosophic soft b^{**} open in $\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{\mathrm{j:j}\in\lambda}\right]$. Hence, $\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{\mathrm{j:j}\in\lambda}\right]$ is neutrosophic soft $\mathfrak{b}_2{}^{**}$space. Conversely, let $\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{\mathrm{j:j}\in\lambda}\right]$is neutrosophic soft $\mathfrak{b}_2{}^{**}$space and let $\left(\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{\mathrm{j:j}\in\lambda}\right]\right)^{\beta}$ be an arbitrary neutrosophic soft coordinate space of $\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{\mathrm{j:j}\in\lambda}\right]$. We must show that $\mathrm{j:j}\in\lambda^{\beta}$ is neutrosophic soft $\mathfrak{b}_2{}^{**}$space. Let $\left({\mathfrak{t}_2}^{m//}{}_{(\mathfrak{p}_1//,\mathfrak{p}_2//,\mathfrak{p}_3//)},\varpi\right)=\langle\left({\mathfrak{t}_2}^{m//}{}_{(\mathfrak{p}_1//,\mathfrak{p}_2//,\mathfrak{p}_3//)},\varpi\right)^{\mathrm{j:j}\in\lambda}\rangle$ be neutrosophic soft fixed point of $\times\left\{\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{\mathrm{j}}\right\}$. Let $(\mathfrak{f}_{3\varpi})$ be neutrosophic soft subset of $\times\left\{\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{\partial}\right\}$ consisting of all points of the form $\left({\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)},\varpi\right)=\langle\left({\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)},\varpi\right)^{\mathrm{j:j}\in\lambda}\rangle$ such that $\left({\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)},\varpi\right)^{\mathrm{j}}=\left({\mathfrak{t}_2}^{m//}{}_{(\mathfrak{p}_1//,\mathfrak{p}_2//,\mathfrak{p}_3//)},\varpi\right)^{\alpha}$ if $\alpha\succ\beta$ and $\left({\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)},\varpi\right)^{\beta}$ may be any neutrosophic soft point of $\left(\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{\mathrm{j:j}\in\lambda}\right]\right)^{\beta}$. Let $\mathfrak{H}^{\left({\mathfrak{t}_2}^{m//}{}_{(\mathfrak{p}_1//,\mathfrak{p}_2//,\mathfrak{p}_3//)},\varpi\right)}$ be the restriction of the neutrosophic soft projection mapping $\mathfrak{H}^{\left({\mathfrak{t}_2}^{m//}{}_{(\mathfrak{p}_1//,\mathfrak{p}_2//,\mathfrak{p}_3//)},\varpi\right)}:\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{\mathrm{j}}\right]\rightarrow$

$\left(\times\left\{\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{\mathrm{j:j}\in\lambda}\right\}\right)^{\beta}$ to $\left({\mathfrak{t}_2}^{m//}{}_{(\mathfrak{p}_1//,\mathfrak{p}_2//,\mathfrak{p}_3//)},\varpi\right)$ such that

$\mathfrak{H}^{\left({\mathfrak{t}_2}^{m//}{}_{(\mathfrak{p}_1//,\mathfrak{p}_2//,\mathfrak{p}_3//)},\varpi\right)}:\left({\mathfrak{t}_2}^{m//}{}_{(\mathfrak{p}_1//,\mathfrak{p}_2//,\mathfrak{p}_3//)},\varpi\right)\rightarrow$

$\left(\times\left\{\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{\mathrm{j:j}\in\lambda}\right\}\right)^{\beta}:\mathfrak{H}^{\left({\mathfrak{t}_2}^{m//}{}_{(\mathfrak{p}_1//,\mathfrak{p}_2//,\mathfrak{p}_3//)},\varpi\right)}\left(\left(\left({\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)},\varpi\right)\right)=\right.$

$\mathfrak{H}^{\beta}\left(\left(\left({\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)},\varpi\right)\right)=\left({\mathfrak{t}_2}^{m//}{}_{(\mathfrak{p}_1//,\mathfrak{p}_2//,\mathfrak{p}_3//)},\varpi\right)^{\beta}\right.$

$\forall\left({\mathfrak{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)},\varpi\right)\in\left({\mathfrak{t}_2}^{m//}{}_{(\mathfrak{p}_1//,\mathfrak{p}_2//,\mathfrak{p}_3//)},\varpi\right)$. Then, $\mathfrak{H}^{\left({\mathfrak{t}_2}^{m//}{}_{(\mathfrak{p}_1//,\mathfrak{p}_2//,\mathfrak{p}_3//)},\varpi\right)}$ is clearly neutrosophic soft one-one and neutrosophic soft onto. Also, the projection mapping $\mathfrak{H}^{\beta}$ being neutrosophic soft continuous, is a restriction $\mathfrak{H}^{\left({\mathfrak{t}_2}^{m//}{}_{(\mathfrak{p}_1//,\mathfrak{p}_2//,\mathfrak{p}_3//)},\varpi\right)}$ is therefore neutrosophic soft continuous. Now, let

$(\not{f}_{1\varpi})$ be any neutrosophic soft b^{**} open set in the neutrosophic soft subspace $\left(\mathbb{t}_2{}^{m//}{}_{(p_1//p_2//p_3//)},\varpi\right)$. Then, $(\not{f}_{1\varpi})=\left(\mathbb{t}_2{}^{m//}{}_{(p_1//p_2//p_3//)},\varpi\right)\cap(\not{f}_{4\varpi})$ for some basic neutrosophic soft b^{**} open set $(\not{f}_{4\varpi})$ in $\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j}\right]$. Let $(\not{f}_{4\varpi})=\times\{(\not{f}_{1\varpi})^{\alpha}:\alpha\in\Lambda\}$, where, $(\not{f}_{1\varpi})^{\alpha}$ is neutrosophic soft b^{**} open in $\left(\times\left\{\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j:j\in\lambda}\right\}\right)^{\alpha}$. Let $(\not{f}_{4\varpi})=\times[\{(\not{f}_{1\varpi})^{\alpha}:\alpha\tilde{\in}\Lambda\}]$, where $(\not{f}_{1\varpi})^{\alpha}$ is neutrosophic soft b^{**} open in $\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{\partial}\right]$. And consequently, $(\not{f}_{1\varpi})=\left(\mathbb{t}_2{}^{m//}{}_{(p_1//p_2//p_3//)},\varpi\right)\tilde{\cap}(\times[\{(\not{f}_{1\varpi})^{\alpha}:\alpha\tilde{\in}\Lambda\}])$, Where $(\not{f}_{1\varpi})^{\alpha}$ is neutrosophic soft b^{**} open in $\left(\times\left\{\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j:j\in\lambda}\right\}\right)^{\alpha}$ for each $\alpha\tilde{\in}\Lambda$. Thus, either $(\not{f}_{1\varpi})=\widetilde{0_{(\pi,\varpi)}}$ or $(\not{f}_{1\varpi})=\{(\ \mathbb{t}_1{}^{m}{}_{(p_1,p_2,p_3)},\varpi\ )\ \tilde{\in}\ (\ \mathbb{t}_2{}^{m//}{}_{(p_1//p_2//p_3//)},\varpi\ )$: βth coordinate of $(\mathbb{t}_1{}^{m}{}_{(p_1,p_2,p_3)},\varpi)$ in $(\not{f}_{1\varpi})$^β }. Therefore $\mathfrak{H}^{\left(\mathbb{t}_2{}^{m//}{}_{(p_1//p_2//p_3//)},\varpi\right)}((\not{f}_{1\varpi}))=\widetilde{0_{(\pi,\varpi)}}$ or $(\not{f}_{1\varpi})^{\beta}$, each one of which is neutrosophic soft b^{**}open. This shows that the neutrosophic soft image under $\mathfrak{H}^{\left(\mathbb{t}_2{}^{m//}{}_{(p_1//p_2//p_3//)},\varpi\right)}$ of every basic neutrosophic soft b^{**}open set in $\left(\mathbb{t}_2{}^{m//}{}_{(p_1//p_2//p_3//)},\varpi\right)$ is neutrosophic soft b^{**}open and therefore, $\mathfrak{H}^{\left(\mathbb{t}_2{}^{m//}{}_{(p_1//p_2//p_3//)},\varpi\right)}$ is neutrosophic soft $*_b$open. Thus, $\mathfrak{H}^{\left((\mathbb{t}_1{}^{m}{}_{(p_1,p_2,p_3)},\varpi\right)}$, is neutrosophic soft homeomorphism and therefore, $\left(\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j:j\in\lambda}\right]\right)^{\beta}$ is the neutrosophic soft homeomorphic image of $\left(\mathbb{t}_2{}^{m//}{}_{(p_1//p_2//p_3//)},\varpi\right)$. Now, every neutrosophic soft subspace of a neutrosophic soft $\mathfrak{b}_2{}^{**}$space being neutrosophic soft $\mathfrak{b}_2{}^{**}$space, $\left(\mathbb{t}_2{}^{m//}{}_{(p_1//p_2//p_3//)},\varpi\right)$ is neutrosophic soft b^{**}open and therefore, $\mathfrak{H}^{\left(\mathbb{t}_2{}^{m//}{}_{(p_1//p_2//p_3//)},\varpi\right)}{}^{\left(\mathbb{t}_2{}^{m//}{}_{(p_1//p_2//p_3//)},\varpi\right)}$ is neutrosophic soft $\mathfrak{b}_2{}^{**}$space and so its neutrosophic soft homeomorphic image $\left(\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j:j\in\lambda}\right]\right)$ is soft neutrosophic soft $\mathfrak{b}_2{}^{**}$space. Hence, each neutrosophic soft coordinate space is neutrosophic soft $\mathfrak{b}_2{}^{**}$space.

Theorem 4.6 Let $\langle\pi,\tilde{\tau}_1,\varpi\rangle$and $\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle$ be two neutrosophic soft topological spaces on the crisp set π and $\mathfrak{Q}$ respectively. $\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle$be the neutrosophic soft product space, then the product space $\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j:j\in\lambda}\right]$ is neutrosophic soft

b^{**}-regular space iff each neutrosophic soft coordinate space $\left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \mathfrak{Q}, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{j}\right]$ is neutrosophic soft b^{**}regular space.

Proof. Suppose each neutrosophic soft coordinate space $\times\left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \mathfrak{Q}, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{j}\right]$is neutrosophic soft b^{**}-regular space. Let $\left({\mathfrak{t}_1}^{m}{}_{(p_1,p_2,p_3)}, \varpi\right) = \langle\left({\mathfrak{t}_1}^{m}{}_{(p_1,p_2,p_3)}, \varpi\right)^{j:j\in\lambda}\rangle$ be any neutrosophic soft point of the neutrosophic soft product space $\times\left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \mathfrak{Q}, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{j:j\in\lambda}\right]$ and $\mathcal{F}_{1\varpi}$ be any neutrosophic soft b^{**} open set in $\times\left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \mathfrak{Q}, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{j:j\in\lambda}\right]$ such that $\left({\mathfrak{t}_1}^{m}{}_{(p_1,p_2,p_3)}, \varpi\right) \tilde{\in} \mathcal{F}_{1\varpi}$ then there exists a neutrosophic soft b^{**} open set $(\mathcal{F}_{2\varpi})$ in $\times\left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \mathfrak{Q}, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{j:j\in\lambda}\right]$ such that $\left({\mathfrak{t}_1}^{m}{}_{(p_1,p_2,p_3)}, \varpi\right) \tilde{\in} (\mathcal{F}_{2\varpi}) \tilde{\subseteqq} \mathcal{F}_{1\varpi}$. Let, $\times\left\{\left((\mathcal{F}_{1\varpi})\right)^{j:j\in\lambda}\right\}$ is the neutrosophic soft product space such that $(\mathcal{F}_{1\varpi})^{j}$ is neutrosophic soft $**_b$ open in $\times\left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \mathfrak{Q}, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{j}\right]$. Since each $\left\{\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \mathfrak{Q}, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{j}\right\}$ is neutrosophic soft b^{**} regular space and $\left((\mathcal{F}_{1\varpi})\right)^{j}$ is neutrosophic soft b^{**} open in $\times\left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \mathfrak{Q}, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{j}\right]$containing $\left({\mathfrak{t}_1}^{m}{}_{(p_1,p_2,p_3)}, \varpi\right)^{j}$ there exists a neutrosophic soft b^{**} open set $\left((\mathcal{F}_{3\varpi})\right)^{j}$ in $\times\left\{\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \mathfrak{Q}, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{j}\right\}$ such that $\left({\mathfrak{t}_1}^{m}{}_{(p_1,p_2,p_3)}, \varpi\right)^{j} \tilde{\in} \left((\mathcal{F}_{3\varpi})\right)^{j}$ and $\overline{\left((\mathcal{F}_{3\varpi})\right)^{j}} \tilde{\subseteqq} (\mathcal{F}_{1\varpi})^{j}$. Let $\times\left[\left((\mathcal{F}_{3\varpi})\right)^{j:j\in\lambda}\right]$, then$\left[\left((\mathcal{F}_{3\varpi})\right)^{j:j\in\lambda}\right]$ is neutrosophic soft b^{**}open set in $\times\left[\left(\langle \pi, \tilde{\tau}_1, \varpi\rangle * \langle \mathfrak{Q}, \tilde{\mathfrak{F}}_1, \varpi\rangle\right)^{j:j\in\lambda}\right]$ and contains $\left({\mathfrak{t}_1}^{m}{}_{(p_1,p_2,p_3)}, \varpi\right)$. Also, $\overline{\times\left((\mathcal{F}_{3\varpi})\right)^{j:j\in\lambda}} = \times\left[\overline{\left((\mathcal{F}_{3\varpi})\right)^{j:j\in\lambda}} : j \in \Delta\right]$.Further, since $\overline{\left((\mathcal{F}_{3\varpi})\right)^{j}} \tilde{\subseteqq} (\mathcal{F}_{1\varpi})^{j}$ for each j, we have $\times\left[\overline{\left((\mathcal{F}_{3\varpi})\right)^{j:j\in\lambda}} : j \in \Delta\right] \tilde{\subseteqq} \times\left[\left((\mathcal{F}_{3\varpi})\right)^{j:j\in\lambda}\right]$,this

shows that for every neutrosophic soft point $\left(\mathbb{t}_1{}^{m}{}_{(\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3)},\varpi\right)\tilde{\in}\times\left\{\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j:j\in\lambda}\right\}$ and every neutrosophic soft b^{**} open set $(\mathscr{f}_{1\varpi})$ containing $\left(\mathbb{t}_1{}^{m}{}_{(\mathfrak{p}_1\mathfrak{p}_2)},\varpi\right)$ there exists a neutrosophic soft b^{**} open set $\times\left[\overline{\left((\mathscr{f}_{3\varpi})\right)^{j:j\in\lambda}}:j\in\Delta\right]$ in $\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j:j\in\lambda}\right]$ such that $\left(\mathbb{t}_1{}^{m}{}_{(\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3)},\varpi\right)\tilde{\in}\times\left[\left((\mathscr{f}_{3\varpi})\right)^{j:j\in\lambda}\right]$ and $\overline{\times\left((\mathscr{f}_{3\varpi})\right)^{j:j\in\lambda}}\,\tilde{\in}\times\left[\left((\mathscr{f}_{3\varpi})\right)^{j:j\in\lambda}\right]$. Hence, $\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j:j\in\lambda}\right]$ is neutrosophic soft b^{**}-regular. Conversely, let the non-empty neutrosophic soft product space $\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j:j\in\lambda}\right]$ be neutrosophic soft b^{**}-regular and let $\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j:j\in\lambda}\right]^{\beta}$ be an arbitrary neutrosophic soft coordinate space. Then, we must show that it is a neutrosophic soft b^{**}-regular. Let $\left(\mathbb{t}_1{}^{m}{}_{(\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3)},\varpi\right)^{\beta}$ be any neutrosophic soft point of $\left(\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j:j\in\lambda}\right]\right)^{\beta}$ and let $(\mathscr{f}_{1\varpi})^{\beta}$ be any neutrosophic soft b^{**} open in $\left(\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j:j\in\lambda}\right]\right)^{\beta}$ such that $\left(\mathbb{t}_1{}^{m}{}_{(\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3)},\varpi\right)^{\beta}\tilde{\in}(\mathscr{f}_{1\varpi})^{\beta}$. now, choose, soft element $\left(\mathbb{t}_1{}^{m}{}_{(\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3)},\varpi\right)$ in $\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j:j\in\lambda}\right]$ whose βth coordinate in $\left(\mathbb{t}_1{}^{m}{}_{(\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3)},\varpi\right)^{\beta}$. Let $(\mathscr{f}_{1\varpi})=\mathfrak{H}^{\beta-1}(\left(\mathbb{t}_1{}^{m}{}_{(\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3)},\varpi\right)^{i}$. Then, $\left(\mathbb{t}_1{}^{m}{}_{(\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3)},\varpi\right)\in(\mathscr{f}_{1\varpi})$ and by neutrosophic soft continuity of $\mathfrak{H}^{\beta}$, $(\mathscr{f}_{1\varpi})$ is neutrosophic soft b^{**} open in $\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j:j\in\lambda}\right]$. Since, $\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle*\langle\mathfrak{Q},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j:j\in\lambda}\right]$is neutrosophic soft b^{**}-regular space so there exists a neutrosophic s o ft b^{**} open set $\times\left[\overline{\left((\mathscr{f}_{3\varpi})\right)^{j:j\in\lambda}}\right]$ and where each $\left((\mathscr{f}_{3\varpi})\right)^{j}$ is neutrosophic soft b^{**} open in

$\left(\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle * \langle\Omega,\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j:j\in\lambda}\right]\right)^{\beta}$ such that $\left(\mathfrak{t}_1{}^{m}{}_{(p_1,p_2,p_3)},\varpi\right)\tilde{\in}\times\left\{\left((\mathfrak{f}_{3\varpi})\right)^{j:j\in\lambda}\right\}$ and $\overline{\left[\left((\mathfrak{f}_{3\varpi})\right)^{j:j\in\lambda}\right]}\tilde{\in}(\mathfrak{f}_{1\varpi})$. Now $\left(\mathfrak{t}_1{}^{m}{}_{(p_1,p_2,p_3)},\varpi\right)\tilde{\in}\overline{\left[\left((\mathfrak{f}_{3\varpi})\right)^{j:j\in\lambda}\right]}\tilde{\in}(\mathfrak{f}_{1\varpi})$ this implies that $\left(\mathfrak{t}_1{}^{m}{}_{(p_1,p_2,p_3)},\varpi\right)\tilde{\in}(\mathfrak{f}_{1\varpi})=\mathfrak{H}^{\beta-1}((\mathfrak{f}_{1\varpi})^{\beta})$ implies that $\left(\mathfrak{t}_1{}^{m}{}_{(p_1,p_2,p_3)},\varpi\right)^{\beta}\tilde{\in}(\mathfrak{f}_{1\varpi})$, Moreover, $\overline{\left[\left((\mathfrak{f}_{3\varpi})\right)^{j:j\in\lambda}\right]}=\left[\left((\mathfrak{f}_{3\varpi})\right)^{j:j\in\lambda}\right]$ and $\overline{\left[\left((\mathfrak{f}_{3\varpi})\right)^{j:j\in\lambda}\right]}\tilde{\in}(\mathfrak{f}_{1\varpi})^{\beta}$ This shows that $\left(\times\left\{\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle * \langle\Omega,\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j:j\in\lambda}\right\}\right)^{\beta}$ is neutrosophic soft b^{**} regular and hence, each coordinate space is neutrosophic soft b^{**}regular.

Theorem 4.7 Let $\langle\pi,\tilde{\tau}_1,\varpi\rangle$ and $\langle\Omega,\tilde{\mathfrak{F}}_1,\varpi\rangle$ be two neutrosophic soft topological spaces on the crisp set π and Ωrespectively. $\langle\pi,\tilde{\tau}_1,\varpi\rangle * \langle\Omega,\tilde{\mathfrak{F}}_1,\varpi\rangle$be the soft product space, then the product space $\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle * \langle\Omega,\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j:j\in\lambda}\right]$ is neutrosophic soft b^{**}close regular space if each neutrosophic soft coordinate space $\left\{\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle * \langle\Omega,\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j}\right\}$ is neutrosophic soft close b^{**}- space.

Proof. Let each soft coordinate space $\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle * \langle\Omega,\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j}\right]$ is neutrosophic soft b^{**}close regular space. Then, we must show that the neutrosophic soft product space $\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle * \langle\Omega,\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j:j\in\lambda}\right]$. Let $\mathfrak{f}_{1\varpi}$ be any member of the usual neutrosophic soft subbase for the neutrosophic soft product topology and let $\left(\mathfrak{t}_1{}^{m}{}_{(p_1,p_2,p_3)},\varpi\right)=\langle\left(\mathfrak{t}_1{}^{m}{}_{(p_1,p_2,p_3)},\varpi\right)^{j:j\in\lambda}\rangle$ be any neutrosophic soft point in $\mathfrak{f}_{1\varpi}$ Then $\mathfrak{f}_{1\varpi}=\mathfrak{H}^{\beta^{-1}}\left((\mathfrak{f}_{2\varpi})^{\beta}\right)$ is neutrosophic soft b^{**} open in $\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle * \langle\Omega,\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{\beta}\right]$ and contains $\left(\mathfrak{t}_1{}^{m}{}_{(p_1,p_2,p_3)},\varpi\right)^{\beta}$). Since, $\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle * \langle\Omega,\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{\beta}\right]$is neutrosophic soft close regular b^{**} -space there exists a neutrosophic soft mapping

$f:\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle * \langle\mathfrak{L},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{\beta}\right] \longrightarrow [\tilde{0},\tilde{1}]\times[\tilde{0},\tilde{1}]$ such that $f\left(\left({\mathbb{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)},\varpi\right)^{\beta}\right)=\widetilde{0_{(\pi,\varpi)}}$ and

$f\left({\mathbb{t}_2}^{m'}{}_{(\mathfrak{p}_1/,\mathfrak{p}_2/,\mathfrak{p}_3/)},\varpi\right)=\widetilde{1_{(\pi,\varpi)}}.\ \forall\ {\mathbb{t}_2}^{m'}{}_{(\mathfrak{p}_1/,\mathfrak{p}_2/,\mathfrak{p}_3/)}\ \tilde{\in}\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle * \langle\mathfrak{L},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{\beta}\right]-(\mathfrak{f}_{2\varpi})^{\beta}.$

Since $\mathfrak{H}^{\beta}$ is neutrosophic soft continuous and f is neutrosophic soft continuous, so the neutrosophic soft composite mapping $(fo\mathfrak{H}^{\beta}):\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle * \langle\mathfrak{L},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j;j\in\lambda}\right]\longrightarrow[\tilde{0},\tilde{1}]\times[\tilde{0},\tilde{1}]$ is neutrosophic soft continuous.

Now, if $\left({\mathbb{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)},\varpi\right)\tilde{\in}\mathfrak{f}_{1\varpi}$, then $\left({\mathbb{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)},\varpi\right)\tilde{\in}\mathfrak{H}^{\beta^{-1}}((\mathfrak{f}_{1\varpi})^{\beta})$

$\Longrightarrow\mathfrak{H}^{\beta}\left({\mathbb{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)},\varpi\right)\in(\mathfrak{f}_{2\varpi})^{\beta}\Longrightarrow f\left[\mathfrak{H}^{\beta}\left(\left({\mathbb{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)},\varpi\right)\right)\right]=f\left(\left({\mathbb{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)},\varpi\right)^{\beta}\right)=\widetilde{0_{(\pi,\varpi)}}$

$\Longrightarrow(fo\mathfrak{H}^{\beta})\left({\mathbb{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)},\varpi\right)=\widetilde{0_{(\pi,\varpi)}}.$ Again, if

$\left({\mathbb{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)},\varpi\right)\tilde{\in}\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle * \langle\mathfrak{L},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j;j\in\lambda}\right]-\mathfrak{f}_{1\varpi}$, then $\left({\mathbb{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)},\varpi\right)\tilde{\in}\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle * \langle\mathfrak{L},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j;j\in\lambda}\right]-\mathfrak{f}_{1\varpi}\Longrightarrow\left({\mathbb{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)},\varpi\right)\tilde{\in}\mathfrak{H}^{\beta^{-1}}(\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle * \langle\mathfrak{L},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{\beta}\right]-\mathfrak{H}^{\beta^{-1}}((\mathfrak{f}_{2\varpi})^{\beta})$

$\Longrightarrow\left({\mathbb{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)},\varpi\right)\in\mathfrak{H}^{\beta^{-1}}(\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle * \langle\mathfrak{L},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{\beta}\right]-(\mathfrak{f}_{2\varpi})^{\beta})\Longrightarrow$

$\mathfrak{H}^{\beta}\left(\left({\mathbb{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)},\varpi\right)\right)\tilde{\in}\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle * \langle\mathfrak{L},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{\beta}\right]-(\mathfrak{f}_{2\varpi})^{\beta})\Longrightarrow$

$f\left[\mathfrak{H}^{\beta}\left(\left({\mathbb{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)},\varpi\right)\right)\right]=\widetilde{1_{(\pi,\varpi)}}\Longrightarrow(fo\mathfrak{H}^{\beta})\left(\left({\mathbb{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)},\varpi\right)\right)=\widetilde{1_{(\pi,\varpi)}}$

Thus,

$$f\left[\mathfrak{H}^{\beta}\left(\left({\mathbb{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)},\varpi\right)\right)\right]=\begin{bmatrix}\widetilde{0_{(\pi,\varpi)}}=\text{if}\left({\mathbb{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)},\varpi\right)\in(\mathfrak{f}_{2\varpi})^{\beta})\\ \widetilde{0_{(\pi,\varpi)}}=\text{if}\left({\mathbb{t}_1}^{m}{}_{(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)},\varpi\right)\in\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle * \langle\mathfrak{L},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j;j\in\lambda}\right]-\mathfrak{f}_{1\varpi}.\end{bmatrix}$$

Hence, $\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle * \langle\mathfrak{L},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j;j\in\lambda}\right]$ is neutrosophic close b^{**}regular space and conversely, let the neutrosophic soft product space $\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle * \langle\mathfrak{L},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j;j\in\lambda}\right]$ be neutrosophic soft b^{**}close

regular b^{**}- space and let $\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle * \langle\mathfrak{L},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j:j\in\lambda}\right]$ be an arbitrary neutrosophic soft coordinate space. Then by continuing, we can show that $\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle * \langle\mathfrak{L},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j:j\in\lambda}\right]$ is the neutrosophic soft homeomorphic image of a neutrosophic soft subspace of $\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle * \langle\mathfrak{L},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j:j\in\lambda}\right]$. Now, every soft subspace of a neutrosophic soft b^{**} close regular space being a neutrosophic soft b^{**}close regular space and neutrosophic soft homeomorphic image of a neutrosophic soft b^{**}close regular space being neutrosophic soft b^{**}close regular space it follows that $\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle * \langle\mathfrak{L},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j:j\in\lambda}\right]$ is neutrosophic soft b^{**}close regular space. Hence, each coordinate space of $\times\left[\left(\langle\pi,\tilde{\tau}_1,\varpi\rangle * \langle\mathfrak{L},\tilde{\mathfrak{F}}_1,\varpi\rangle\right)^{j:j\in\lambda}\right]$ is neutrosophic soft b^{**}close regular space.

Definition 4.8 Let $\hbar: \mathrm{NSS}(\mathbb{U}) \to [0,1]$ be a mapping, where $NSS(\mathbb{U})$ represents the set of all $(NSSs)$ on $\mathbb{U}$. For $\langle\widetilde{Q_1},\varpi\rangle \in NSS(\mathrm{U}), \hbar\langle\widetilde{Q_1},\varpi\rangle$is called the entropy of $\langle\widetilde{Q_1},\varpi\rangle$ if it satisfies the following conditions:

1) $\hbar\hbar\langle\widetilde{Q_1},\varpi\rangle = 0 \Leftrightarrow \forall \mathfrak{p} \in \varpi, \mathfrak{t} \in \mathrm{U}, \mathbb{T}_{Q_1(\mathfrak{p})}(\mathfrak{t}) = 0, \mathbb{I}_{Q_1(\mathfrak{p})}(\mathfrak{t}) = 0$ and $\mathbb{F}_{Q_1(\mathfrak{p})}(\mathfrak{t}) = 1$ or

   $\mathbb{T}_{Q_1(\mathfrak{p})}(\mathfrak{t}) = 1, \mathbb{I}_{Q_1(\mathfrak{p})}(\mathfrak{t}) = 1$ and $\mathbb{F}_{Q_1(\mathfrak{p})}(\mathfrak{t}) = 0$

2) $\hbar\langle\widetilde{Q_1},\varpi\rangle = 1\ \forall\ \mathfrak{p} \in \varpi, \mathfrak{t} \in \mathbb{U}, \mathbb{T}_{Q_1(\mathfrak{p})}(\mathfrak{t}) = \mathbb{I}_{Q_1(\mathfrak{p})}(\mathfrak{t}) = \mathbb{F}_{Q_1(\mathfrak{p})}(\mathfrak{t}) = 0.5$

3) $\hbar\langle\widetilde{Q_1},\varpi\rangle = \hbar\langle\widetilde{Q_1},\varpi\rangle^c$

4) $\forall \mathfrak{p} \in \varpi, \mathfrak{t} \in \mathbb{U}$ when $\langle\widetilde{Q_1},\varpi\rangle \subseteq \langle\widetilde{Q_2},\varpi\rangle$ and $\mathbb{T}_{\widetilde{Q_2}(\mathfrak{p})}(\mathfrak{t}) \leq \mathbb{F}_{Q_2(\mathfrak{p})}(\mathfrak{t}), \mathbb{I}_{Q_1(\mathfrak{p})}(\mathfrak{t}) \leq \mathbb{I}_{Q_2(\mathfrak{p})}(\mathfrak{t})$if

   $\mathbb{I}_{Q_2(\mathfrak{p})}(\mathfrak{t})) \leq 0.5$ or

   $\langle\widetilde{Q_1},\varpi\rangle \supseteq \langle\widetilde{Q_2},\varpi\rangle$, and $\mathbb{T}_{\widetilde{Q_2}(\mathfrak{p})}(\mathfrak{t}) \geq \mathbb{F}_{\widetilde{Q_2}(\mathfrak{p})}(\mathfrak{t})$, then $\hbar\langle\widetilde{Q_1},\varpi\rangle \leq \hbar\langle\widetilde{Q_2},\varpi\rangle$, $\mathbb{I}_{Q_1(\mathfrak{p})}(\mathfrak{t}) \geq$

   $\mathbb{I}_{Q_2(\mathfrak{p})}(\mathfrak{t})$if $\mathbb{I}_{Q_2(\mathfrak{p})}(\mathfrak{t}) \geq 0.5$ then $\langle\widetilde{Q_1},\varpi\rangle \subseteq \langle\widetilde{Q_2},\varpi\rangle$

Example 4.9 In order to obtain an effective risk supervision in the field of engineering, certain risks are classified along with some parameters and these risks are evaluated by the team of experts (engineers). Assume that there is a set of three experts (engineers) assessing the five different kinds of risks with the set of parameters.

Let $\mathbb{U}$ denote the set of risks $\mathbb{U} = \{\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3, \mathfrak{t}_4, \mathfrak{p}_5\}$ and let ϖ denote the set of parameters $\varpi = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_4, \mathfrak{p}_5\}$.

The neutrosophic soft set

$\mathfrak{f}_{1\varpi}$describes the assessment of engineer 1.

$$\mathfrak{f}_{1\varpi}(\mathfrak{p}_1)(\mathfrak{t}_1) = \langle 07 \times 10^{-1}, 02 \times 10^{-1}, 01 \times 10^{-1}\rangle, \mathfrak{f}_{1\varpi}(\mathfrak{p}_2)(\mathfrak{t}_1) = \langle 05 \times 10^{-1}, 05 \times 10^{-1}, 05 \times 10^{-1}\rangle,$$

$$\mathfrak{f}_{1\varpi}(\mathfrak{p}_3)(\mathfrak{t}_1) = \langle 07 \times 10^{-1}, 02 \times 10^{-1}, 01 \times 10^{-1}\rangle, \mathfrak{f}_{1\varpi}(\mathfrak{p}_4)(\mathfrak{t}_1) = \langle 06 \times 10^{-1}, 07 \times 10^{-1}, 08 \times 10^{-1}\rangle,$$

$$\mathfrak{f}_{1\varpi}(\mathfrak{p}_5)(\mathfrak{t}_1) = \langle 07 \times 10^{-1}, 05 \times 10^{-1}, 04 \times 10^{-1}\rangle.$$

$$\mathfrak{f}_{1\varpi}(\mathfrak{p}_1)(\mathfrak{t}_2) = \langle 08 \times 10^{-1}, 03 \times 10^{-1}, 04 \times 10^{-1}\rangle, \mathfrak{f}_{1\varpi}(\mathfrak{p}_2)(\mathfrak{t}_2) = \langle 07 \times 10^{-1}, 03 \times 10^{-1}, 02 \times 10^{-1}\rangle,$$

$$\mathfrak{f}_{1\varpi}(\mathfrak{p}_3)(\mathfrak{t}_2) = \langle 06 \times 10^{-1}, 08 \times 10^{-1}, 09 \times 10^{-1}\rangle, \mathfrak{f}_{1\varpi}(\mathfrak{p}_4)(\mathfrak{t}_2) = \langle 08 \times 10^{-1}, 01 \times 10^{-1}, 09 \times 10^{-1}\rangle,$$

$$\mathfrak{f}_{1\varpi}(\mathfrak{p}_5)(\mathfrak{t}_2) = \langle 08 \times 10^{-1}, 02 \times 10^{-1}, 01 \times 10^{-1}\rangle.$$

$$\mathfrak{f}_{1\varpi}(\mathfrak{p}_1)(\mathfrak{t}_3) = \langle 04 \times 10^{-1}, 06 \times 10^{-1}, 02 \times 10^{-1}\rangle, \mathfrak{f}_{1\varpi}(\mathfrak{p}_2)(\mathfrak{t}_3) = \langle 03 \times 10^{-1}, 07 \times 10^{-1}, 02 \times 10^{-1}\rangle,$$

$$\mathfrak{f}_{1\varpi}(\mathfrak{p}_3)(\mathfrak{t}_3) = \langle 02 \times 10^{-1}, 09 \times 10^{-1}, 02 \times 10^{-1}\rangle, \mathfrak{f}_{1\varpi}(\mathfrak{p}_4)(\mathfrak{t}_3) = \langle 04 \times 10^{-1}, 06 \times 10^{-1}, 05 \times 10^{-1}\rangle$$

$$\mathfrak{f}_{1\varpi}(\mathfrak{p}_5)(\mathfrak{t}_3) = \langle 03 \times 10^{-1}, 08 \times 10^{-1}, 07 \times 10^{-1}\rangle.$$

$$\mathcal{F}_{1\varpi}(\mathfrak{p}_1)(\mathbb{t}_4) = \langle 04\times10^{-1}, 05\times10^{-1}, 03\times10^{-1}\rangle, \mathcal{F}_{1\varpi}(\mathfrak{p}_2)(\mathbb{t}_4) = \langle 03\times10^{-1}, 06\times10^{-1}, 07\times10^{-}$$
$$\mathcal{F}_{1\varpi}(\mathfrak{p}_3)(\mathbb{t}_4) = \langle 03\times10^{-1}, 05\times10^{-1}, 04\times10^{-1}\rangle, \mathcal{F}_{1\varpi}(\mathfrak{p}_4)(\mathbb{t}_4) = \langle 03\times10^{-1}, 04\times10^{-1}, 05\times10^{-}$$
$$\mathcal{F}_{1\varpi}(\mathfrak{p}_5)(\mathbb{t}_4) = \langle 04\times10^{-1}, 08\times10^{-1}, 03\times10^{-1}\rangle.$$
$$\mathcal{F}_{1\varpi}(\mathfrak{p}_1)(\mathbb{t}_5) = \langle 03\times10^{-1}, 02\times10^{-1}, 07\times10^{-1}\rangle, \mathcal{F}_{1\varpi}(\mathfrak{p}_2)(\mathbb{t}_5) = \langle 02\times10^{-1}, 07\times10^{-1}, 01\times10^{-}$$
$$\mathcal{F}_{1\varpi}(\mathfrak{p}_3)(\mathbb{t}_5) = \langle 03\times10^{-1}, 05\times10^{-1}, 04\times10^{-1}\rangle, \mathcal{F}_{1\varpi}(\mathfrak{p}_4)(\mathbb{t}_5) = \langle 03\times10^{-1}, 06\times10^{-1}, 09\times10^{-}$$
$$\mathcal{F}_{1\varpi}(\mathfrak{p}_5)(\mathbb{t}_5) = \langle 04\times10^{-1}, 02\times10^{-1}, 01\times10^{-1}\rangle.$$

The neutrosophic soft set $\mathcal{F}_{2\varpi}$ describes the assessment of engineer 2.

$$\mathcal{F}_{2\varpi}(\mathfrak{p}_1)(\mathbb{t}_1) = \langle 04\times10^{-1}, 03\times10^{-1}, 02\times10^{-1}\rangle, \mathcal{F}_{2\varpi}(\mathfrak{p}_2)(\mathbb{t}_1) = \langle 03\times10^{-1}, 01\times10^{-1}, 01\times10^{-}$$
$$\mathcal{F}_{2\varpi}(\mathfrak{p}_3)(\mathbb{t}_1) = \langle 05\times10^{-1}, 05\times10^{-1}, 09\times10^{-1}\rangle, \mathcal{F}_{2\varpi}(e_4)(\mathbb{t}_1) = \langle 04\times10^{-1}, 05\times10^{-1}, 04\times10^{-}$$
$$\mathcal{F}_{2\varpi}(\mathfrak{p}_5)(\mathbb{t}_1) = \langle 04\times10^{-1}, 02\times10^{-1}, 01\times10^{-1}\rangle$$

$$\mathcal{F}_{2\varpi}(\mathfrak{p}_1)(\mathbb{t}_2) = \langle 03\times10^{-1}, 04\times10^{-1}, 05\times10^{-1}\rangle, \mathcal{F}_{2\varpi}(\mathfrak{p}_2)(\mathbb{t}_2)$$
$$= \langle 03\times10^{-1}, 04\times10^{-1}, 02\times10^{-1}\rangle$$

$$\mathcal{F}_{2\varpi}(\mathfrak{p}_3)(\mathbb{t}_2) = \langle 03\times10^{-1}, 06\times10^{-1}, 07\times10^{-1}\rangle, \mathcal{F}_{2\varpi}(\mathfrak{p}_4)(\mathbb{t}_2)$$
$$= \langle 04\times10^{-1}, 09\times10^{-1}, 01\times10^{-1}\rangle$$

$$\mathcal{F}_{2\varpi}(\mathfrak{p}_5)(\mathbb{t}_2) = \langle 03\times10^{-1}, 02\times10^{-1}, 01\times10^{-1}\rangle$$

$$\mathcal{F}_{2\varpi}(\mathfrak{p}_1)(\mathbb{t}_3) = \langle 04\times10^{-1}, 06\times10^{-1}, 05\times10^{-1}\rangle, \mathcal{F}_{2\varpi}(\mathfrak{p}_2)(\mathbb{t}_3)$$
$$= \langle 03\times10^{-1}, 09\times10^{-1}, 08\times10^{-1}\rangle \mathcal{F}_{2\varpi}(\mathfrak{p}_3)(\mathbb{t}_3)$$
$$= \langle 03\times10^{-1}, 08\times10^{-1}, 01\times10^{-1}\rangle, \mathcal{F}_{2\varpi}(\mathfrak{p}_4)(\mathbb{t}_3)$$
$$= \langle 04\times10^{-1}, 07\times10^{-1}, 08\times10^{-1}\rangle,$$

$$\mathcal{F}_{2\varpi}(\mathfrak{p}_5)(\mathbb{t}_3) = \langle 02\times10^{-1}, 04\times10^{-1}, 06\times10^{-1}\rangle$$

$$\mathcal{F}_{2\varpi}(\mathfrak{p}_1)(\mathbb{t}_4) = \langle 03\times10^{-1}, 02\times10^{-1}, 01\times10^{-1}\rangle, \mathcal{F}_{2\varpi}(\mathfrak{p}_2)(\mathbb{t}_4)$$
$$= \langle 03\times10^{-1}, 01\times10^{-1}, 02\times10^{-1}\rangle$$

$$\mathcal{F}_{2\varpi}(\mathfrak{p}_3)(\mathbb{t}_4) = \langle 03\times10^{-1}, 02\times10^{-1}, 05\times10^{-1}\rangle, \mathcal{F}_{2\varpi}(\mathfrak{p}_4)(\mathbb{t}_4)$$
$$= \langle 05\times10^{-1}, 03\times10^{-1}, 04\times10^{-1}\rangle, \mathcal{F}_{2\varpi}(\mathfrak{p}_5)(\mathbb{t}_4)$$
$$= \langle 04\times10^{-1}, 06\times10^{-1}, 08\times10^{-1}\rangle$$

$$\digamma_{2\varpi}(\mathfrak{p}_1)(\mathbb{t}_5) = \langle 04 \times 10^{-1}, 06 \times 10^{-1}, 08 \times 10^{-1} \rangle, \digamma_{2\varpi}(\mathfrak{p}_2)(\mathbb{t}_5) = \langle 03 \times 10^{-1}, 07 \times 10^{-1}, 08 \times 10^{-1} \rangle$$

$$\digamma_{2\varpi}(\mathfrak{p}_3)(\mathbb{t}_5) = \langle 04 \times 10^{-1}, 02 \times 10^{-1}, 06 \times 10^{-1} \rangle, \digamma_{2\varpi}(\mathfrak{p}_4)(\mathbb{t}_5) = \langle 04 \times 10^{-1}, 02 \times 10^{-1}, 01 \times 10^{-1} \rangle$$

$$\digamma_{2\varpi}(\mathfrak{p}_5)(\mathbb{t}_5) = \langle 03 \times 10^{-1}, 04 \times 10^{-1}, 08 \times 10^{-1} \rangle$$

The neutrosophic soft set $\digamma_{3\varpi}$ describes the assessment of engineer 3.

$$\digamma_{3\varpi}(\mathfrak{p}_1)(\mathbb{t}_1) = \langle 04 \times 10^{-1}, 05 \times 10^{-1}, 01 \times 10^{-1} \rangle, \digamma_{3\varpi}(\mathfrak{p}_2)(\mathbb{t}_1) = \langle 03 \times 10^{-1}, 01 \times 10^{-1}, 01 \times 10^{-1} \rangle,$$

$$\digamma_{3\varpi}(\mathfrak{p}_3)(\mathbb{t}_1) = \langle 03 \times 10^{-1}, 02 \times 10^{-1}, 03 \times 10^{-1} \rangle, \digamma_{3\varpi}(\mathfrak{p}_4)(\mathbb{t}_1) = \langle 03 \times 10^{-1}, 04 \times 10^{-1}, 07 \times 10^{-1} \rangle,$$

$$\digamma_{3\varpi}(\mathfrak{p}_5)(\mathbb{t}_1) = \langle 05 \times 10^{-1}, 05 \times 10^{-1}, 01 \times 10^{-1} \rangle$$

$$\digamma_{3\varpi}(\mathfrak{p}_1)(\mathbb{t}_2) = \langle 03 \times 10^{-1}, 04 \times 10^{-1}, 05 \times 10^{-1} \rangle, \digamma_{3\varpi}(\mathfrak{p}_2)(\mathbb{t}_2) = \langle 02 \times 10^{-1}, 01 \times 10^{-1}, 03 \times 10^{-1} \rangle,$$

$$\digamma_{3\varpi}(\mathfrak{p}_3)(\mathbb{t}_2) = \langle 03 \times 10^{-1}, 02 \times 10^{-1}, 01 \times 10^{-1} \rangle, \digamma_{3\varpi}(\mathfrak{p}_4)(\mathbb{t}_2) = \langle 03 \times 10^{-1}, 05 \times 10^{-1}, 06 \times 10^{-1} \rangle,$$

$$\digamma_{3\varpi}(\mathfrak{p}_5)(\mathbb{t}_2) = \langle 05 \times 10^{-1}, 04 \times 10^{-1}, 01 \times 10^{-1} \rangle$$

$$\digamma_{3\varpi}(\mathfrak{p}_1)(\mathbb{t}_3) = \langle 03 \times 10^{-1}, 07 \times 10^{-1}, 09 \times 10^{-1} \rangle, \digamma_{3\varpi}(\mathfrak{p}_2)(\mathbb{t}_3) = \langle 03 \times 10^{-1}, 01 \times 10^{-1}, 05 \times 10^{-1} \rangle,$$

$$\begin{aligned}\mathscr{f}_{3\varpi}(\mathfrak{p}_3)(\mathbb{t}_3) &= \langle 03\times10^{-1}, 05\times10^{-1}, 04\times10^{-1}\rangle, \mathscr{f}_{3\varpi}(\mathfrak{p}_4)(\mathbb{t}_3)\\ &= \langle 03\times10^{-1}, 09\times10^{-1}, 08\times10^{-1}\rangle,\end{aligned}$$

$$\begin{aligned}\mathscr{f}_{3\varpi}(\mathfrak{p}_5)(\mathbb{t}_3) &= \langle 03\times10^{-1}, 01\times10^{-1}, 07\times10^{-1}\rangle, \mathscr{f}_{3\varpi}(\mathfrak{p}_1)(\mathbb{t}_4)\\ &= \langle 03\times10^{-1}, 02\times10^{-1}, 06\times10^{-1}\rangle, \mathscr{f}_{3\varpi}(\mathfrak{p}_2)(\mathbb{t}_4)\\ &= \langle 03\times10^{-1}, 03\times10^{-1}, 02\times10^{-1}\rangle,\end{aligned}$$

$$\begin{aligned}\mathscr{f}_{3\varpi}(\mathfrak{p}_3)(\mathbb{t}_4) &= \langle 03\times10^{-1}, 05\times10^{-1}, 06\times10^{-1}\rangle, \mathscr{f}_{3\varpi}(\mathfrak{p}_4)(\mathbb{t}_4)\\ &= \langle 03\times10^{-1}, 08\times10^{-1}, 09\times10^{-1}\rangle,\end{aligned}$$

$$\mathscr{f}_{3\varpi}(e_5)(\mathbb{t}_4) = \langle 03\times10^{-1}, 01\times10^{-1}, 08\times10^{-1}\rangle$$

$$\begin{aligned}\mathscr{f}_{3\varpi}(e_1)(\mathbb{t}_5) &= \langle 04\times10^{-1}, 05\times10^{-1}, 09\times10^{-1}\rangle, \mathscr{f}_{3\varpi}(e_2)(\mathbb{t}_5)\\ &= \langle 04\times10^{-1}, 05\times10^{-1}, 09\times10^{-1}\rangle,\end{aligned}$$

$$\begin{aligned}\mathscr{f}_{3\varpi}(e_3)(\mathbb{t}_5) &= \langle 04\times10^{-1}, 05\times10^{-1}, 07\times10^{-1}\rangle, \mathscr{f}_{3\varpi}(e_4)(\mathbb{t}_5)\\ &= \langle 03\times10^{-1}, 07\times10^{-1}, 08\times10^{-1}\rangle,\end{aligned}$$

$$\mathscr{f}_{3\varpi}(e_5)(\mathbb{t}_5) = \langle 03\times10^{-1}, 02\times10^{-1}, 06\times10^{-1}\rangle$$

Now

$$\begin{aligned}&\mathscr{h}_1\langle\mathscr{f}_{1\varpi}\rangle = 00906\times10^{-5}, \mathscr{h}_2\langle\mathscr{f}_{1\varpi}\rangle = 01148\times10^{-4}, \mathscr{h}_3\langle\mathscr{f}_{1\varpi}\rangle = 00984\times10^{-4},\\ &\mathscr{h}_4\langle\mathscr{f}_{1\varpi}\rangle = 01143\times10^{-4}, \mathscr{h}_5\langle\mathscr{f}_{1\varpi}\rangle = 00746\times10^{-4}\end{aligned}$$

$$\begin{aligned}\mathscr{h}_1\langle\mathscr{f}_{2\varpi}\rangle = 01069\times10^{-4}, \mathscr{h}_2\langle\mathscr{f}_{2\varpi}\rangle = 00646\times10^{-4}, \mathscr{h}_3\langle\mathscr{f}_{2\varpi}\rangle = 00923\times10^{-4}\\ \mathscr{h}_4\langle\mathscr{f}_{2\varpi}\rangle = 00918\times10^{-4}, \mathscr{h}_5\langle\mathscr{f}_{2\varpi}\rangle = 00857\times10^{-4}\\ \mathscr{h}_1\langle\mathscr{f}_{3\varpi}\rangle = 01031\times10^{-4}, \mathscr{h}_2\langle\mathscr{f}_{3\varpi}\rangle = 00742\times10^{-4}, \mathscr{h}_3\langle\mathscr{f}_{3\varpi}\rangle = 01222\times10^{-4}\\ \mathscr{h}_4\langle\mathscr{f}_{3\varpi}\rangle = 00822\times10^{-4}, \mathscr{h}_5\langle\mathscr{f}_{3\varpi}\rangle = 00706\times10^{-4}\end{aligned}$$

Therefore,

$$\mathscr{h}(\langle\mathscr{f}_{1\varpi}\rangle = 00985\times10^{-4}, \mathscr{h}\langle\mathscr{f}_{2\varpi}\rangle = 0.0883\times10^{-4}, \mathscr{h}\langle\mathscr{f}_{3\varpi}\rangle = 00905\times10^{-4}.$$

Entropy is an significant idea for measuring ambiguous information. The less ambiguity

information has the larger possibility to select the best. From the totaling we have $\hbar\langle \mathbb{f}_{2\varpi} \rangle \leq \hbar\langle \mathbb{f}_{3\varpi} \rangle \leq \hbar\langle \mathbb{f}_{21\varpi} \rangle$ and therefore, the engineer 2 has larger opportunity to make the decision on risk supervision than engineer 1 and engineer 3. According to engineer 2 $\hbar_1\langle \mathbb{f}_{2\varpi} \rangle = 01069 \times 10^{-4}$ has the largest entropy value between the risks. Hence, engineering risk has to be minimized to have an effective risk supervision system.

Conclusion

Neutrosophic soft set is considered to be the most important set in mathematics. This study concerns the concept of new type of neutrosophic soft open sets in neutrosophic soft bi topological space. Eight new neutrosophic soft open sets are introduced. Among these neutrosophic soft b^{**} open set is chosen to generate the results. In addition, some results are studied in terms of neutrosophic soft interior and neutrosophic soft closures with respect to soft points. Finally, the most important structure known as entropy is discussed and on the basis of this structures an example is given which is related to engineering which goes like this, in order to obtain an effective risk supervision in the field of engineering, certain risks are classified along with some parameters and these risks are evaluated by the team of experts (engineers). In upcoming time, we will try to introduce new structures which will play important role in decision making problem as in similarity measures. In addition to this, we will try to see the pairwise structures and some engineering problems in [44, 45, 46, 47, 49, 54,55, 56] with respect new generliased open sets with respect to soft points and crisp points of the space.

Acknowledgement: We would like to extend our heartfelt and profound gratitude to **Dr. Arif Mehmood**, the principal investigator of our collaborative project. His extraordinary brilliance and insightful guidance have been the cornerstone of our success. Dr. Mehmood has not only illuminated our path with remarkable clarity but

has also inspired us to embody professionalism and uphold our core values, even in the face of significant challenges. His unwavering support and steadfast commitment to our team have been truly invaluable. Without his relentless dedication and encouragement, the ambitious goals of this project would have remained unfulfilled. We are deeply appreciative of his exceptional leadership and mentorship throughout this transformative journey.

Author Contributions: All authors read and approved the final manuscript.

Funding Statement: The authors received no specific funding for this study.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

Availability of Data and Materials: All the data and materials are provided in the manuscript.

References

[1] L. A. Zadeh, Fuzzy sets, Information and Control, vol. 8, no. 3, pp. 338-353, 1965.

[2] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning (I), Information Science, vol. 8, no. 3, pp. 199-249, 1975.

[3] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning (II), Information Science, vol. 8, no. 4, pp. 301-357, 1975.

[4] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning (III), Information Science, vol. 9, no. 1, pp. 43-80, 1975.

[5] L. A. Zadeh, Toward a generalized theory of uncertainty (GTU): an outline, Information Sciences, vol. 172, no. 1-2, pp. 1-40, 2005.

[6] J. Ye, A multicriteria decision-making method using aggregation operators for simplified neutrosophic set, Journal of Intelligent & Fuzzy Systems, vol. 26, no. 5, pp. 2459-2466, 2014.

[7] C. Liu and Y. Luo, Correlated aggregation operators for simplified neutrosophic set and their application in multi-attribute group decision making, Journal of Intelligent & Fuzzy Systems, vol. 30, no. 3, pp. 1755-1761, 2016.

[8] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems, vol. 20, no. 1, pp. 87-96, 1986.

[9] K. T. Atanassov and G. Gargov, Interval valued intuitionistic fuzzy sets, Fuzzy Sets and Systems, vol. 31, no. 3, pp. 343-349, 1989.

[10] F. Smarandache, Neutrosophy: Neutrosophic Probability, Set and Logic. Rehoboth, DE: American Research Press, 1999.

[11] F. Smarandache, A Unifying Field in Logics: Neutrosophic Logic: Neutrosophy, Neutrosophic Set and Neutrosophic Probability. Rehoboth, DE: American Research Press, 2003.

[12] H. Wang, F. Smarandache, Y. Zhang, and R. Sunderraman, Single valued neutrosophic sets, Multi-space Multistruction, no. 4, pp. 410-413, 2010.

[13] H. Wang, F. Smarandache, Y. Zhang, and R. Sunderraman, Interval Neutrosophic Sets and Logic: Theory and Applications in Computing. Phoenix, AZ: Hexis, 2005.

[14] J. Ye, Vector similarity measures of simplified neutrosophic sets and their application in multi criteria decision making, International Journal of Fuzzy Systems, vol. 16, no. 2, pp. 204-211, 2014.

[15] C. Li and, S. Luo, The weighted distance measure based method to neutrosophic multi-attribute group decision making, Mathematical Problems in Engineering, vol. 2016, and article ID 3145341, 2016.

[16] P. Liu and L. Shi, The generalized hybrid weighted average operator based on interval neutrosophic hesitant set and its application to multiple attribute decision making," Neural Computing and Applications, vol. 26, no. 2, pp. 457-471, 2015.

[17] D. Molodtsov. Soft set theory - first results. Comput. Math. Appl. 37(4–5), 19–31 (1999)

[18] P.K. Maji, Biswas, R., Roy, A.R.: Soft set theory. Comput. Math. Appl. 45(4–5), 555–562 (2003)

[19] P.K. Maji., Biswas, R., Roy, A.R.: Fuzzy soft sets. J. Fuzzy Math. 9(3), 589–602 (2001)

[20] F. Wang., Li, X., Chen, X.: Hesitant fuzzy soft set and its applications in multi criteria decision making. J. Appl. Math. 643785, 10 (2014)

[21] D. Pei, D. Miao.: From soft sets to information systems. IEEE International Conference on Granular Computing, 2,617-621((2005)

[22]. S.J. John, Soft Sets. In: Soft Sets. Studies in Fuzziness and Soft Computing, vol 400. Springer, Cham (2021). https://doi.org/10.1007/978-3-030-57654-7_1

[23]. S.J. John, Topological Structures of Soft Sets. In: Soft Sets. Studies in Fuzziness and Soft Computing, vol 400. Springer, Cham (2021). https://doi.org/10.1007/978-3-030-57654-7_3

[24]. S.J. John, Hybrid Structures Involving Soft Sets. In: Soft Sets. Studies in Fuzziness and Soft Computing, vol 400. Springer, Cham (2021). https://doi.org/10.1007/978-3-030-57654-7_5

[25] T. M. Al-sham, D. Ljubisa and R. Kocinac, Nearly soft menger spaces, Journal of Mathematics, pp.1-9, 2020.

[26] T. M. Al-sham, New Soft Structure, Infra soft topological spaces, Mathematical Problems in Engineering, pp. 1-12,2021.

[27] T. M. Al-sham, E. A. Tabl and B. A. Asaad, Weak forms of soft separation axioms and fixed soft points, Fuzzy Information & Engineering, vol.12, no.4, pp.509-528, 2020.

[28] T. M. Al-sham, A. El-Sayed and A. Tabi, Connectedness and local connectedness on infra soft topological spaces, Mathematics, vol. 9, pp.1-15, 2021.

[29] T. M. Al-shami, Homeomorphism and quotient mappings in infra soft topological spaces, Journal of Mathematics, pp.1-10, 2021.

[30] T. M. Al-sham, Infra soft compact spaces and application to fixed point theorem, Journal of Function Spaces, pp.1-9, 2021.

[31] B. A. Asaad, T. M. Al-shami and A. Mhemdi, Bioperators on soft topological spaces, AIMS Mathematics, vol.6, no.11, pp.12471-12490, 2021.

[32] T. Y. Ozturk, C. G. Aras and S. Bayramov, A new approach to operations on neutrosophic soft sets and to neutrosophic soft topological spaces, Communications in Mathematics & Applications, vol.10, no.3, pp.481- 493, 2019.

[33] T. Y. Ozturk, E. Karatas, A. Yolcu, On neutrosophic soft continuous mappings, Turkish Journal of Mathematics, vol. 45,no.1,pp.81-95, 2021.

[34] C. Gunduz, T. Y. Ozturk, S. Bayramov, Separation axioms on neutrosophic soft topological spaces, Turkish Journal of Mathematics, vol. 43, no. 1, pp. 498-510, 2019.

[35] T. Y. Ozturk, Some structures on neutrosophic topological spaces, Applied Mathematics & Nonlinear Sciences, vol.6, no.1, pp.467- 478, 2021.

[36] A. Mehmood, M. Aslam, M. I, Khan, H. Qureshi, C. Park and et al., A new attempt to neutrosophic soft bi-topological spaces, Computer Modeling in Engineering and Sciences, vol.10, no.3, pp.1565-1585, 2022.

[37] I. Deli and S. Broumi, Neutrosophic soft relations and some properties, Annals of Fuzzy Mathematics & Informatics, vol. 9, no.1, pp. 169-182, 2015.

[38] T. Bera and N. K. Mahapatra, Introduction to neutrosophic soft topological space, Opsearch, vol. 54, no. 4, pp.841-867, 2017.

[39] P. K. Maji, Neutrosophic soft set, Annals of Fuzzy Mathematics and Informatics, vol 5, no.1, pp. 157-168, 2013.

[40] A. Mehmood, S. Abdullah, M. M. Al-Shomrani, M.M.I. Khan, O. Thinnukool, Some results in neutrosophic soft open sets, Journal of Function Spaces, vol. 9, no.1, pp. 1-15, 2021.

[41] A. Mehmood, F. Afzal, S. Abdullah, M.I. Khan and S. Gul, An absolutely new attempt to vague soft bi-topological spaces basis at newly defined operations regarding vague soft points, Mathematical Problems in Engineering 2021, Article ID 440854, 11.

[42] A. Mehmood, S. Abdullah and C. Park, A new approach to vague soft bi-topological spaces, Computer Modeling in Engineering & Sciences 131 (2021), 411–428.

[43] A. Mehmood, S.A. Ghour, M. Ishfaq and F. Afzal, A look at soft neutrosophic spaces, J Intell Fuzzy Syst (2021), 6473–6494

[44] A. Mehmood, F. Afzal, M.I. Khan, S. Abdullah and S. Gul, Neutrosophic soft quad structures concerning neutrosophic soft points, Mathematical Problems in Engineering 2022, Article ID 8483632, 12.

[45] A. Mehmood, S.A. Ghour, F. Afzal, G. Nordo, N. Saleem, Comprehensive note on characterization of neutrosophic soft p-open sets in neutrosophic soft quad-topological space, J Intell Fuzzy Syst (2022), 1519–1540.

[46] P. Arulpandy , M. Trinita Pricilla, Bipolar neutrosophic soft contra generalized pre-continuous and contra generalized α-continuous mappings, International Journal of Neutrosophic Science, Vol. 18 , No. 1 , (2022) : 144-157 (Doi : https://doi.org/10.54216/IJNS.180112)

[47] M . Mohan, M .Krishnaswamy, K-algebras on quadripartitioned single valued neutrosophic sets. Journal of Fuzzy Extension and Applications, 1(4), (2020): 325-339. doi: 10.22105/jfea.2020.254945.1026

[48] M. Abobala, A study of maximal and minimal ideals of n-refined neutrosophic rings. Journal of Fuzzy Extension and Applications, 2(1), (2021): 16-22. doi: 10.22105/jfea.2021.270647.1072

[49] V. Kandasamy , I. Kandasamy , F. Smarandache, Neutro Algebra of Substructures of the Semigroups built using Zn and Z+, International Journal of Neutrosophic Science, Vol. 18 , No. 3 , (2022) : 135-156 (Doi : https://doi.org/10.54216/IJNS.1803012)

[50] F. Smarandache, Extension of Soft Set to Hyper soft Set, and then to Plithogenic Hyper soft Set, Neutrosophic Sets and Systems, vol. 22, 2018, pp. 168-170. DOI: 10.5281/zenodo.2159754 http://fs.unm.edu/NSS/ExtensionOfSoftSetToHypersoftSet.

[51] F. Smarandache, Extension of Soft Set to Hyper soft Set, and then to Plithogenic Hyper soft Set (revisited), Octogon Mathematical Magazine, vol. 27, no. 1, April 2019, pp. 413-418.

[52] A Vadivel, M. Seenivasan, C. J. Sundar, An Introduction to δ-open sets in a Neutrosophic Topological Spaces, Journal of Physics: Conference Series , 1724 (2021) 012011, doi:10.1088/1742-6596/1724/1/012011.

[53] A. Acikgoz, F. Esenbel, Neutrosophic Soft δ−Topology and Neutrosophic Soft Compactness, AIP Conference Proceedings 2183, 030002 (2019); https://doi.org/10.1063/1.5136106 Published Online: 06, 2019

[54] Singh, P.K, Turiyam set a fourth dimension data representation, Journal of Applied Mathematics and Physics, 9 (2021) 1821-1828. https://doi.org/10.4236/jamp.2021.97116

[55] Singh, P.K, Data with Turiyam Set for Fourth Dimension Quantum Information Processing, Journal of Neutrosophic and Fuzzy Systems, Vol.1, No. 01, PP. 09-23, 2021

[56] C. Granados, A note on Anti Geometry and Neutro Geometry and their application to real life, Neutrosophic Sets and Systems, Vol. 49, 2022.

[57] Hayat, K., Ali, M.I., Karaaslan, F. B.Y. Cao, M. shah. Design concept evaluation using soft sets based on acceptable and satisfactory levels: an integrated TOPSIS and Shannon entropy. Soft Comput **24**, 2229–2263 (2020). https://doi.org/10.1007/s00500-019-04055-7.

[58] Hayat. K, Y. Bing. Y. Cao, A.I. Muhammad, K. Faruk, Q. Zejian, Characterizations of Certain Types of Type 2 Soft Graphs, Discrete Dynamics in Nature and Society, vol. 2018, Article ID 8535703, 15 pages, 2018. https://doi.org/10.1155/2018/8535703

Received: June 28, 2024. Accepted: Oct 11, 2024

Neutrosophic Sets and Systems, Vol. 75, 2025
University of New Mexico

Deciphering the Geometric Bonferroni Mean Operator in Pythagorean Neutrosophic Sets Framework

Mohammad Shafiq bin Mohammad Kamari [1], Zahari Bin Md. Rodzi [2,3,*], R.H. Al-Obaidi[4], Faisal Al-Sharqi[5], Ashraf Al-Quran[6], Rawan A. shlaka[7]

[1] College of Computing, Informatics and Mathematics Studies, MARA University of Technology Negeri Sembilan Branch, Seremban Campus, Seremban, Negeri Sembilan, 70300, Malaysia; shafiqkamari922@gmail.com
[2] College of Computing, Informatics and Mathematics Studies, MARA University of Technology Negeri Sembilan Branch, Seremban Campus, Seremban, Negeri Sembilan, 70300, Malaysia; zahari@uitm.edu.my
[3] Faculty of Science and Technology, National University of Malaysia, Bandar Baru Bangi, Selangor, 43600, Malaysia; zahari@uitm.edu.my
[4] Fuel and Energy Techniques Engineering Department, College of Engineering and Technologies, Al-mustaqbal University, 51001, Babylon, Iraq, raidh.h.salman@uomus.edu.iq
[5] Department of Mathematics, Faculty of Education for Pure Sciences, University of Anbar, Ramadi, Anbar, Iraq, faisal.ghazi@uoanbar.edu.iq
[6] Department of Basic Sciences, Preparatory Year, King Faisal University, Al-Ahsa 31982, Saudi Arabia, aalquran@kfu.edu.sa
[7] College of Pharmacy, National University of Science and Technology, Dhi Qar, Iraq, rawan-a.shlaka@nust.edu.iq

* Correspondence: zahari@uitm.edu.my

Abstract: The Geometric Bonferroni Mean (GBM), is an extension of The Bonferroni mean (BM), that combines both BM and the geometric mean, allowing for the representation of correlations among the combined factors while acknowledging the inherent uncertainty within the decision-making process. Within the framework of Pythagorean neutrosophic set (PNS) that encompasses truth, indeterminacy, and falsity-membership degrees, each criterion can be integrated into a unified PNS value, portraying the overall evaluation of that criterion by employing the Geometric Bonferroni mean. This study aims to enhance decision-making in Pythagorean neutrosophic framework by introducing an aggregation operator to PNS using the Geometric Bonferroni Mean. Additionally, it proposes a normalized approach to resolve decision-making quandaries within the realm of PNS, striving for improved solutions. The novel Pythagorean Neutrosophic Normalized Weighted Geometric Bonferroni Mean (PNNWGBM) aggregating operator has been tested in a case of multi-criteria decision-making (MCDM) problem involving the selection of Halal products suppliers with several criteria. The result shows that this aggregating operator is offering dependable and pragmatic method for intricate decision-making challenges and able to effectively tackle uncertainty and ambiguity in MCDM problem.

Keywords: aggregating operator; Bonferroni Mean (BM); Geometric Bonferroni Mean (GBM); Pythagorean neutrosophic set (PNS); multi-criteria decision-making (MCDM).

1. Introduction

Zadeh [1] proposed the concept of fuzzy sets as a category of entities characterized by a spectrum of membership grades. He broadened the concepts of inclusion, union, intersection, complement, relation, convexity, and others to apply to these sets, and elucidated several properties associated with these concepts within the framework of fuzzy sets. Intuitionistic Fuzzy Sets (IFS), first

introduced by Atanassov [2], assigns both membership and non-membership functions to elements of a universe, ensuring their combined sum is less than or equal to one. Consequently, IFS offers a more precise and definitive description compared to fuzzy sets. However, it's limited to handling incomplete and uncertain information, unable to address the indeterminate and inconsistent information frequently encountered in real-world scenarios. Hence, as an expansion of fuzzy sets and IFS, Smarandache introduced neutrosophic sets (NS) in 1995 and published his results in 1998 [3]. Smarandache's definition outlines that a NS, denoted as A within a universal set X, is distinguished by three distinct functions: a truth-membership function, denoted as $T_A(x)$, an indeterminacy-membership function, denoted as $I_A(x)$, and a falsity-membership function, denoted as $F_A(x)$. The main strength of the neutrosophic set lies in its enhancement of fuzzy set theory by integrating membership, non-membership, and indeterminacy parameters, which are crucial for effectively managing uncertainty in the decision-making process. Smarandache [4] and H. Wang et al. [5] additionally introduced the concept of a single-valued neutrosophic set (SVNS) through adjustments to the established conditions such that $T_A(x)$, $I_A(x)$ and $F_A(x) \in [0,1]$ and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$, which are better suited for addressing scientific and engineering problems.

Throughout the years, numerous extensions of neutrosophic sets have been developed by other researchers because of their broad range of descriptive scenarios frequently encountered in various real-life situations. Example of neutrosophic set extensions are interval neutrosophic set [6], simplified neutrosophic set [7], neutrosophic soft set [8], multi-valued neutrosophic set [9] and rough neutrosophic set [10]. Over the past decade, a wealth of intriguing studies on neutrosophic sets has emerged across diverse domains within multi-criteria decision-making (MCDM), demonstrating their relevance and impact [11]-[17].

Another novel expansion of neutrosophic set is the Pythagorean neutrosophic sets (PNS). Jansi et al. [18] extends the theory of correlation coefficient from neutrosophic sets (NS) to Pythagorean neutrosophic sets (PNS), where 'T' and 'F' represent dependent neutrosophic components. This extension relaxes the constraint condition requiring the square sum of membership, non-membership, and indeterminacy to be less than two. The Pythagorean constraint helps in better modeling and representing complex situations where the interplay between truth, indeterminacy, and falsity is more intricate. This can be particularly useful in scenarios with high degrees of uncertainty or where traditional neutrosophic sets might be too rigid. Hence, PNS can provide more accurate and refined decision-making capabilities. They allow for more sophisticated aggregation and comparison techniques, leading to potentially better outcomes in decision-making processes involving uncertain or vague information. This makes PNS a valuable tool in fields where robust decision-making under uncertainty is essential [19].

In multi-criteria decision-making (MCDM), an aggregating operator is a mathematical function or method used to combine multiple criteria or attributes into a single composite score or decision value. The purpose of these operators is to synthesize the diverse information provided by the different criteria to facilitate decision-making. Originally introduced as an enhancement of the arithmetic mean, the Bonferroni Mean (BM) is an aggregating operator celebrated for its unique ability to factor in the significance and interplay between element pairs during aggregation [20]. The geometric mean serves as an aggregation operator in various fields, particularly in situations where multiplication or compounding of values is relevant. As an aggregation operator, it combines

multiple values into a single representative value that retains essential information from the original dataset. Xia et al. [21] introduces Geometric Bonferroni Mean (GBM), an integration of geometric mean with BM. The Geometric Bonferroni Mean (GBM) extends the concept of BM by applying geometric aggregation functions, offering a more flexible approach to handling multiplicative interactions among variables. The Geometric Bonferroni Mean (GBM) specifically aggregates multiple criteria in decision-making by capturing their interrelationships, enhancing the accuracy and robustness of evaluations, particularly in complex scenarios with interdependent attributes. Moreover, within the realm of Pythagorean fuzzy environments, extensions like the Pythagorean fuzzy GBM operator have emerged to delineate the connections between parameters and explore their unique characteristics [22]. These advancements underscore the flexibility and versatility of the GBM concept across different decision-making contexts. To increase its adaptability and usefulness, the GBM has been expanded through various modifications. For example, the introduction of weighted GBM (WGBM) operators assign weights to indicate the relative importance of certain criteria or attributes, enhancing the aggregation precision. Additionally, the GBM aggregating operator harbors vast potential for application across a multitude of domains within the realm of fuzzy and neutrosophic sets [23]-[27]. In addition, many research works have addressed the significance of Pythagorean fuzzy environments with some mathematical technicals, for example, Edalatpanah [32-34] discussed the impact of some applications of Pythagorean fuzzy in feature selection. Dirik and others [35-39] used Pythagorean fuzzy to develop decision models. Mohammed et al. [40,41] combined the above model with developed topological concepts in topological spaces. Al-sharqi et al [42] introduced the MCDM method for decision-making based on multi-mathematical structures like complex fuzzy structure [43,44], fuzzy graph structure [45,46], and some algebraic structures [47,48]. Al-Quran et al and other [49-52] proposed approaches for the selection of MCDM technology by using the fuzzy set and its extension method,

The motivation for this study arises from the capabilities of the Pythagorean neutrosophic set, which has garnered significant attention from researchers in MCDM techniques due to its impressive performance. A new aggregating operator that combines the strengths of the Geometric Bonferroni Mean with the versatile capabilities of Pythagorean neutrosophic sets would be a valuable addition to the field of multi-criteria decision-making. To the best of the authors' knowledge, such an aggregating operator has not yet been explored, thereby addressing a current gap in the research.

The main objectives of this study are (i) To develop Pythagorean Neutrosophic Geometric Bonferroni Mean (PNGBM), a novel aggregating operator that integrates PNS methodology with the classic GBM aggregating operator. (ii) To develop PNNWGBM, a normalized weighted Geometric Bonferroni Mean aggregating operator within the PNS framework. (iii) To test the applicability of the developed aggregating operator in a multi-criteria decision-making (MCDM) problem involving the selection of Halal product suppliers.

The paper is structured as follows: Section 2 introduces fundamental PNS theories. Section 3 thoroughly explains the proposed methodology. Section 4 demonstrates the application of the suggested methodology through a case study focusing on the selection of Halal products suppliers with several criteria for the decision makers. Lastly, Section 5 functions as the concluding remarks.

2. Experimental Section

2.1 Preliminaries

This section covers the fundamental theories essential for the development of PNNWGBM.

Definition 1. [20] Let $p, q \geq 0$, and a_i be a collection of positive real numbers $(i = 1,2, \dots, n)$, then BM is defined as

$$BM^{p,q}(a_1, a_2, \dots, a_n) = \left[\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_i^p a_j^q \right]^{\frac{1}{p+q}} \tag{1}$$

Definition 2. [21] Let $p, q > 0$, and a_i be a set of non-negative numbers $(i = 1,2, \dots, n)$. Then:

$$GB^{p,q}(a_1, a_2, \dots, a_n) = \frac{1}{p+q} \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (pa_i + qa_j)^{\frac{1}{n(n-1)}} \tag{2}$$

Equation (2) is called Geometric Bonferroni Mean (GBM). The GBM possesses the following characteristics:

1. $GB^{p,q}(0,0, \dots, 0) = 0$.
2. $GB^{p,q}(a, a, \dots, a) = a \; if \; a_i = a$, for all i.
3. $GB^{p,q}(a_1, a_2, \dots, a_n) \geq GB^{p,q}(d_1, d_2, \dots, d_n), i.e., GB^{p,q}$ is monotonic, if $a_i \geq d_i$, for all i.
4. $min_i\{a_i\} \leq GB^{p,q}(a_1, a_2, \dots, a_n) \leq max_i\{a_i\}$.

Definition 3. [28] Let $p, q > 0$, and a_i be a set of non-negative numbers $(i = 1,2, \dots, n)$. w_i indicates the importance degree of a_i, satisfying $w_i > 0$ $(i = 1,2, \dots, n)$, $\sum_{j=1}^{n} \omega_i = 1$, for $j \neq i$, v_{ij} indicates the importance degree of a_i for a_j, satisfying $v_{ij} > 0$ $(j \neq i, j = 1,2, \dots, n)$, $\sum_{j=1}^{n} v_{ij} = 1$. Then we call

$$WGBM^{p,q}(a_1, a_2, \dots, a_n) = \frac{1}{p+q} \prod_{i=1}^{n} \left(pa_i + \prod_{j \neq i}^{n} (qa_j)^{v_{ij}} \right)^{w_i} \tag{3}$$

the weighted Geometric Bonferroni Mean (WGBM).

Definition 4. [29] Let X be a non-empty set or a universe. We define a Pythagorean fuzzy set A as

$$A = \{(x, \tau_A(x), \eta_A(x)) | x \in X\} \tag{4}$$

Where $\tau_A(x), \eta_A(x) \in [0,1]$ indicate the truth membership and false membership respectively for each element $x \in X$ to the set A, and $0 \leq {\tau_A}^2(x) + {\eta_A}^2(x) \leq 1$ for each $x \in X$. The indeterminacy membership is given by $\xi_A(x) = \sqrt{1 - {\tau_A}^2(x) - {\eta_A}^2(x)}$.

Definition 5. [5] Let X be a universe or non-empty set. A single valued neutrosophic set β in X is given by:

$$\beta = \{(x, \tau_\beta(x), \xi_\beta(x), \eta_\beta(x)) | x \in X\} \tag{5}$$

Where $\tau_\beta(x), \xi_\beta(x), \eta_\beta(x) \in [0,1]$ and with no limitations on the sum of the components where $0 \leq \tau_\beta(x) + \xi_\beta(x) + \eta_\beta(x) \leq 3$.

Definition 6. [30] Let X be a universe or non-empty set. A Pythagorean neutrosophic set with $\tau_A(x)$ and $\eta_A(x)$ are dependent neutrosophic components that is given by:

$$A = \{(x, \tau_A(x), \xi_A(x), \eta_A(x)) | x \in X\} \tag{6}$$

Where τ_A represent the degree of membership, ξ_A represent the degree of indeterminacy and η_A represent the degree of non-membership respectively such that $\tau_A(x), \xi_A(x), \eta_A(x) \in [0,1]$ and satisfying

$$0 \leq (\tau_A(x))^2 + (\eta_A(x))^2 \leq 1 \tag{7}$$

$$0 \leq (\tau_A(x))^2 + (\xi_A(x))^2 + (\eta_A(x))^2 \leq 2 \tag{8}$$

Definition 7. [31] Let $x_1 = (\tau_{x_1}, \xi_{x_1}, \eta_{x_1})$, $x_2 = (\tau_{x_2}, \xi_{x_2}, \eta_{x_2})$ and $x = (\tau_x, \xi_x, \eta_x)$ are any two PNSs, then the following definitions apply to the operating rules for PNSs, which include addition, multiplication, scalar multiplication, and power operations:

i. $x_1 \oplus x_2 = \left(\sqrt{{\tau_{x_1}}^2 + {\tau_{x_2}}^2 - {\tau_{x_1}}^2 {\tau_{x_2}}^2}, \xi_{x_1}\xi_{x_2}, \eta_{x_1}\eta_{x_2}\right)$ (9)

ii. $x_1 \otimes x_2 = \left(\tau_{x_1}\tau_{x_2}, \xi_{x_1} + \xi_{x_2} - \xi_{x_1}\xi_{x_2}\sqrt{{\eta_{x_1}}^2 + {\eta_{x_2}}^2 - {\eta_{x_1}}^2 {\eta_{x_2}}^2}\right)$ (10)

iii. $\mu x = \left(\sqrt{1 - (1 - {\tau_x}^2)^\mu}, {\xi_x}^\mu, {\eta_x}^\mu\right)$ where $\mu \in \Re$ and $\mu \geq 0$ (11)

iv. $x^\mu = \left({\tau_x}^\mu, 1 - (1 - \xi_x)^\mu, \sqrt{1 - (1 - {\eta_x}^2)^\mu}\right)$ where $\mu \in \Re$ and $\mu \geq 0$. (12)

2.2 Proposed Method

In this section, our goal is to enhance the capabilities of the GBM operator to accommodate situations where PNS are employed as input parameters. Therefore, our study involves implementing the PNGBM operator within the Pythagorean neutrosophic framework.

Definition 8. Let $p, q \geq 0$ and $x_i = (\tau_i(x), \xi_i(x), \eta_i(x))$ be a set of non-negative numbers $(i = 1,2, \dots, n)$. Then:

$$PNGBM^{p,q}(x_1, x_2, \dots, x_n) = \frac{1}{p+q}\left(\bigotimes_{\substack{i,j=1 \\ i \neq j}}^{n} \left(px_i \oplus qx_j\right)^{\frac{1}{n(n-1)}}\right) \tag{13}$$

Equation (13) was formed based on the operating rules of PNS as mentioned in Definition 7.

Proposition 1. Let $p, q \geq 0$ with $x_i = (\tau_i(x), \xi_i(x), \eta_i(x))$ where a PNS set consist of $(i = 1,2,3, \dots, n)$. For any $i, j \; and \; i \neq j$:

$$px_i \oplus qx_j = \left(\sqrt{1-(1-{\tau_i}^2)^p\left(1-{\tau_j}^2\right)^q}, {\xi_i}^p \, {\xi_j}^q, {\eta_i}^p {\eta_j}^q\right) \tag{14}$$

Proof Let two PNS sets $x_i = (\tau_i(x), \xi_i(x), \eta_i(x))$ and $x_j = (\tau_j(x), \xi_j(x), \eta_j(x))$.
Using the operating rules (iii) in Definition 7, we have

$$px_i = \left(\sqrt{1-(1-{\tau_i}^2)^p}, {\xi_i}^p, {\eta_i}^p\right)$$

$$qx_j = \left(\sqrt{1-\left(1-{\tau_j}^2\right)^q}, {\xi_j}^q, {\eta_j}^q\right)$$

By commencing the operating rules (i) based on Definition 7, we get:

$$px_i \oplus qx_j = \left(\sqrt{1-(1-{\tau_i}^2)^p}, {\xi_i}^p, {\eta_i}^p\right) \oplus \left(\sqrt{1-\left(1-{\tau_j}^2\right)^q}, {\xi_j}^q, {\eta_j}^q\right)$$

$$= \left(\sqrt{1-(1-{\tau_i}^2)^p + 1 - \left(1-{\tau_j}^2\right)^q - (1-(1-{\tau_i}^2)^p)\left(1-\left(1-{\tau_j}^2\right)^q\right)}, {\xi_i}^p \, {\xi_j}^q, {\eta_i}^p {\eta_j}^q\right)$$

$$= \left(\sqrt{1-(1-{\tau_i}^2)^p\left(1-{\tau_j}^2\right)^q}, {\xi_i}^p \, {\xi_j}^q, {\eta_i}^p {\eta_j}^q\right)$$

Thus, Proposition 1 holds.

Proposition 2. Let $p, q \geq 0$ and $x_i = (\tau_i(x), \xi_i(x), \eta_i(x))$ where a PNS set consist of $(i = 1,2,3, \dots, n)$. For any $i, j \; and \; i \neq j$:

$$\left(px_i \oplus qx_j\right) \otimes \left(px_j \oplus qx_i\right) = \begin{pmatrix} \sqrt{\left(1-(1-{\tau_i}^2)^p\left(1-{\tau_j}^2\right)^q\right)\left(1-\left(1-{\tau_j}^2\right)^p(1-{\tau_i}^2)^q\right)}, \\ 1-\left(1-{\xi_i}^p \, {\xi_j}^q\right)\left(1-{\xi_j}^p \, {\xi_i}^q\right), \\ \sqrt{1-\left(1-\left({\eta_i}^{2p}{\eta_j}^{2q}\right)\right)\left(1-\left({\eta_j}^{2p}{\eta_i}^{2q}\right)\right)} \end{pmatrix} \tag{15}$$

Proof By referring to the result from Proposition 1, we get:

$$px_i \oplus qx_j = \left(\sqrt{1-(1-{\tau_i}^2)^p\left(1-{\tau_j}^2\right)^q}, {\xi_i}^p \, {\xi_j}^q, {\eta_i}^p {\eta_j}^q\right) \text{ and}$$

$$px_j \oplus qx_i = \left(\sqrt{1-\left(1-{\tau_j}^2\right)^p(1-{\tau_i}^2)^q}, {\xi_j}^p \, {\xi_i}^q, {\eta_j}^p {\eta_i}^q\right)$$

By commencing the operating rules (ii) based on Definition 7, we get:

$$(px_i \oplus qx_j)\otimes(px_j \oplus qx_i) = \begin{pmatrix} \sqrt{1-(1-{\tau_i}^2)^p\left(1-{\tau_j}^2\right)^q}\sqrt{1-\left(1-{\tau_j}^2\right)^p(1-{\tau_i}^2)^q}, \\ {\xi_i}^p\,{\xi_j}^q+{\xi_j}^p\,{\xi_i}^q-\left({\xi_i}^p\,{\xi_j}^q\right)\left({\xi_j}^p\,{\xi_i}^q\right), \\ \sqrt{\left({\eta_i}^p{\eta_j}^q\right)^2+\left({\eta_j}^p{\eta_i}^q\right)^2-\left({\eta_i}^p{\eta_j}^q\right)^2\left({\eta_j}^p{\eta_i}^q\right)^2} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\left(1-(1-{\tau_i}^2)^p\left(1-{\tau_j}^2\right)^q\right)\left(1-\left(1-{\tau_j}^2\right)^p(1-{\tau_i}^2)^q\right)}, \\ 1-\left(1-{\xi_i}^p\,{\xi_j}^q\right)\left(1-{\xi_j}^p\,{\xi_i}^q\right), \\ \sqrt{1-\left(1-\left({\eta_i}^p{\eta_j}^q\right)^2\right)\left(1-\left({\eta_j}^p{\eta_i}^q\right)^2\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\left(1-(1-{\tau_i}^2)^p\left(1-{\tau_j}^2\right)^q\right)\left(1-\left(1-{\tau_j}^2\right)^p(1-{\tau_i}^2)^q\right)}, \\ 1-\left(1-{\xi_i}^p\,{\xi_j}^q\right)\left(1-{\xi_j}^p\,{\xi_i}^q\right), \\ \sqrt{1-\left(1-\left({\eta_i}^{2p}{\eta_j}^{2q}\right)\right)\left(1-\left({\eta_j}^{2p}{\eta_i}^{2q}\right)\right)} \end{pmatrix}$$

Therefore, Proposition 2 is valid.

Proposition 3. Let $p, q \geq 0$ and $x_i = (\tau_i(x), \xi_i(x), \eta_i(x))$ where a PNS set consist of $(i = 1,2,3,\dots,n)$. Provided the value of f at which $1 \leq f < n$, we have:

$$\bigotimes_{i=1}^{f}(px_i \oplus qx_{f+1}) = \begin{pmatrix} \sqrt{\prod_{i=1}^{f}\left(1-(1-{\tau_i}^2)^p\left(1-{\tau_{f+1}}^2\right)^q\right)}, 1-\prod_{i=1}^{f}\left(1-{\xi_i}^p\,{\xi_{f+1}}^q\right), \\ \sqrt{1-\prod_{i=1}^{f}\left(1-\left({\eta_i}^{2p}{\eta_{f+1}}^{2q}\right)\right)} \end{pmatrix} \quad (16)$$

Proof Referring to Equation (14) in Proposition 1, when $f = 2$, we obtain:

$$px_i \oplus qx_{2+1} = \left(\sqrt{1-(1-{\tau_i}^2)^p(1-{\tau_{2+1}}^2)^q}, {\xi_i}^p\,{\xi_{2+1}}^q, {\eta_i}^p{\eta_{2+1}}^q\right)$$

$$= \left(\sqrt{1-(1-{\tau_i}^2)^p(1-{\tau_3}^2)^q}, {\xi_i}^p\,{\xi_3}^q, {\eta_i}^p{\eta_3}^q\right)$$

Then, $\bigotimes_{i=1}^{2}(px_i \oplus qx_3) = (px_1 \oplus qx_3)\otimes(px_2 \oplus qx_3)$

$$= \begin{pmatrix} \sqrt{(1-(1-{\tau_1}^2)^p(1-{\tau_3}^2)^q)(1-(1-{\tau_2}^2)^p(1-{\tau_3}^2)^q)}, \\ 1-(1-{\xi_1}^p\,{\xi_3}^q)(1-{\xi_2}^p\,{\xi_3}^q), \\ \sqrt{1-\left(1-({\eta_1}^{2p}{\eta_3}^{2q})\right)\left(1-({\eta_2}^{2p}{\eta_3}^{2q})\right)} \end{pmatrix}$$

By referring Equation (15), let $f = f_0$ to obtain the following general form:

$$\bigotimes_{i=1}^{f_0}\left(px_i \oplus qx_{f_0+1}\right) = \left(\begin{array}{c}\sqrt{\prod_{i=1}^{f_0}\left(1-(1-\tau_i{}^2)^p\left(1-\tau_{f_0+1}{}^2\right)^q\right)},\ 1-\prod_{i=1}^{f_0}\left(1-\xi_i{}^p\,\xi_{f_0+1}{}^q\right),\\ \sqrt{1-\prod_{i=1}^{f_0}\left(1-\left(\eta_i{}^{2p}\eta_{f_0+1}{}^{2q}\right)\right)}\end{array}\right)$$

Thus, Equation (16) as stated in Proposition 3 is applicable when $f = f_0$.

Next, let $f = f_0 + 1$

$$\bigotimes_{i=1}^{f_0+1}\left(px_i \oplus qx_{f_0+1+1}\right) = \bigotimes_{i=1}^{f_0+1}\left(px_i \oplus qx_{f_0+2}\right)$$

$$= \bigotimes_{i=1}^{f_0}\left(px_i \oplus qx_{f_0+2}\right)\otimes\left(px_{f_0+1} \oplus qx_{f_0+2}\right)$$

$$= \left(\begin{array}{c}\sqrt{\prod_{i=1}^{f_0}\left(1-(1-\tau_i{}^2)^p\left(1-\tau_{f_0+1}{}^2\right)^q\right)},\ 1-\prod_{i=1}^{f_0}\left(1-\xi_i{}^p\,\xi_{f_0+1}{}^q\right),\\ \sqrt{1-\prod_{i=1}^{f_0}\left(1-\left(\eta_i{}^{2p}\eta_{f_0+1}{}^{2q}\right)\right)}\end{array}\right)\otimes$$

$$\left(\sqrt{1-\left(1-\tau_{f_0+1}{}^2\right)^p\left(1-\tau_{f_0+2}{}^2\right)^q},\ \xi_{f_0+1}{}^p\,\xi_{f_0+2}{}^q,\ \eta_{f_0+1}{}^p\eta_{f_0+2}{}^q\right)$$

$$= \left(\begin{array}{c}\sqrt{\prod_{i=1}^{f_0}\left(1-(1-\tau_i{}^2)^p\left(1-\tau_{f_0+2}{}^2\right)^q\right)},\ 1-\prod_{i=1}^{f_0}\left(1-\xi_i{}^p\,\xi_{f_0+2}{}^q\right),\\ \sqrt{1-\prod_{i=1}^{f_0}\left(1-\left(\eta_i{}^{2p}\eta_{f_0+2}{}^{2q}\right)\right)}\end{array}\right)$$

This proof also applies to the case where $f = f_0 + 1$. Thus, Proposition 3 remains valid. Consequently, we can infer the following Proposition 4 directly from Proposition 3 as stated earlier.

Proposition 4. Let $p, q \geq 0$ and $x_i = (\tau_i(x), \xi_i(x), \eta_i(x))$ where a PNS set consist of $(i = 1,2,3,\ldots,n)$. Provided the value of f at which $1 \leq f < n$, we have:

$$\bigotimes_{j=1}^{f}\left(px_{f+1} \oplus qx_j\right) = \left(\begin{array}{c}\sqrt{\prod_{i=1}^{f}\left(1-\left(1-\tau_{f+1}{}^2\right)^p\left(1-\tau_j{}^2\right)^q\right)},\ 1-\prod_{i=1}^{f}\left(1-\xi_{f+1}{}^p\,\xi_j{}^q\right),\\ \sqrt{1-\prod_{i=1}^{f}\left(1-\left(\eta_{f+1}{}^{2p}\eta_j{}^{2q}\right)\right)}\end{array}\right) \quad (17)$$

Proposition 5. Let $p, q \geq 0$ and $x_i = (\tau_i(x), \xi_i(x), \eta_i(x))$ where a PNS set consist of $(i = 1,2,3, \dots, n)$. For any $i, j \; and \; i \neq j$:

$$\bigotimes_{\substack{i,j=1 \\ i \neq j}}^{n} \left(px_i \oplus qx_j\right) = \begin{pmatrix} \sqrt{\prod_{i=1}^{n}\left(1-(1-{\tau_i}^2)^p\left(1-{\tau_j}^2\right)^q\right)}, 1-\prod_{i=1}^{n}\left(1-{\xi_i}^p\,{\xi_j}^q\right), \\ \sqrt{1-\prod_{i=1}^{n}\left(1-\left({\eta_i}^{2p}{\eta_j}^{2q}\right)\right)} \end{pmatrix} \tag{18}$$

Proof Proposition 5 can be proven as follows by combining the results from Propositions 2 to 4. By using Equation (15) and letting $n = 2$, we get:

$$\bigotimes_{\substack{i,j=1 \\ i \neq j}}^{2} \left(px_i \oplus qx_j\right) = (px_1 \oplus qx_2) \otimes (px_2 \oplus qx_1)$$

$$= \begin{pmatrix} \sqrt{(1-(1-{\tau_1}^2)^p(1-{\tau_2}^2)^q)(1-(1-{\tau_2}^2)^p(1-{\tau_1}^2)^q)}, \\ 1-(1-{\xi_1}^p\,{\xi_2}^q)(1-{\xi_2}^p\,{\xi_1}^q), \\ \sqrt{1-\left(1-({\eta_1}^{2p}{\eta_2}^{2q})\right)\left(1-({\eta_2}^{2p}{\eta_1}^{2q})\right)} \end{pmatrix}$$

and if $n = f$, the equation becomes:

$$\bigotimes_{\substack{i,j=1 \\ i \neq j}}^{f} \left(px_i \oplus qx_j\right) = \begin{pmatrix} \sqrt{\prod_{\substack{i,j=1 \\ i \neq j}}^{f}\left(1-(1-{\tau_i}^2)^p\left(1-{\tau_j}^2\right)^q\right)}, 1-\prod_{\substack{i,j=1 \\ i \neq j}}^{f}\left(1-{\xi_i}^p\,{\xi_j}^q\right), \\ \sqrt{1-\prod_{\substack{i,j=1 \\ i \neq j}}^{f}\left(1-{\eta_i}^{2p}{\eta_j}^{2q}\right)} \end{pmatrix}$$

Next, let $n = f + 1$

$$\bigotimes_{\substack{i,j=1 \\ i \neq j}}^{f+1} \left(px_i \oplus qx_j\right) = \bigotimes_{\substack{i,j=1 \\ i \neq j}}^{f} \left(px_i \oplus qx_j\right) \bigotimes_{i=1}^{f} \left(px_i \oplus qx_{f+1}\right) \bigotimes_{j=1}^{f} \left(px_{f+1} \oplus qx_j\right)$$

where $\bigotimes_{i=1}^{f} \left(px_i \oplus qx_{f+1}\right)$ can be referred from Proposition 3, which is

$$\left(\sqrt{\prod_{i=1}^{f}\left(1-(1-{\tau_i}^2)^p\left(1-{\tau_{f+1}}^2\right)^q\right)}, 1-\prod_{i=1}^{f}\left(1-{\xi_i}^p\,{\xi_{f+1}}^q\right), \sqrt{1-\prod_{i=1}^{f}\left(1-\left({\eta_i}^{2p}{\eta_{f+1}}^{2q}\right)\right)} \right)$$

and $\bigotimes_{j=1}^{f}\left(px_{f+1}\oplus qx_j\right)$ can be referred from Proposition 4, which is

$$\left(\sqrt{\prod_{i=1}^{f}\left(1-\left(1-{\tau_{f+1}}^2\right)^p\left(1-{\tau_j}^2\right)^q\right)}, 1-\prod_{i=1}^{f}\left(1-{\xi_{f+1}}^p\,{\xi_j}^q\right), \sqrt{1-\prod_{i=1}^{f}\left(1-\left({\eta_{f+1}}^{2p}{\eta_j}^{2q}\right)\right)} \right)$$

Therefore, operations are performed for $\bigotimes_{\substack{i,j=1\\ i\neq j}}^{f+1}\left(px_i\oplus qx_j\right)$:

$$\bigotimes_{\substack{i,j=1\\ i\neq j}}^{f+1}\left(px_i\oplus qx_j\right)=\left(\sqrt{\prod_{i=1}^{f+1}\left(1-(1-{\tau_i}^2)^p\left(1-{\tau_j}^2\right)^q\right)}, 1-\prod_{i=1}^{f+1}\left(1-{\xi_i}^p\,{\xi_j}^q\right), \sqrt{1-\prod_{i=1}^{f+1}\left(1-\left({\eta_i}^{2p}{\eta_j}^{2q}\right)\right)} \right)$$

Given that Equation (18) remains applicable for $n=f+1$. Thus, Proposition 5 holds.

Proposition 6. Let $p,q\geq 0$ and $x_i=(\tau_i(x),\xi_i(x),\eta_i(x))$ where a PNS set consist of $(i=1,2,3,\dots,n)$. For any $i,j\ and\ i\neq j$:

$$\frac{1}{p+q}\left(\bigotimes_{\substack{i,j=1\\ i\neq j}}^{n}\left(px_i\oplus qx_j\right)\right)=\left(\sqrt{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-{\tau_i}^2)^p\left(1-{\tau_j}^2\right)^q\right)^{\frac{1}{p+q}}}, 1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-{\xi_i}^p\,{\xi_j}^q\right)^{\frac{1}{p+q}}, \sqrt{1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-{\eta_i}^{2p}{\eta_j}^{2q}\right)^{\frac{1}{p+q}}} \right) \tag{19}$$

Proof Referring to Proposition 5, we possess the following:

$$\bigotimes_{\substack{i,j=1 \\ i\neq j}}^{n} \left(px_i \oplus qx_j\right) = \left(\begin{array}{c} \sqrt{\prod_{i=1}^{n}\left(1-(1-\tau_i{}^2)^p\left(1-\tau_j{}^2\right)^q\right)}, 1-\prod_{i=1}^{n}\left(1-\xi_i{}^p\,\xi_j{}^q\right), \\ \sqrt{1-\prod_{i=1}^{n}\left(1-\left(\eta_i{}^{2p}\eta_j{}^{2q}\right)\right)} \end{array} \right)$$

By commencing the operating rules (iii) based on Definition 6, we get:

$$\frac{1}{p+q}\left(\bigotimes_{\substack{i,j=1 \\ i\neq j}}^{n} \left(px_i \oplus qx_j\right)\right) = \left(\begin{array}{c} \sqrt{\prod_{\substack{i,j=1 \\ i\neq j}}^{n}\left(1-(1-\tau_i{}^2)^p\left(1-\tau_j{}^2\right)^q\right)^{\frac{1}{p+q}}}, \\ 1-\prod_{\substack{i,j=1 \\ i\neq j}}^{n}\left(1-\xi_i{}^p\,\xi_j{}^q\right)^{\frac{1}{p+q}}, \\ \sqrt{1-\prod_{\substack{i,j=1 \\ i\neq j}}^{n}\left(1-\eta_i{}^{2p}\eta_j{}^{2q}\right)^{\frac{1}{p+q}}} \end{array} \right)$$

Hence, Proposition 6 holds. Subsequently, through the application of PNGBM as outlined in Definition 8, we embark on deducing the PNNWGBM operator in the following steps:

Definition 9. Let $p,q \geq 0$ and $x_i = (\tau_i(x), \xi_i(x), \eta_i(x))$ where a PNS set consist of $(i = 1,2,3,\dots,n)$, then PNNWGBM (Pythagorean Neutrosophic Normalized Weighted Geometric Bonferroni Mean) is defined for all sets of PNS as follows:

$$PNNWGBM^{p,q}(x_1, x_2, \dots, x_n) = \left(\bigotimes_{\substack{i,j=1 \\ i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}\left(px_i \oplus qx_j\right)^{\frac{1}{n(n-1)}}\right) \tag{20}$$

where the weight vector, $W = (\omega_1, \omega_2, \omega_3, \dots, \omega_n)^T$ signifies the level of importance of x_i for $(i = 1,2,3,\dots,n)$ with the condition $\sum_{j=1}^{n}\omega_j = 1$ and $\omega_j \in [0,1]$.

Theorem 1. Let $p,q \geq 0$ and $x_i = (\tau_i(x), \xi_i(x), \eta_i(x))$ where a PNS set consist of $(i = 1,2,3,\dots,n)$, the value computed by the PNNWGBM operator in Equation (20) represents a **Pythagorean neutrosophic number** with the following components:

$$\tau_i(x) = \left(\sqrt{\prod_{\substack{i,j=1 \\ i\neq j}}^{n}\left(1-(1-\tau_i{}^2)^p\left(1-\tau_j{}^2\right)^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}\right)^{\frac{1}{n(n-1)}}, \xi_i(x) = 1-\left(\prod_{\substack{i,j=1 \\ i\neq j}}^{n}\left(1-\xi_i{}^p\,\xi_j{}^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{n(n-1)}},$$

$$and\ \eta_i(x) = \sqrt{1-\left(\prod_{\substack{i,j=1 \\ i\neq j}}^{n}\left(1-\left(\eta_i{}^{2p}\eta_j{}^{2q}\right)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{n(n-1)}}}.$$

Proof Theorem 1 can be established by using the derived equations from Propositions 1 to 6:

We possess $px_i \oplus qx_j = \left(\sqrt{1-(1-{\tau_i}^2)^p\left(1-{\tau_j}^2\right)^q}, {\xi_i}^p\,{\xi_j}^q, {\eta_i}^p{\eta_j}^q\right)$, then

$$\frac{\omega_i\omega_j}{1-\omega_i}\left(px_i \oplus qx_j\right) = \left(\sqrt{\left(1-(1-{\tau_i}^2)^p\left(1-{\tau_j}^2\right)^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}, \left({\xi_i}^p\,{\xi_j}^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}, \left({\eta_i}^p{\eta_j}^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)$$

Next, we get:

$$\bigotimes_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}\left(px_i \oplus qx_j\right) = \left(\begin{array}{c} \sqrt{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-{\tau_i}^2)^p\left(1-{\tau_j}^2\right)^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}, \\ 1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-{\xi_i}^p\,{\xi_j}^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}, \\ \sqrt{1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left({\eta_i}^{2p}{\eta_j}^{2q}\right)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}} \end{array}\right)$$

Therefore,

$$PNNWGBM^{p,q}(x_1, x_2, \ldots, x_n) = \left(\bigotimes_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}\left(px_i \oplus qx_j\right)^{\frac{1}{n(n-1)}}\right)$$

$$= \left(\begin{array}{c} \left(\sqrt{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-{\tau_i}^2)^p\left(1-{\tau_j}^2\right)^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}\right)^{\frac{1}{n(n-1)}}, \\ 1-\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-{\xi_i}^p\,{\xi_j}^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)\right)^{\frac{1}{n(n-1)}}, \\ \sqrt{1-\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left({\eta_i}^{2p}{\eta_j}^{2q}\right)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)\right)^{\frac{1}{n(n-1)}}} \end{array}\right)$$

Furthermore, we can simplify the equation to get:

$$PNNWGBM^{p,q}(x_1, x_2, \dots, x_n) = \left(\begin{array}{c} \left(\sqrt{\prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1 - \tau_i{}^2)^p \left(1 - \tau_j{}^2\right)^q\right)^{\frac{\omega_i \omega_j}{1-\omega_i}}} \right)^{\frac{1}{n(n-1)}}, \\ 1 - \left(\prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \xi_i{}^p \, \xi_j{}^q\right)^{\frac{\omega_i \omega_j}{1-\omega_i}} \right)^{\frac{1}{n(n-1)}}, \\ \sqrt{1 - \left(\prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(\eta_i{}^{2p} \eta_j{}^{2q}\right)\right)^{\frac{\omega_i \omega_j}{1-\omega_i}} \right)^{\frac{1}{n(n-1)}}} \end{array} \right) \quad (21)$$

$$where\ 0 \leq \left(\sqrt{\prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1 - \tau_i{}^2)^p \left(1 - \tau_j{}^2\right)^q\right)^{\frac{\omega_i \omega_j}{1-\omega_i}}} \right)^{\frac{1}{n(n-1)}} \leq 1,$$

$$0 \leq 1 - \left(\prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \xi_i{}^p \, \xi_j{}^q\right)^{\frac{\omega_i \omega_j}{1-\omega_i}} \right)^{\frac{1}{n(n-1)}} \leq 1 \ and\ 0 \leq \sqrt{1 - \left(\prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(\eta_i{}^{2p} \eta_j{}^{2q}\right)\right)^{\frac{\omega_i \omega_j}{1-\omega_i}} \right)^{\frac{1}{n(n-1)}}} \leq 1$$

satisfying Equation (7) and (8). Hence, this validates Theorem 1. Building on the established Theorem 1, we can additionally deduce the important properties of the $PNGBM^{p,q}$, namely reducibility, commutativity, idempotency, monotonicity, and boundedness.

Theorem 2. Reducibility: Let $\omega = \left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)$ then

$$PNNWGBM^{p,q}(x_1, x_2, \dots, x_n) = PNGBM^{p,q}(x_1, x_2, \dots, x_n) \quad (22)$$

Proof Considering $\omega = \left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)$, then in line with Definition 9, it follows that

$$\begin{aligned} PNNWGBM^{p,q}(x_1, x_2, \dots, x_n) &= \left(\bigotimes_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1 - \omega_i} \left(p x_i \oplus q x_j\right)^{\frac{1}{n(n-1)}} \right) \\ &= \left(\bigotimes_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{1}{p+q} \left(p x_i \oplus q x_j\right)^{\frac{1}{n(n-1)}} \right) \\ &= \frac{1}{p+q} \left(\bigotimes_{\substack{i,j=1 \\ i \neq j}}^{n} \left(p x_i \oplus q x_j\right)^{\frac{1}{n(n-1)}} \right) \\ &= PNGBM^{p,q}(x_1, x_2, \dots, x_n) \end{aligned}$$

thus, completing the proof of Theorem 2.

Theorem 3. Idempotency: Let $x_i = x$ where $(i = 1,2,3,\dots,n)$ then

$$PNNWGBM^{p,q}(x_1, x_2, \dots, x_n) = x \tag{23}$$

Proof Given that $x_i = x$ for every i, it follows that

$$PNNWGBM^{p,q}(x_1, x_2, \dots, x_n) = \left(\bigotimes_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} \left(p x_i \oplus q x_j \right)^{\frac{1}{n(n-1)}} \right)$$

$$= \left(\bigotimes_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} \left(px \oplus qx \right)^{\frac{1}{n(n-1)}} \right)$$

$$= \left(\bigotimes_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} \left(x(p \oplus q) \right)^{\frac{1}{n(n-1)}} \right)$$

$$= x \left(\bigotimes_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} \right)^{\frac{1}{n(n-1)}} = x$$

thus, concluding the proof of Theorem 3.

Theorem 4. Commutativity: $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$ is any permutation of Pythagorean neutrosophic numbers $(x_1, x_2, \dots, x_n)$. For any $p, q > 0$,

$$PNNWGBM^{p,q}(x_1, x_2, \dots, x_n) = PNNWGBM^{p,q}(\widetilde{x_1}, \widetilde{x_2}, \dots, \widetilde{x_n}) \tag{24}$$

Proof Let $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$ be any permutation of Pythagorean neutrosophic numbers $(x_1, x_2, \dots, x_n)$. Then

$$PNNWGBM^{p,q}(x_1, x_2, \dots, x_n) = \left(\bigotimes_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} \left(p x_i \oplus q x_j \right)^{\frac{1}{n(n-1)}} \right)$$

$$= \left(\bigotimes_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} \left(p \tilde{x}_i \oplus q \tilde{x}_j \right)^{\frac{1}{n(n-1)}} \right)$$

thus, completing the proof of Theorem 4.

Theorem 5. Monotonicity: Let $x_i = (\tau_i(x), \xi_i(x), \eta_i(x))$ where $(i = 1,2,3,\dots,n)$ and $y_i = (\tau_i(y), \xi_i(y), \eta_i(y))$ where $(i = 1,2,3,\dots,n)$ be two collections of PNSs. For any i, $\tau_i(x) \geq \tau_i(y)$, $\xi_i(x) \leq \xi_i(y)$ and $\eta_i(x) \leq \eta_i(y)$, then

$$PNNWGBM^{p,q}(x_1, x_2, \dots, x_n) \geq PNNWGBM^{p,q}(y_1, y_2, \dots, y_n) \tag{25}$$

Proof For the degree of truth, we have $\tau_i(x) \geq \tau_i(y)$ for all i and $p, q > 0$. Thus, $p\tau_i(x) \geq p\tau_i(y)$ and $q\tau_i(x) \geq q\tau_i(y)$. Following that

$$1 - p\tau_i(x)\, q\tau_j(x) \leq 1 - p\tau_i(y)\, q\tau_j(y)$$

$$\prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - p\tau_i(x)\, q\tau_j(x)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}} \leq \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - p\tau_i(y)\, q\tau_j(y)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}$$

$$\sqrt{\left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - p\tau_i(x)\, q\tau_j(x)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{p+q}}} \geq \sqrt{\left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - p\tau_i(y)\, q\tau_j(y)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{p+q}}}$$

For the degree of indeterminacy, since $\xi_i(x) \leq \xi_i(y)$ for all i and $p, q > 0$, then we get $\left(1 - p\xi_i(x)\right) \geq \left(1 - p\xi_i(y)\right)$ and $\left(1 - q\xi_j(x)\right) \geq \left(1 - q\xi_j(y)\right)$. Therefore,

$$\left(1 - p\xi_i(x)\right)\left(1 - q\xi_j(x)\right) \geq \left(1 - p\xi_i(y)\right)\left(1 - q\xi_j(y)\right)$$

$$\prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - p\xi_i(x)\right)\left(1 - q\xi_j(x)\right)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}} \leq \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - p\xi_i(y)\right)\left(1 - q\xi_j(y)\right)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}$$

$$\left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - p\xi_i(x)\right)\left(1 - q\xi_j(x)\right)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{p+q}} \geq \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - p\xi_i(y)\right)\left(1 - q\xi_j(y)\right)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{p+q}}$$

$$1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - p\xi_i(x)\right)\left(1 - q\xi_j(x)\right)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{p+q}} \leq 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - p\xi_i(y)\right)\left(1 - q\xi_j(y)\right)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{p+q}}$$

Likewise, for the degree of falsity, we can observe that:

$$\sqrt{1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - p\xi_i(x)\right)\left(1 - q\xi_j(x)\right)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{p+q}}} \leq \sqrt{1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - p\xi_i(y)\right)\left(1 - q\xi_j(y)\right)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{p+q}}}$$

Hence, considering that $\tau_i(x) \geq \tau_i(y)$, $\xi_i(x) \leq \xi_i(y)$ and $\eta_i(x) \leq \eta_i(y)$, we managed to conclude that $x \geq y$, thereby concluding the proof of Theorem 5.

Theorem 6. Boundedness: Let $x_i = (\tau_i(x), \xi_i(x), \eta_i(x))$ where a PNS set consist of $(i = 1,2,3, \dots, n)$ and $x^- = (min_i\{\tau_i\}, max_i\{\xi_i\}, max_i\{\eta_i\})$, $x^+ = (max_i\{\tau_i\}, min_i\{\xi_i\}, min_i\{\eta_i\})$, then

$$x^- \leq PNNWGBM^{p,q}(x_1, x_2, \dots, x_n) \leq x^+ \tag{26}$$

Proof Since $x_i \geq x^-$, using Theorem 3 and 5 as a basis, we obtain:

$$PNNWGBM^{p,q}(x_1, x_2, \dots, x_n) \geq PNNWGBM^{p,q}(x^-, x^-, \dots, x^-) = x^-$$

In a similar manner, we can derive:

$$PNNWGBM^{p,q}(x_1, x_2, \dots, x_n) \leq PNNWGBM^{p,q}(x^+, x^+, \dots, x^+) = x^+$$

Thus, boundedness is obtained, and Theorem 6 proof has been concluded.

3. Results and Discussion

This section may be divided by subheadings. It should provide a concise and precise description of the experimental results, their interpretation as well as the experimental conclusions that can be drawn.

3.1. The Multi-Criteria Decision-Making Method Based on PNNWGBM Operator

In this section, we introduce a Multiple Criteria Decision-Making (MCDM) problem within a Pythagorean neutrosophic setting. Let $S = \{S_1, S_2, S_3, \dots, S_n\}$ as the list of suppliers and $C = \{C_1, C_2, C_3, \dots, C_n\}$ as the list of criteria. Then ω_j is the weight assigned to criterion C_j for $j = 1,2, \dots, m$ such that $0 \leq \omega_j \leq 1$ and $\sum_{j=1}^{n} \omega_j = 1$. We propose the PNNWGBM operator to consolidate the overall criteria for each supplier into a singular, aggregated preference. The computational steps for this method are detailed below.

Step 1. Construct the decision matrix in the form of a Pythagorean neutrosophic set (PNS), denoted as $(x_{ij})_{m\times n} = (\tau_{ij}(x), \xi_{ij}(x), \eta_{ij}(x))_{m\times n}$. A direct-relation matrix, incorporating the criteria score for each supplier, is formed. Afterward, each criterion is converted to Pythagorean neutrosophic numbers. The rating scale for PNS numbers is determined by using seven linguistic scores ranging from negligible to exceptionally significant effect, employing Pythagorean neutrosophic linguistic variables, as detailed in Table 1.

Table 1. The new Pythagorean neutrosophic linguistic variable [31].

Score	Linguistic Variable	Rating Scale in Pythagorean Neutrosophic Set

1	No Effect	$\langle 0.10, 0.80, 0.90\rangle$
2	Low Effect	$\langle 0.20, 0.70, 0.80\rangle$
3	Medium Low Effect	$\langle 0.35, 0.60, 0.60\rangle$
4	Medium Effect	$\langle 0.50, 0.40, 0.45\ \rangle$
5	Medium High Effect	$\langle 0.65, 0.30, 0.25\rangle$
6	High Effect	$\langle 0.80, 0.20, 0.15\rangle$
7	Very High Effect	$\langle 0.90, 0.10, 0.10\rangle$

Step 2. Each criterion for each supplier is aggregated into a unified value by using PNNWGBM aggregating operator and the weight vector $\omega = (\omega_1, \omega_2, \omega_3, \dots, \omega_m)^n$ that represents the importance of each criterion x_i with i varying from 1 to n. The weight vector satisfies the condition $\sum_{j=1}^{n} \omega_j = 1$ and $\omega_j \in [0,1]$.

$$PNNWGBM^{p,q}\left(a_{ij}^1, a_{ij}^2, \dots, a_{ij}^m\right) = \left(\left(\sqrt{\prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(1-(1-{\tau_i}^2)^p \left(1-{\tau_j}^2\right)^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}} \right)^{\frac{1}{n(n-1)}}, \right.$$

$$1-\left(\prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(1-{\xi_i}^p\, {\xi_j}^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}} \right)^{\frac{1}{n(n-1)}}, \quad \left. \sqrt{1-\left(\prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(1-\left({\eta_i}^{2p}{\eta_j}^{2q}\right)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}} \right)^{\frac{1}{n(n-1)}}} \right)$$

Step 3. The aggregated values should adhere to the PNS number conditions outlined in Definition 6 such that

$$0 \le \left(\tau_A(x)\right)^2 + (\eta_A(x))^2 \le 1$$
$$0 \le \left(\tau_A(x)\right)^2 + \left(\xi_A(x)\right)^2 + (\eta_A(x))^2 \le 2$$

Step 4. Deneutrosophicate the PNS numbers, $(\tau_A(x), \xi_A(x), \eta_A(x))$ into a crisp value by using the following formula:

$$K = \frac{\tau_A(x) + \xi_A(x) + \eta_A(x)}{3}$$

Step 5. According to the crisp value, each supplier is ranked from highest to lowest value which represents the most ideal supplier for the decision makers to the least ideal.

3.2. Illustrative Example

To illustrate the feasibility of the proposed method, we present an example of a multi-criteria decision-making problem. A company wants to choose a Halal products supplier for their business.

Then there are four different suppliers to choose from. Each of these suppliers is going to be evaluated based on four criteria which are, (C_1) Quality of Products, (C_2) Product Variety, (C_3) Cost and Pricing, (C_4) Customer Service and Support and (C_5) Location and Delivery Options.

Step 1. By using the five criteria $C_i(i = 1,2,3,4,5)$ and four potential Halal products suppliers $S_i(i = 1,2,3,4)$, the 4×5 initial direct-relation matrix, X is obtained Table 2. These values then are transformed into Pythagorean neutrosophic numbers, which encompass the degrees of truth, indeterminacy, and falsity.

Table 2. Initial direct-relation matrix, X.

	C1	**C2**	**C3**	**C4**	**C5**
S1	6	3	1	7	7
S2	5	3	6	6	1
S3	5	4	5	6	1
S4	2	7	3	4	2

Step 2. The aggregated value was computed using the PNNWGBM operator from Equation (21) to depict the criterion selection for each supplier. The decision-makers employ a weighting vector $w = (0.25, 0.18, 0.32, 0.10, 0.15)$ and the result obtained is shown in Table 3.

Table 3. The aggregated value using PNNWGBM operator.

	PNNWGBM
S1	(0.9867,0.0091,0.055)
S2	(0.9872,0.0082,0.0417)
S3	(0.9856,0.008,0.0377)
S4	(0.9712,0.0165,0.0901)

Step 3. The aggregated PNS set has been verified and it satisfies the conditions outlined in Definition 6 where $0 \leq \left(\tau_A(x)\right)^2 + (\eta_A(x))^2 \leq 1$ and $0 \leq \left(\tau_A(x)\right)^2 + \left(\xi_A(x)\right)^2 + (\eta_A(x))^2 \leq 2$ as shown in Table 4.

Table 4. PNS number verification.

	$\boldsymbol{\tau}$	$\boldsymbol{\xi}$	$\boldsymbol{\eta}$	$\boldsymbol{\tau^2 + \eta^2}$	$\boldsymbol{\tau^2 + \xi^2 + \eta^2}$
S1	0.98673	0.00909	0.05501	0.97666	0.97674
S2	0.98718	0.00823	0.04171	0.97626	0.97633
S3	0.98557	0.00795	0.03768	0.97277	0.97284
S4	0.97118	0.01652	0.09007	0.9513	0.95157

Step 4. The aggregated PNS set has been deneutrosophicated into a crisp value to represent the overall criterion for each supplier.

Table 5. Crisp value.

S1	0.3503
S2	0.3457
S3	0.3437
S4	0.3593

Step 5. Table 6 presents the ranking of the suppliers according to the crisp value that has been aggregated by PNNWGBM operator. Therefore, Supplier 4 is the most recommended alternative for the company to choose.

Table 6. The rankings of suppliers.

	PNNWGBM	Rank
S1	0.3503	2
S2	0.3457	3
S3	0.3437	4
S4	0.3593	1

4. Conclusions

This paper has discussed the application of Geometric Bonferroni Mean (GBM) operator to Pythagorean neutrosophic set framework. The primary aim was to introduce and verify a novel normalized weighted Geometric Bonferroni Mean, termed PNNWGBM tailored for Pythagorean neutrosophic sets. The integration of GBM into PNS setting provides a new reliable means for decision makers in the MCDM problems, offering a nuanced approach to capturing interactions among variables in decision-making processes. The proposed PNNWGBM method exhibited promising results, validated through the illustrative example of case study pertaining to Halal products supplier selection. Moreover, this study contributes to existing literature by broadening the applications of the Geometric Bonferroni Mean operator and assessing its efficacy in Pythagorean neutrosophic settings. The findings underscore the potential of PNNWGBM to be applied to existing MCDM methodologies such as TOPSIS, AHP, PROMETHEE and DEMATEL.

The advantages of this newly developed PNNWGBM aggregating operator compared to existing operators lie in Pythagorean neutrosophic sets, which provide a more flexible framework for representing uncertainty than traditional fuzzy sets. By incorporating Pythagorean neutrosophic sets, the new operator can better capture and process degrees of truth, indeterminacy, and falsity, leading to more accurate decision-making. Moreover, the combination of PNS methodology with the GBM aggregating operator enables effective aggregation of information while considering interdependencies and correlations between different criteria, resulting in a more comprehensive assessment. Lastly, this new operator can be applied to a wide range of multi-criteria decision-making (MCDM) problems in fields such as engineering, economics, and social sciences. Its ability to handle complex and uncertain information makes it suitable for real-world applications where traditional methods may fall short.

Future research for the GBM operator in Pythagorean neutrosophic set theory encompasses algorithm development, decision-making applications, uncertainty modelling, integration with other operators, real-world applications, extensions to fuzzy and neutrosophic sets, theoretical analysis, robustness and sensitivity analysis, machine learning integration, and comparative studies. These investigations aim to enhance computational efficiency, explore practical applications, understand theoretical properties, and assess performance in uncertain environments, ultimately advancing the

utility and understanding of this operator. Overall, this study lays the groundwork for utilizing the Geometric Bonferroni Mean aggregation operator in Pythagorean neutrosophic decision-making.

The integration of the Geometric Bonferroni Mean (GBM) with Pythagorean neutrosophic set, despite its advantages, has limitations including increased computational complexity, sensitivity to parameter selection, and high data quality requirements. It can also suffer from reduced interpretability and scalability issues as the number of criteria and interactions increase. The subjectivity in defining neutrosophic membership functions introduces potential biases, and there are limited practical applications and case studies validating this approach. Additionally, validating the outcomes can be challenging due to the abstract and complex nature of the method.

Funding: We would like to acknowledge the Ministry of Higher Education Malaysia for their sponsorship of the Fundamental Research Grant Scheme (Project Code: FRGS/1/2023/STG06/UITM/02/5). This financial support has been crucial in advancing our research efforts, and we are grateful for their assistance.

Conflicts of Interest: The authors declare no conflict of interest.

References

[1] L. A. Zadeh, "Fuzzy Sets," *Information and Control 8*, pp. 338–353, 1965.

[2] K. T. Atanassov, "Intuitionistic Fuzzy Sets," *Fuzzy Sets and Systems 20*, pp. 87–96, 1986.

[3] F. Smarandache, *Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis*. 1998.

[4] F. Smarandache, *A unifying field in logics : Neutrosophic logic*. 1999.

[5] H. Wang, F. Smarandache, Y. Zhang, and R. Sunderraman, "Single Valued Neutrosophic Sets," *Infinite study*, vol. 12, no. 20110, 2010, [Online]. Available: https://www.researchgate.net/publication/262034557

[6] H. Wang, F. Smarandache, Y.-Q. Zhang, and R. Sunderraman, "Interval neutrosophic sets and logic: theory and applications in computing: Theory and applications in computing (Vol. 5). Infinite Study.

[7] J. Ye, "A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets," *Journal of Intelligent and Fuzzy Systems*, vol. 26, no. 5, pp. 2459–2466, 2014, doi: 10.3233/IFS-130916.

[8] P. K. Maji, "Neutrosophic soft set," *Annals of Fuzzy Mathematics and Informatics*, vol. 5, no. 1, pp. 157–168, 2013, [Online]. Available: http://www.afmi.or.kr@FMIhttp://www.kyungmoon.com

[9] J. Q. Wang and X. E. Li, "An application of the TODIM method with multi-valued neutrosophic set," *Control Decis*, vol. 30, pp. 1139–1142, 2015.

[10] S. Broumi, F. Smarandache, and M. Dhar, "Said Broumi, and Florentin Smarandache, Mamoni Dhar, Rough Neutrosophic Sets Rough Neutrosophic Sets," 2014.

[11] V. Uluçay, I. Deli, and M. Şahin, "Similarity measures of bipolar neutrosophic sets and their application to multiple criteria decision making," *Neural Comput Appl*, vol. 29, no. 3, pp. 739–748, Feb. 2018, doi: 10.1007/s00521-016-2479-1.

[12] V. Christianto and F. Smarandache, "A Refined Neutrosophic Components into Subcomponents with Plausible Applications to Long Term Energy Planning Predominated by Renewable Energy," *Plithogenic Logic and Computation*, vol. 1, pp. 54–60, Jan. 2024, doi: 10.61356/j.plc.2024.19060.

[13] A. Siraj, T. Fatima, D. Afzal, K. Naeem, and F. Karaaslan, "Pythagorean m-polar Fuzzy Neutrosophic Topology with Applications," *Neutrosophic Sets and Systems*, vol. 48, pp. 251–290, 2022.

[14] A. Siraj, K. Naeem, and B. Said, "Pythagorean m-polar Fuzzy Neutrosophic Metric Spaces," *Neutrosophic Sets and Systems*, vol. 53, pp. 562–579, 2023.

[15] A. Darvesh *et al.*, "Time for a New Player in Business Analytics: An MCDM Scheme Based on One-Dimensional Uncertain Linguistic Interval-Valued Neutrosophic Fuzzy Data Adil Darvesh et al., Time for a New Player in Business Analytics: An MCDM Scheme Based on One-Dimensional Uncertain Linguistic Interval-Valued Neutrosophic Fuzzy Data," 2023.

[16] V. Uluçay, "Some concepts on interval-valued refined neutrosophic sets and their applications," *J Ambient Intell Humaniz Comput*, vol. 12, no. 7, pp. 7857–7872, 2021.

[17] V. Uluçay, "Q-neutrosophic soft graphs in operations management and communication network," *Soft comput*, vol. 25, no. 13, pp. 8441–8459, 2021.

[18] R. Jansi, K. Mohana, and F. Smarandache, "Correlation Measure for Pythagorean Neutrosophic Fuzzy Sets with T and F as Dependent Neutrosophic Components," *Neutrosophic Sets and Systems*, vol. 30, 2019.

[19] R. K. Saini, A. Ahirwar, F. Smarandache, and M. Kushwaha, "Multi-Criteria Group Decision-Making with ELECTRE-III Method for Selection of Female Spouse in Pythagorean Neutrosophic Environment," *Plithogenic Logic and Computation*, vol. 1, pp. 36–53, Jan. 2024, doi: 10.61356/j.plc.2024.18960.

[20] C. Bonferroni, *Sulle medie multiple di potenze. Bollettino Dell'Unione Matematica Italiana*, vol. 5(3–4). 1950.

[21] M. Xia, Z. Xu, and B. Zhu, "Geometric Bonferroni means with their application in multi-criteria decision making," *Knowl Based Syst*, vol. 40, pp. 88–100, Mar. 2013, doi: 10.1016/j.knosys.2012.11.013.

[22] D. Liang, Zeshui Xu, and Adjei Peter Darko, "Projection Model for Fusing the Information of Pythagorean Fuzzy Multicriteria Group Decision Making Based on Geometric Bonferroni Mean," *International Journal of Intelligent Systems*, vol. 32, no. 9, pp. 966–987, 2017.

[23] Z. Xu and R. R. Yager, "Some geometric aggregation operators based on intuitionistic fuzzy sets," *Int J Gen Syst*, vol. 35, no. 4, pp. 417–433, Aug. 2006, doi: 10.1080/03081070600574353.

[24] Q. H. B. Hasnan *et al.*, "Revolutionizing Multi-Criteria Decision Making with the Triangular Fuzzy Geometry Bonferroni Mean Operator (TFGBM)," *Science and Technology Indonesia*, vol. 9, no. 1, pp. 1–6, Jan. 2024, doi: 10.26554/sti.2024.9.1.1-6.

[25] Z. M. Rodzi and A. G. Ahmad, "Application of parameterized hesitant fuzzy soft set theory in decision making," *Mathematics and Statistics*, vol. 8, no. 3, pp. 244–253, 2020, doi: 10.13189/ms.2020.080302.

[26] Z. M. Rodzi, M. N. H. B. Kamarulamirin, N. Q. B. A. Halim, and Y. B. Ya'akop, "The DEMATEL Approach to Analyzing the Factors Influencing University Students' Purchase of Smartphones," in *2021 2nd International Conference on Artificial Intelligence and Data Sciences, AiDAS 2021*, Institute of Electrical and Electronics Engineers Inc., Sep. 2021. doi: 10.1109/AiDAS53897.2021.9574256.

[27] D. Ajay, S. Broumi, and J. Aldring, "An MCDM Method under Neutrosophic Cubic Fuzzy Sets with Geometric Bonferroni Mean Operator," 2020.

[28] D. Li, W. Zeng, and J. Li, "Geometric Bonferroni Mean Operators," *International Journal of Intelligent Systems*, vol. 31, no. 12, pp. 1181–1197, Dec. 2016, doi: 10.1002/int.21822.

[29] R. R. Yager, "Pythagorean Membership Grades in Multicriteria Decision Making," *IEEE Transactions on Fuzzy Systems*, vol. 22, no. 4, pp. 958–965, 2014.

[30] R. Radha, A. Stanis, and A. Mary, "Neutrosophic Pythagorean Soft Set With T and F as Dependent Neutrosophic Components," 2021.

[31] J. N. Ismail, Z. Md Rodzi, F. Al-Sharqi, A. Al-Quran, H. Hashim, and N. H. Sulaiman, "Algebraic Operations on Pythagorean neutrosophic sets (PNS): Extending Applicability and Decision-Making Capabilities," *International Journal of Neutrosophic Science*, vol. 21, no. 4, pp. 127–134, 2023, doi: 10.54216/IJNS.210412.

[32] Edalatpanah, S. A. (2020). Neutrosophic structured element. Expert systems, 37(5), e12542.

[33] Akram, M., Ullah, I., Allahviranloo, T., & Edalatpanah, S. A. (2021). LR-type fully Pythagorean fuzzy linear programming problems with equality constraints. Journal of Intelligent & Fuzzy Systems, 41(1), 1975-1992.

[34] Radha, R., Mary, A. S. A., Broumi, S., Jafari, S., & Edalatpanah, S. A. (2023). Effectiveness on impact of COVID vaccines on correlation coefficients of pentapartitioned neutrosophic pythagorean statistics. In Cognitive Intelligence with Neutrosophic Statistics in Bioinformatics (pp. 335-356). Academic Press.

[35] Dirik, M. (2023). Fire extinguishers based on acoustic oscillations in airflow using fuzzy classification. Journal of fuzzy extension and applications, 4(3), 217-234.

[36] Smarandache, F. (2024). The dynamic interplay of opposites in Zoroastrianism. Journal of Fuzzy Extension and Applications, 5(1), 1-15.

[37] Ur Rahman A, Saeed M, Saeed MH, Zebari DA, Albahar M, Abdulkareem KH, Al-Waisy AS, Mohammed MA. A Framework for Susceptibility Analysis of Brain Tumours Based on Uncertain Analytical Cum Algorithmic Modeling. Bioengineering. 2023; 10(2):147

[38] El-Morsy, S. (2023). Stock portfolio optimization using Pythagorean fuzzy numbers. Journal of Operational and Strategic Analytics, 1(1), 8-13.

[39] Adak, A. K., Kumar, D., & Edalatpanah, S. A. (2024). Some new operations on Pythagorean fuzzy sets. Uncertainty Discourse and Applications, 1(1), 11-19.

[40] Khaled, Z. A., & Mohammed, F. M. (2023). An Introduction to M-Open Sets in Fuzzy Neutrosophic Topological Spaces. International Journal of Neutrosophic Science, (4), 36-6.

[41] Ali, T. M., & Mohammed, F. M. (2022). Some Perfectly Continuous Functions via Fuzzy Neutrosophic Topological Spaces. International Journal of Neutrosophic Science, 18(4), 174-183.

[42] F. Al-Sharqi, A. Al-Quran and Z. M. Rodzi, Multi-Attribute Group Decision-Making Based on Aggregation Operator and Score Function of Bipolar Neutrosophic Hypersoft Environment, Neutrosophic Sets and Systems, 61(1), 465-492, 2023.

[43] M. U. Romdhini, F. Al-Sharqi, A. Nawawi, A. Al-Quran and H. Rashmanlou, Signless Laplacian Energyof Interval-Valued Fuzzy Graph and its Applications, Sains Malaysiana 52(7), 2127-2137, 2023

[44] F. Al-Sharqi, Y. Al-Qudah and N. Alotaibi, Decision-making techniques based on similarity measures of possibility neutrosophic soft expert sets. Neutrosophic Sets and Systems, 55(1) (2023), 358-382.

[45] Z. bin M. Rodzi et al., "Integrated Single-Valued Neutrosophic Normalized Weighted Bonferroni Mean (SVNNWBM)-DEMATEL for Analyzing the Key Barriers to Halal Certification Adoption in Malaysia," Int. J. Neutrosophic Sci., vol. 21, no. 3, pp. 106–114, 2023.

[46] Abed, M. M., Hassan, N., & Al-Sharqi, F. (2022). On neutrosophic multiplication module. Neutrosophic Sets and Systems, 49(1), 198-208.

[47] Ashraf Al-Quran, Faisal Al-Sharqi, Zahari Md. Rodzi, Mona Aladil, Rawan A. shlaka, Mamika Ujianita Romdhini, Mohammad K. Tahat, Obadah Said Solaiman. (2023). The Algebraic Structures of Q-Complex Neutrosophic Soft Sets Associated with Groups and Subgroups. International Journal of Neutrosophic Science, 22 (1), 60-76.

[48] A. Al-Quran, F. Al-Sharqi, A. U. Rahman and Z. M. Rodzi, The q-rung orthopair fuzzy-valued neutrosophic sets: Axiomatic properties, aggregation operators and applications. AIMS Mathematics, 9(2), 5038-5070, 2024.

[49] Al-Quran, A., Hassan, N., & Alkhazaleh, S. (2019). Fuzzy parameterized complex neutrosophic soft expert set for decision under uncertainty. Symmetry, 11(3), 382.

[50] A. Al-Quran, F. Al-Sharqi, K. Ullah, M. U. Romdhini, M. Balti and M. Alomai, Bipolar fuzzy hypersoft set and its application in decision making, International Journal of Neutrosophic Science, vol. 20, no. 4, pp. 65-77, 2023.

[51] Rahman, A. U., Saeed, M., Mohammed, M. A., Abdulkareem, K. H., Nedoma, J., & Martinek, R. (2023). An innovative mathematical approach to the evaluation of susceptibility in liver disorder based on fuzzy parameterized complex fuzzy hypersoft set. Biomedical signal processing and control, 86, 105-204.

[52] Rahman, A. U., Saeed, M., Mohammed, M. A., Abdulkareem, K. H., Nedoma, J., & Martinek, R. (2023). Fppsv-NHSS: Fuzzy parameterized possibility single valued neutrosophic hypersoft set to site selection for solid waste management. Applied soft computing, 140, 110273.

Received: June 27, 2024. Accepted: Oct 12, 2024

NSS

Neutrosophic Sets and Systems, Vol. 75, 2025

University of New Mexico

Similarity -Based Pattern Recognition for Disease Symptom Extraction and Characterization

Priya Mathews[1], Lovelymol Sebastian[2] and Baiju Thankachan[3,*]

[1]Department of Mathematics, St. Thomas College Kozhencherry, Pathanamthitta, Kerala,India-689641; priyamathews91@gmail.com

[2]Department of Mathematics, MES College Nedumkandam, Idukki, Kerala, India.; lovelymaths95@gmail.com

[3]Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, India-576104.; baiju.t@manipal.edu

*Correspondence: baiju.t@manipal.edu; Tel.: +91-8217610206

Abstract. Neutrosophic Fuzzy Sets (NFS) expand upon classical fuzzy sets in the field of fuzzy set theory by including measures of truth, indeterminacy, and falsity. This paper thoroughly examines the creation and assessment of similarity measures for Single-Valued Neutrosophic Fuzzy Sets (SVNFS). The similarity measure is a crucial metric that quantifies the extent of similarity between two sets. It finds extensive application in various fields such as pattern recognition, medical diagnosis, and decision-making challenges. Nevertheless, the current similarity measures of Neutrosophic Fuzzy Sets(NFS) suffer from limited practicality and interpretation, and do not yield highly reliable outcomes. In order to tackle this issues,We provide a variety of new similarity measures, including the Hausdorff similarity measure, Membership-grade based similarity measure, and Trigonometric Hausdorff similarity measure specifically designed for Neutrosophic Fuzzy Sets(NFS). We conduct a comparison of their performance against existing measures. We verify the efficacy of these approaches by conducting thorough theoretical research and practical trials, showcasing their suitability in pattern recognition. The findings demonstrate substantial enhancements in precision and resilience, offering vital tools for academics and practitioners working with intricate and unpredictable data. The results of our research provide a foundation for future progress in the Neutrosophic Fuzzy Set theory and its practical use in several areas.

Keywords: Nuetrosophic Fuzzy Sets; Single Valued Nuetrosophic Fuzzy Sets; Hausdorff similarity measure between Nuetrosophic Fuzzy Sets; Enhanced cosine similarity measure between Nuetrosophic Fuzzy Sets

1. Introduction

The fuzzy sets (FS) introduced by Zadeh [2] [6]have a wide variety of application in different fields of study. The fuzzy set handles uncertainty and ambiguity that the classical set cannot handle. To handle uncertain, incomplete, imprecise and inconsistent information,Smarandache [7]popularized the notion of a neutrosophic set (NS) from a philosophical perspective, It expands upon the principles of classical sets and fuzzy sets. In the neutrosophic set, a truth

membership function $\mathcal{T}(x)$, an indeterminacy membership function $\mathcal{I}(x)$, and a falsity membership function $\mathcal{F}(x)$ are characterized independently, [3] where $\mathcal{T}(x)$, $\mathcal{I}(x)$, and $\mathcal{F}(x)$ are real standard or non-standard subsets of] -0,1 + [, such that $\mathcal{T} : X \rightarrow]-0, 1+[, \mathcal{I} : X \rightarrow]-0, 1+[, and \mathcal{F} : X \rightarrow]-0, 1+[$. Thus, the sum of $\mathcal{T}(x), \mathcal{I}(x), and \mathcal{F}(x)$ satisfies the condition $-0 \leq sup\mathcal{T}(x) + sup\mathcal{I}(x) + sup\mathcal{F}(x) \leq 3+$ [7].The neutrosophic set possesses the advantage of representing imprecise and conflicting information, a capability that fuzzy sets (FSs) lack in addressing imprecise and inconsistent information. The neutrosophic set possesses philosophical applications; however, its use in engineering is problematic. Consequently, the specified range of $\mathcal{T}(x)$, $\mathcal{I}(x)$, and $\mathcal{F}(x)$ can be constrained to the real standard unit interval [0, 1] for practical engineering applications [8]. Consequently, a single-valued neutrosophic set (SVNS) [9] was established as a subfamily of a neutrosophic set. S. Das et al. created the concept of a neutrosophic fuzzy set (NFS) by integrating fuzzy set (FS) with neutrosophic set (NS), resulting in the emergence of novel concepts. Due to the challenges that neutrosophic fuzzy sets (NFS) encounter in addressing certain real-world issues stemming from the non-standard intervals of neutrosophic components, the concept of single-valued neutrosophic fuzzy sets (SVNFS) has been introduced [4]. Additionally, certain set-theoretic operations, numerical illustrations, and distance-based metrics for assessing the similarity of single-valued neutrosophic fuzzy sets (SVNFS) are proposed, and their properties are derived accordingly.

Similarity measurements are crucial in the application of fuzzy sets and neutrosophic sets. The research of similarity measurement is of greatest significance. A primary challenge in fuzzy set theory and neutrosophic set theory is the formulation of distance and similarity measurements. A significant amount of effort has been dedicated to the development of distance and similarity measures for fuzzy sets [10]. A degree of resemblance between sets of single-value neutrosophic is essential for addressing multi-criteria decision-making challenges [11].Numerous research studies have established numerous metrics of resemblance between collections of single-valued neutrosophic data. This work delineated Hausdorff similarity measurements of neutrosophic fuzzy sets, membership-grade-based similarity measures, and Enhanced Cosine similarity measures, and examined their application in medical diagnostics.

This paper is organized as follows: The introductory portion comprises essential concepts and findings necessary for our research. The subsequent sections contain the definitions of the Hausdorff similarity measure, the membership grade-based similarity measure, and the enhanced cosine similarity measure pertaining to neutrosophic fuzzy sets. In the subsequent section, we demonstrate that the newly established similarity metrics fulfill their essential properties. The clinical diagnosis utilizing similarity-based pattern recognition is elucidated in a subsequent section. Concluding remarks are presented in the final section.

2. Related Works

The existing similarity measures for fuzzy sets and intuitionistic fuzzy sets constituted the initial endeavors in formulating similarity measures for neutrosophic sets. Numerous research proposed modifications and expansions of these strategies to accommodate the additional dimension of indeterminacy in neutrosophic sets.Recent studies have introduced novel similarity measures tailored for Neutrosophic Sets (NS). Ye, J. [17] proposed an enhanced cosine similarity metric that incorporates the dimensions of truth, indeterminacy, and falsity in neutrosophic numbers. This metric provides a more comprehensive comparative tool for complex datasets. Combining various similarity measurements has proven to be an effective technique for leveraging the benefits of different metrics. Peng et al. [18] introduced a hybrid similarity measure that integrates the weighted Hamming distance with the Jaccard index. This integration enhances the precision and reliability of similarity assessments within the framework of Neutrosophic Set (NS).Entropy-based metrics have been examined to tackle the intrinsic uncertainty and indeterminacy within Neutrosophic Sets (NS). The similarity measure presented by Broumi et al. [19] is based on entropy and is used to assess the level of uncertainty between sets. This approach provides a fresh perspective on measuring similarity.Geometric methodologies have been modified for Neutrosophic Set(NS) in order to quantify similarity. Yang and Singh [20] proposed a novel geometric distance-based similarity measure that takes into account the spatial distribution of neutrosophic fuzzy elements. This measure has been found to be highly effective in many image processing applications. Significant progress has been made in the use of similarity measurements in medical diagnosis. In their study, Khan et al. [21] devised a similarity measure utilizing the overlap coefficient for Neutrosophic Set(NS). This enhancement resulted in improved precision in disease detection by effectively aligning patient symptoms with corresponding medical conditions.The integration of similarity measurements with machine learning algorithms has become a promising field. In a study conducted by Li et al. [22], it was shown that the combination of similarity measures for Neutrosophic Set(NS) with clustering algorithms improved the effectiveness of clustering algorithms in intricate datasets.
Fuzzy set theory is proficient in managing uncertainty, while neutrosophic set theory proves beneficial in handling indeterminate and inconsistent data. However, if the available knowledge is both unreliable and contradictory, then neither the Fuzzy Set (FS) nor the Neutrosophic Set (NS) can effectively manage it separately. A combination of Fuzzy Set (FS) and Neutrosophic Set (NS) is necessary to accomplish the objective. This particular challenge required the extension of the fuzzy set concept within the framework of NS. In fuzzy set theory, the membership grade is represented by a specific real integer [2]. A wide range of theorems has been formulated to address the problem of ambiguous membership grades, predominantly through diverse extensions of fuzzy sets. When the degree of membership is vague and inconsistent,

it is crucial to delineate the membership grade utilizing neutrosophic components—namely, truth, indeterminacy, and falsity membership values—to accurately represent the uncertain and inconsistent information. Nonetheless, to the best of our knowledge, S. Das et al. [4] were the first to incorporate neutrosophic components with fuzzy membership grades to precisely represent an environment characterized by ambiguity and inconsistency. Furthermore, they introduced the notion of Euclidean distance and similarity metrics, along with their practical applications in decision-making [4]. Our work sought to provide more accurate similarity measures than those already available and to investigate their application in pattern recognition.

3. **Preliminaries**

This section establishes a solid framework for our study by introducing essential definitions and properties.

Definition 3.1. [2] Let X be the universal set, Then a **fuzzy set** F over X is defined by $F = \{(x, \mu_F(x))/x \in X$, where $\mu_F : X \to [0, 1]$ is called the membership function of X. The value $\mu_F(x)$ for each $x \in X$,reflects the degree to which x is a member of the fuzzy set F.

Definition 3.2. [3] Consider the universal set X and $x \in X$. A single-valued neutrosophic set (SVNS) N in X is distinguished by the function of truth membership T_N, the function of indeterminacy membership I_N and the function of falsity membership F_N. For each point x in X, $T_N(x), I_N(x), F_N(x) \in [0, 1]$. Thus, an SVNS N is denoted by $N = \{(x, T_N(x), I_N(x), F_N(x))|x \in X\}$

Definition 3.3. [4] Let X be the universal set. Then an NFS NF on X is defined by $NF = \{(x, \mu_{NF}(x), T_{NF}(x, \mu), I_{NF}(x, \mu), F_{NF}(x, \mu))/x \in X\}$ where each membership value is represented by a membership degree for truth, indeterminacy, and falsity, which are indicated as $T_{NF}(x, \mu), I_{NF}(x, \mu)$ and $F_{NF}(x, \mu)$. Furthermore, T_{NF}, I_{NF} and F_{NF} are existing standard or non-standard subsets of $]0^-, 1^+[$, That is,
$T_{NF} : E \to]0^-, 1^+[$,
$I_{NF} : E \to]0^-, 1^+[$,
$F_{NF} : E \to]0^-, 1^+[$.
where,$0^- \leq Sup(T_{NF}) + Sup(I_{NF}) + Sup(F_{NF}) \leq 3^+$.

Definition 3.4. [4] Let X be a universal set. then Single -Valued Neutrosophic Fuzzy Set SNF on X is defined by $SNF = \{(x, \mu_{SNF}(x), T_{SNF}(x, \mu), I_{SNF}(x, \mu), F_{SNF}(x, \mu))/x \in X\}$, where $T_{SNF}(x, \mu), I_{SNF}(x, \mu), F_{SNF}(x, \mu) \in [0, 1]$,and $0 \leq T_{SNF}(x, \mu)+I_{SNF}(x, \mu)+F_{SNF}(x, \mu) \leq 3$.

Definition 3.5. [1] Consider two sets $S = \{s_1, s_2, ...s_p\}$ and $T = \{t_1, t_2, ..., t_q\}$ The Hausdorff distance between S and T is denoted by $\mathcal{H}(S, T)$ and is defined as

$\mathcal{H}(S,T) = max\{h(S,T), h(T,S)\}$ where, $h(S,T) = \max\limits_{s\in S} \min\limits_{t\in T} d(s,t)$
Here, s and t are elements of S and T, respectively, and $d(s,t)$ is the metric between s and t.

Definition 3.6. [5] Consider two Nuetrosophic Sets K_1 and K_2 on X, then the Hausdorff distance between K_1 and K_2 is denoted by $d_{HNS}(K_1, K_2)$ and is defined as follows:
$d_{HNS}(K_1, K_2) = \frac{1}{n}\sum_{i=1}^{n} \max\{|T_{K_1}(x_i) - T_{K_2}(x_i)|, |I_{K_1}(x_i) - I_{K_2}(x_i)|, |F_{K_1}(x_i) - F_{K_2}(x_i)|\}$

Theorem 3.7. *[5] The Hausdorff Distance between K_1 and K_2 satisfies the following properties*

(1) $d_{HNS}(K_1, K_2) \geq 0$
(2) $d_{HNS}(K_1, K_2) = 0$ *if and only if* $K_1 = K_2$
(3) $d_{HNS}(K_1, K_2) = d_{HNS}(K_2, K_1)$
(4) *If* $K_1 \subseteq K_2 \subseteq K_3$ *then* $d_{HNS}(K_1, K_3) \geq d_{HNS}(K_1, K_2)$ *and* $d_{HNS}(K_1, K_3) \geq d_{HNS}(K_2, K_3)$

Definition 3.8. [16] Let $X = \{x_1, x_2, ..., x_n\}$ be the universe of the discourse. Consider$\kappa_1, \kappa_2 \in \mathcal{K}_{NFS}(X)$, where $\mathcal{K}_{NFS}(X)$ is the family of all neutrosophic fuzzy sets (NFSs) in X.
Let $\kappa_1 = \{(x_i, \mu_{\kappa_1}(x_i), \mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}), \mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}), \mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}))/x_i \in X\}$
$\kappa_2 = \{(x_i, \mu_{\kappa_2}(x_i), \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}), \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}), \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2}))/x_i \in X\}$
The Hausdorff distance between κ_1 and κ_2 is denoted by $d_{HNFS}(\kappa_1, \kappa_2)$ and is defined as follows.
$d_{HNFS}(\kappa_1, \kappa_2) = \sum_{i=1}^{n} \max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_2}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2})|\}$

Definition 3.9. [16] Let $X = \{x_1, x_2, ..., x_n\}$ be the universe of the discourse. Consider$K_1, K_2 \in \mathcal{K}_{NFS}(X)$, where $\mathcal{K}_{NFS}(X)$ is the family of all neutrosophic fuzzy sets (NFS) in X.
Let $K_1 = \{(x_i, \mu_{K_1}(x_i), \mathcal{T}_{K_1}(x_i, \mu_{K_1}), \mathcal{I}_{K_1}(x_i, \mu_{K_1}), \mathcal{F}_{K_1}(x_i, \mu_{K_1}))/x_i \in X\}$
$K_2 = \{(x_i, \mu_{K_2}(x_i), \mathcal{T}_{K_2}(x_i, \mu_{K_2}), \mathcal{I}_{K_2}(x_i, \mu_{K_2}), \mathcal{F}_{K_2}(x_i, \mu_{K_2}))/x_i \in X\}$
The normalized Hausdorff distance between K_1 and K_2 is denoted by $d_{NHNFS}(K_1, K_2)$ and is defined as follows.
$d_{NHNFS}(K_1, K_2) = \frac{1}{n}\sum_{i=1}^{n} \max\{|\mu_{K_1}(x_i) - \mu_{K_2}(x_i)|, |\mathcal{T}_{K_1}(x_i, \mu_{K_1}) - \mathcal{T}_K(x_i, \mu_{K_2})|, |\mathcal{I}_{K_1}(x_i, \mu_{K_1}) - \mathcal{I}_{K_2}(x_i, \mu_{K_2})|, |\mathcal{F}_{K_1}(x_i, \mu_{K_1}) - \mathcal{F}_{K_2}(x_i, \mu_{K_2})|\}$

Definition 3.10. [15] A measure of similarity between two sets of neutrosophic with a single value is a mapping $\mathcal{S} : \mathcal{N}(X) \to [0, 1]$ that satisfies the following properties:

(1) $\mathcal{S}(K_1, K_2) \in [0, 1]$

(2) $\mathcal{S}(K_1, K_2) = 1 \iff K_1 = K_2$

(3) $\mathcal{S}(K_1, K_2) = \mathcal{S}(K_2, K_1)$

(4) $K_1 \subset K_2 \subset K_3 \rightarrow \mathcal{S}(K_1, K_3) \leq \mathcal{S}(K_1, K_2)$ and $\mathcal{S}(K_1, K_3) \leq \mathcal{S}(K_2, K_3)$

$(1) - (4)$ are called Axioms of Similarity.

4. **Similarity Measure for Neutrosophic Fuzzy Sets**

A similarity measure can be used to examine differences between alternatives, making it an effective tool for multiple criteria judgment problems. Using the Hausdorff distance measure and membership grades, this section extends the similarity measure for Fuzzy Sets and Neutrosophic Sets to Neutrosophic Fuzzy Sets. In addition, a brand-new similarity metric based on the combination of Hausdorff distance and cosine similarity is suggested.

4.1. *Hausdorff Similarity Measure For Neutrosophic Fuzzy Sets*

Definition 4.1. Let $X = \{x_1, x_2, ..., x_n\}$ be the universe of discourse. Consider$\kappa_1, \kappa_2 \in \mathcal{K}_{NFS}(X)$, where $\mathcal{K}_{NFS}(X)$ is the group that includes all neutrosophic fuzzy sets (NFSs) in X.

Let $\kappa_1 = \{(x_i, \mu_{\kappa_1}(x_i), \mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}), \mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}), \mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}))/x_i \in X\}$

$\kappa_2 = \{(x_i, \mu_{\kappa_2}(x_i), \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}), \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}), \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2}))/x_i \in X\}$

The Hausdorff Similarity Measure for κ_1 and κ_2 is denoted by $\mathcal{S}_H(\kappa_1, \kappa_2)$ and is defined as follows.

$\mathcal{S}_H(\kappa_1, \kappa_2) = 1 - \frac{1}{n}\sum_{i=1}^{n} \max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_2}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2})|\}$

Theorem 4.2. *The Hausdorff Similarity Measure between two Neutrosophic Fuzzy Sets κ_1 and κ_2 has the following properties.*

(1) $\mathcal{S}_H(\kappa_1, \kappa_2) \in [0, 1]$

(2) $\mathcal{S}_H(\kappa_1, \kappa_2) = 1 \iff \kappa_1 = \kappa_2$

(3) $\mathcal{S}_H(\kappa_1, \kappa_2) = \mathcal{S}_H(\kappa_2, \kappa_1)$

(4) $\kappa_1 \subset \kappa_2 \subset \kappa_3 \rightarrow \mathcal{S}_H(\kappa_1, \kappa_3) \leq \mathcal{S}_H(\kappa_1, \kappa_2)$ *and* $\mathcal{S}_H(\kappa_1, \kappa_3) \leq \mathcal{S}_H(\kappa_2, \kappa_3)$

Proof. (1) We have $\mathcal{S}_H(\kappa_1, \kappa_2) = 1 - \frac{1}{n}\sum_{i=1}^{n} \max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_2}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2})|\}$

From the definition itself it is clear that $\mathcal{S}_H(\kappa_1, \kappa_2) \in [0, 1]$

(2) $\mathcal{S}_H(\kappa_1, \kappa_2) = 1$ implies $\frac{1}{n}\sum_{i=1}^{n} \max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_2}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2})|\} = 0$ that is,

$d_{NHNFS}(\kappa_1, \kappa_2) = 0$. Then by the properties of Hausdorff Distance between NFSs we can conclude that $\kappa_1 = \kappa_2$

(3) $\mathcal{S}_H(\kappa_1, \kappa_2) = 1 - \frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_2}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2})|\}$
$= 1 - \frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_2}(x_i) - \mu_{\kappa_1}(x_i)|, |\mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1})|, |\mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1})|, |\mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1})|\}$
$= \mathcal{S}_H(\kappa_2, \kappa_1)$

(4) Since $\kappa_1 \subseteq \kappa_2 \subseteq \kappa_3$ we have $\mu_{\kappa_1}(x_i) \leq \mu_{\kappa_2}(x_i) \leq \mu_{\kappa_3}(x_i)$,$\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) \leq \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}) \leq \mathcal{T}_{\kappa_3}(x_i, C$,$\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) \leq \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}) \leq \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3}), \mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) \geq \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2}) \geq \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})$.

First, we have to prove that $d_{HNFS}(\kappa_1, \kappa_3) \geq d_{HNFS}(\kappa_1, \kappa_2)$

Case-1

$|\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)| \geq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})| \geq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})| \geq |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

Then, $d_{HNFS}(\kappa_1, \kappa_3) = |\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|$

(a) For all $x_i \in X$

$|\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2})| \leq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|$
$|\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2})| \leq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|$
$|\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2})| \leq |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|$

(b) For all $x_i \in X$

$|\mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|$
$|\mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|$
$|\mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|$

(c) For all $x_i \in X$

$|\mu_{\kappa_1}(x_i) - \mu_{\kappa_2}(x_i)| \leq |\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|$ and $|\mu_{\kappa_2}(x_i) - \mu_{\kappa_3}(x_i)| \leq |\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|$

We can deduce from (1),(2) and (3) that, for all $x_i \in X$

$\frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_2}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2})|\} \leq$
$\frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|\}$
$\rightarrow d_{HNFS}(\kappa_1, \kappa_2) \leq d_{HNFS}(\kappa_1, \kappa_3)$

and

$\frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_2}(x_i) - \mu_{\kappa_3}(x_i)|, |\mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|\} \leq$
$\frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|\}$

$\rightarrow d_{HNFS}(\kappa_2, \kappa_3) \leq d_{HNFS}(\kappa_1, \kappa_3)$

Case-2

$|\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)| \leq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

Then, $d_{HNFS}(\kappa_1, \kappa_3) = |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

(a) For all $x_i \in X$

$|\mu_{\kappa_1}(x_i) - \mu_{\kappa_2}(x_i)| \leq |\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)| \leq |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

$|\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2})| \leq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

$|\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2})| \leq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

(b) For all $x_i \in X$

$|\mu_{\kappa_2}(x_i) - \mu_{\kappa_3}(x_i)| \leq |\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)| \leq |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

$|\mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

$|\mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

(c) For all $x_i \in X$

$|\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2})| \leq |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

and $|\mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

We can deduce from (1),(2) and (3) that, for all $x_i \in X$

$\frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_2}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2})|\} \leq \frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|\}$

$\rightarrow d_{HNFS}(\kappa_1, \kappa_2) \leq d_{HNFS}(\kappa_1, \kappa_3)$ and

$\frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_2}(x_i) - \mu_{\kappa_3}(x_i)|, |\mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|\} \leq \frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|\}$

$\rightarrow d_{HNFS}(\kappa_2, \kappa_3) \leq d_{HNFS}(\kappa_1, \kappa_3)$

Case-3

$|\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)| \leq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

Then, $d_{HNFS}(\kappa_1, \kappa_3) = |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

(a) For all $x_i \in X |\mu_{\kappa_1}(x_i) - \mu_{\kappa_2}(x_i)| \leq |\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)| \leq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|$
$|\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2})| \leq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|$
$|\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2})| \leq |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

(b) For all $x_i \in X$
$|\mu_{\kappa_2}(x_i) - \mu_{\kappa_3}(x_i)| \leq |\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)| \leq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|$
$|\mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|$
$|\mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

(c) For all $x_i \in X$
$|\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2})| \leq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|$
and $|\mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

We can deduce from (1),(2) and (3) that, for all $x_i \in X$

$\frac{1}{n}\sum_{i=1}^{n} \max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_2}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2})|\} \leq$
$\frac{1}{n}\sum_{i=1}^{n} \max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|\}$
$\rightarrow d_{HNFS}(\kappa_1, \kappa_2) \leq d_{HNFS}(\kappa_1, \kappa_3)$

and

$\frac{1}{n}\sum_{i=1}^{n} \max\{|\mu_{\kappa_2}(x_i) - \mu_{\kappa_3}(x_i)|, |\mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|\} \leq$
$\frac{1}{n}\sum_{i=1}^{n} \max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|\}$
$\rightarrow d_{HNFS}(\kappa_2, \kappa_3) \leq d_{HNFS}(\kappa_1, \kappa_3)$

Case-4

$|\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)| \leq |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

Then, $d_{HNFS}(\kappa_1, \kappa_3) = |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

(a) For all $x_i \in X |\mu_{\kappa_1}(x_i) - \mu_{\kappa_2}(x_i)| \leq |\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)| \leq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|$
$|\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2})| \leq |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|$
$|\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2})| \leq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

(b) For all $x_i \in X$ $|\mu_{\kappa_2}(x_i) - \mu_{\kappa_3}(x_i)| \leq |\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)| \leq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|$
$|\mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|$
$|\mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

(c) For all $x_i \in X$
$|\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2})| \leq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|$
and $|\mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

We can deduce from (1),(2) and (3) that, for all $x_i \in X$

$\frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_2}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2})|\} \leq$
$\frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|\}$
$\rightarrow d_{HNFS}(\kappa_1, \kappa_2) \leq d_{HNFS}(\kappa_1, \kappa_3)$
and
$\frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_2}(x_i) - \mu_{\kappa_3}(x_i)|, |\mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|\} \leq$
$\frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|\}$
$\rightarrow d_{HNFS}(\kappa_2, \kappa_3) \leq d_{HNFS}(\kappa_1, \kappa_3)$

Thus, in all possible cases, we have proved $d_{HNFS}(\kappa_1, \kappa_2) \leq d_{HNFS}(\kappa_1, \kappa_3)$ and $d_{HNFS}(\kappa_2, \kappa_3) \leq d_{HNFS}(\kappa_1, \kappa_3)$. Therefore, $1 - d_{HNFS}(\kappa_1, \kappa_2) \geq 1 - d_{HNFS}(\kappa_1, \kappa_3)$ and $1 - d_{HNFS}(\kappa_2, \kappa_3) \geq 1 - d_{HNFS}(\kappa_1, \kappa_3)$. $\rightarrow \mathcal{S}(\kappa_1, \kappa_3) \leq \mathcal{S}(\kappa_1, \kappa_2)$ and $\mathcal{S}(\kappa_1, \kappa_3) \leq \mathcal{S}(\kappa_2, \kappa_3)$ Therefore, the Hausdorff similarity measure, as defined above, is a measure of similarity between two neutrosophic fuzzy sets. □

4.2. *Membership Grade-based Similarity Measure for Neutrosophic Fuzzy Sets*

We are trying to define the similarity measure between two neptrosophic fuzzy sets based on the membership grades in this section.

Definition 4.3. Let $X = \{x_1, x_2, ..., x_n\}$ be the universe of discourse. Consider$\kappa_1, \kappa_2 \in \mathcal{K}_{NFS}(X)$, where $\mathcal{K}_{NFS}(X)$ is the group that includes all neutrosophic fuzzy sets (NFSs) in

X.

Let $\kappa_1 = \{(x_i, \mu_{\kappa_1}(x_i), \mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}), \mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}), \mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}))/x_i \in X\}$

$\kappa_2 = \{(x_i, \mu_{\kappa_2}(x_i), \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}), \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}), \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2}))/x_i \in X\}$

The Similarity Measure based on the membership grades for κ_1 and κ_2 is denoted by $\mathcal{S}_M(\kappa_1, \kappa_2)$ and is defined as follows.

$\mathcal{S}_M(\kappa_1, \kappa_2) = \frac{\sum_{i=1}^{n} min(\mu_{\kappa_1}(x_i),\mu_{\kappa_2}(x_i))+min(\mathcal{T}_{\kappa_1}(x_i),\mathcal{T}_{\kappa_2}(x_i))+min(\mathcal{I}_{\kappa_1}(x_i),\mathcal{I}_{\kappa_2}(x_i))+min(\mathcal{F}_{\kappa_1}(x_i),\mathcal{F}_{\kappa_2}(x_i))}{\sum_{i=1}^{n} max(\mu_{\kappa_1}(x_i),\mu_{\kappa_2}(x_i))+max(\mathcal{T}_{\kappa_1}(x_i),\mathcal{T}_{\kappa_2}(x_i))+max(\mathcal{I}_{\kappa_1}(x_i),\mathcal{I}_{\kappa_2}(x_i))+max(\mathcal{F}_{\kappa_1}(x_i),\mathcal{F}_{\kappa_2}(x_i))}$

Theorem 4.4. *The similarity measure based on membership grades between two neutrosophic fuzzy sets κ_1 and κ_2 has the following properties.*

(1) $\mathcal{S}_M(\kappa_1, \kappa_2) \in [0, 1]$

(2) $\mathcal{S}_M(\kappa_1, \kappa_2) = 1 \iff \kappa_1 = \kappa_2$

(3) $\mathcal{S}_M(\kappa_1, \kappa_2) = \mathcal{S}_M(\kappa_2, \kappa_1)$

(4) $\kappa_1 \subset \kappa_2 \subset \kappa_3 \rightarrow \mathcal{S}_M(\kappa_1, \kappa_3) \leq \mathcal{S}_M(\kappa_1, \kappa_2)$ *and* $\mathcal{S}_M(\kappa_1, \kappa_3) \leq \mathcal{S}_M(\kappa_2, \kappa_3)$

Proof. Properties (1) and (3) hold directly from the definition itself. We have to prove properties (2) and (4)

2. If $\kappa_1 = \kappa_2$ then by definition itself it is clear that $\mathcal{S}_M(\kappa_1, \kappa_2) = 1$

Conversely, let $\mathcal{S}_M(\kappa_1, \kappa_2) = 1$

$\rightarrow \frac{\sum_{i=1}^{n} min(\mu_{\kappa_1}(x_i),\mu_{\kappa_2}(x_i))+min(\mathcal{T}_{\kappa_1}(x_i),\mathcal{T}_{\kappa_2}(x_i))+min(\mathcal{I}_{\kappa_1}(x_i),\mathcal{I}_{\kappa_2}(x_i))+min(\mathcal{F}_{\kappa_1}(x_i),\mathcal{F}_{\kappa_2}(x_i))}{\sum_{i=1}^{n} max(\mu_{\kappa_1}(x_i),\mu_{\kappa_2}(x_i))+max(\mathcal{T}_{\kappa_1}(x_i),\mathcal{T}_{\kappa_2}(x_i))+max(\mathcal{I}_{\kappa_1}(x_i),\mathcal{I}_{\kappa_2}(x_i))+max(\mathcal{F}_{\kappa_1}(x_i),\mathcal{F}_{\kappa_2}(x_i))} = 1$

$\rightarrow \sum_{i=1}^{n} min(\mu_{\kappa_1}(x_i), \mu_{\kappa_2}(x_i)) + min(\mathcal{T}_{\kappa_1}(x_i), \mathcal{T}_{\kappa_2}(x_i)) + min(\mathcal{I}_{\kappa_1}(x_i), \mathcal{I}_{\kappa_2}(x_i)) + min(\mathcal{F}_{\kappa_1}(x_i), \mathcal{F}_{\kappa_2}(x_i)) =$
$\sum_{i=1}^{n} max(\mu_{\kappa_1}(x_i), \mu_{\kappa_2}(x_i)) + max(\mathcal{T}_{\kappa_1}(x_i), \mathcal{T}_{\kappa_2}(x_i)) + max(\mathcal{I}_{\kappa_1}(x_i), \mathcal{I}_{\kappa_2}(x_i)) + max(\mathcal{F}_{\kappa_1}(x_i), \mathcal{F}_{\kappa_2}(x_i))$

$\rightarrow \sum_{i=1}^{n} [min(\mu_{\kappa_1}(x_i), \mu_{\kappa_2}(x_i)) - max(\mu_{\kappa_1}(x_i), \mu_{\kappa_2}(x_i))] + [min(\mathcal{T}_{\kappa_1}(x_i), \mathcal{T}_{\kappa_2}(x_i)) - max(\mathcal{T}_{\kappa_1}(x_i), \mathcal{T}_{\kappa_2}(x_i))] + [min(\mathcal{I}_{\kappa_1}(x_i), \mathcal{I}_{\kappa_2}(x_i)) - max(\mathcal{I}_{\kappa_1}(x_i), \mathcal{I}_{\kappa_2}(x_i))] + [min(\mathcal{F}_{\kappa_1}(x_i), \mathcal{F}_{\kappa_2}(x_i)) - max(\mathcal{F}_{\kappa_1}(x_i), \mathcal{F}_{\kappa_2}(x_i))] = 0$

$\rightarrow [min(\mu_{\kappa_1}(x_i), \mu_{\kappa_2}(x_i)) - max(\mu_{\kappa_1}(x_i), \mu_{\kappa_2}(x_i))] = 0$
$[min(\mathcal{T}_{\kappa_1}(x_i), \mathcal{T}_{\kappa_2}(x_i)) - max(\mathcal{T}_{\kappa_1}(x_i), \mathcal{T}_{\kappa_2}(x_i))] = 0$
$[min(\mathcal{I}_{\kappa_1}(x_i), \mathcal{I}_{\kappa_2}(x_i)) - max(\mathcal{I}_{\kappa_1}(x_i), \mathcal{I}_{\kappa_2}(x_i))] = 0$
$[min(\mathcal{F}_{\kappa_1}(x_i), \mathcal{F}_{\kappa_2}(x_i)) - max(\mathcal{F}_{\kappa_1}(x_i), \mathcal{F}_{\kappa_2}(x_i))] = 0$, for each x.Therefore, $\mu_{\kappa_1}(x) = \mu_{\kappa_2}(x)$,$\mathcal{T}_{\kappa_1}(x) = \mathcal{T}_{\kappa_2}(x)$,$\mathcal{I}_{\kappa_1}(x) = \mathcal{I}_{\kappa_2}(x)$, $\mathcal{F}_{\kappa_1}(x) = \mathcal{F}_{\kappa_2}(x)$ for each x.

$\rightarrow \kappa_1 = \kappa_2$

4. Let $\kappa_1 \subset \kappa_2 \subset \kappa_3$, Then

$\mu_{\kappa_1}(x_i) \leq \mu_{\kappa_2}(x_i) \leq \mu_{\kappa_3}(x_i)$,$\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) \leq \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}) \leq \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})$,$\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) \leq \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}) \leq \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})$,

$\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) \geq \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2}) \geq \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})$.

Therefore, we have

$\mu_{\kappa_1}(x_i) + \mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) + \mathcal{I}_{\kappa_1}(x_i), \mu_{\kappa_1}) + \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2}) \geq \mu_{\kappa_1}(x_i) + \mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) + \mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) + \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})$

$\mu_{\kappa_2}(x_i) + \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}) + \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}) + \mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) \leq \mu_{\kappa_3}(x_i) + \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3}) + \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3}) + \mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1})$

$\mathcal{S}_M(\kappa_1, \kappa_2) = \frac{\sum_{i=1}^{n} \mu_{\kappa_1(x_i)} + \mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) + \mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) + \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2})}{\sum_{i=1}^{n} \mu_{\kappa_2(x_i)} + \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}) + \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}) + \mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1})}$

$\geq \frac{\sum_{i=1}^{n} \mu_{\kappa_1(x_i)} + \mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) + \mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) + \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})}{\sum_{i=1}^{n} \mu_{\kappa_3(x_i)} + \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3}) + \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3}) + \mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1})} = \mathcal{S}_M(\kappa_1, \kappa_3)$

Similarly, we have

$\mu_{\kappa_2}(x_i) + \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}) + \mathcal{I}_{\kappa_2}(x_i), \mu_{\kappa_2}) + \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3}) \geq \mu_{\kappa_1}(x_i) + \mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) + \mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) + \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})$

$\mu_{\kappa_3}(x_i) + \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3}) + \mathcal{I}_{\kappa_3}(x_i), \mu_{\kappa_3}) + \mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) \geq \mu_{\kappa_3}(x_i) + \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3}) + \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3}) + \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2})$

$\mathcal{S}_M(\kappa_2, \kappa_3) = \frac{\sum_{i=1}^{n} \mu_{\kappa_2(x_i)} + \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}) + \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}) + \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})}{\sum_{i=1}^{n} \mu_{\kappa_3(x_i)} + \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3}) + \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3}) + \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2})}$

$\geq \frac{\sum_{i=1}^{n} \mu_{\kappa_1(x_i)} + \mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) + \mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) + \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})}{\sum_{i=1}^{n} \mu_{\kappa_3(x_i)} + \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3}) + \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3}) + \mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1})} = \mathcal{S}_M(\kappa_1, \kappa_3)$

Therefore, the membership grade-based similarity measure, as defined above, is a measure of similarity between two neutrosophic fuzzy sets. □

5. **Enhanced Cosine Similarity Measure for Single Valued Neutrosophic Fuzzy Sets**

5.1. *Cosine Similarity*

Definition 5.1. Cosine similarity is a basic angle-based measure of similarity. Here, we are comparing two n-dimensional vectors for the similarity between them, expressed as a cosine. Assesses how similar things are. The comparison between two vectors is based only on their direction and omits the effect of their distance from each other. Consider the vectors $P = (p_1, p_2, ..., p_n)$ and $Q = (q_1, q_2, ..., q_n)$, the cosine similarity [12] is defined as

$$\cos\theta = \frac{\sum_{i=1}^{n} x_i y_i}{\sqrt{\sum_{i=1}^{n} x_i^2} \sqrt{\sum_{i=1}^{n} y_i^2}}$$

The measure of cosine similarity between two fuzzy sets κ_1 and κ_2 [13] is defined as follows.

$$C_F(\kappa_1.\kappa_2) = \frac{\sum_{i=1}^{n} \mu_{\kappa_1}(x_i)\mu_{\kappa_2}(x_i)}{\sqrt{\sum_{i=1}^{n}(\mu_{\kappa_1}(x_i))^2}\sqrt{\sum_{i=1}^{n}(\mu_{\kappa_2}(x_i))^2}}$$

The cosine similarity measure among two single valued neutrosophic sets κ_1 and κ_2 [14] is defined as follows

$$C_{SVNS}(\kappa_1, \kappa_2) = \frac{\sum_{i=1}^{n} \mathcal{T}_{\kappa_1}(x_i)\mathcal{T}_{\kappa_2}(x_i)+\mathcal{I}_{\kappa_1}(x_i)\mathcal{I}_{\kappa_2}(x_i)+\mathcal{F}_{\kappa_1}(x_i)\mathcal{F}_{\kappa_2}(x_i)}{\sqrt{\sum_{i=1}^{n} \mathcal{T}_{\kappa_1}(x_i)^2+\mathcal{I}_{\kappa_1}(x_i)^2+\mathcal{F}_{\kappa_1}(x_i)^2}\sqrt{\sum_{i=1}^{n} \mathcal{T}_{\kappa_2}(x_i)^2+\mathcal{I}_{\kappa_2}(x_i)^2+\mathcal{F}_{\kappa_2}(x_i)^2}}$$

Based on the above definitions, we can define the measure of cosine similarity between two single-valued neutrosophic fuzzy sets κ_1 and κ_2 as follows.

$$C_{SVNFS}(\kappa_1, \kappa_2) = \frac{1}{n}\sum_{i=1}^{n} \frac{\mu_{\kappa_1}(x_i)\mu_{\kappa_2}(x_i)+\mathcal{T}_{\kappa_1}(x_i,\mu)\mathcal{T}_{\kappa_2}(x_i,\mu)+\mathcal{I}_{\kappa_1}(x_i,\mu)\mathcal{I}_{\kappa_2}(x_i,\mu)+\mathcal{F}_{\kappa_1}(x_i,\mu)\mathcal{F}_{\kappa_2}(x_i,\mu)}{\sqrt{\mu_{\kappa_1}(x_i)^2+\mathcal{T}_{\kappa_1}(x_i,\mu)^2+\mathcal{I}_{\kappa_1}(x_i,\mu)^2+\mathcal{F}_{\kappa_1}(x_i,\mu)^2}\sqrt{\mu_{\kappa_2}(x_i)^2+\mathcal{T}_{\kappa_2}(x_i,\mu)^2+\mathcal{I}_{\kappa_2}(x_i,\mu)^2+\mathcal{F}_{\kappa_2}(x_i,\mu)^2}}$$

But the above-defined similarity measure sometimes fails to satisfy the basic properties of a similarity measure, that is,

If 1. $\mathcal{T}_{\kappa_1}(x_i,\mu) = 2\mathcal{T}_{\kappa_2}(x_i,\mu), \mathcal{I}_{\kappa_1}(x_i,\mu) = 2\mathcal{I}_{\kappa_2}(x_i,\mu), \mathcal{F}_{\kappa_1}(x_i,\mu) = 2\mathcal{F}_{\kappa_2}(x_i,\mu)$

or

$2.\mathcal{T}_{\kappa_1}(x_i,\mu) = \mathcal{T}_{\kappa_2}(x_i,\mu), 2\mathcal{I}_{\kappa_1}(x_i,\mu) = \mathcal{I}_{\kappa_2}(x_i,\mu), 2\mathcal{F}_{\kappa_1}(x_i,\mu) = \mathcal{F}_{\kappa_2}(x_i,\mu)$

That is, when $\kappa_1 \neq \kappa_2$, $C_{SVNFS}(\kappa_1, \kappa_2) = 1$, which means that the cosine similarity measure defined above does not satisfy the necessary condition for a similarity measure. Therefore, based on the improved cosine similarity measure proposed by [15] we are going to define the Enhanced Cosine Similarity Measure for Neutrosophic Fuzzy Sets(EC_{SVNFS}) as follows.

5.2. *Enhanced Cosine Similarity Measure between Neutrosophic Fuzzy Sets*

Definition 5.2. Let $X = \{x_1, x_2, ..., x_n\}$ be the universe of discourse. Consider$\kappa_1, \kappa_2 \in \mathcal{K}_{NFS}(X)$, where $\mathcal{K}_{NFS}(X)$ is the group that includes all neutrosophic fuzzy sets (NFSs) in X.

Let $\kappa_1 = \{(x_i, \mu_{\kappa_1}(x_i), \mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}), \mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}), \mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}))/x_i \in X\}$

$\kappa_2 = \{(x_i, \mu_{\kappa_2}(x_i), \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}), \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}), \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2}))/x_i \in X\}$

Then, $EC_{SVNFS}(\kappa_1, \kappa_2) = \frac{1}{2}[C_{SVNFS}(\kappa_1, \kappa_2) + 1 - d_{NHNFS}(\kappa_1, \kappa_2)]$

Theorem 5.3. *The Enhanced Cosine Similarity Measure between two Neutrosophic Fuzzy Sets κ_1 and κ_2 has the following properties.*

(1) $EC_{SVNFS}(\kappa_1, \kappa_2) \in [0, 1]$
(2) $EC_{SVNFS}(\kappa_1, \kappa_2) = 1 \iff \kappa_1 = \kappa_2$
(3) $EC_{SVNFS}(\kappa_1, \kappa_2) = EC_{SVNFS}(\kappa_2, \kappa_1)$

Proof. (1) Since $d_{NHNFS}(A, B)$ is the normalized Hausdorff distance between (A, B), we have $0 \leq d_{NHNFS}(A, B) \leq 1$ Also by [15] we have $0 \leq C_{SNFS}(A, B) \leq 1$, Therefore $0 \leq \frac{1}{2}[C_{SNFS}(A, B) + 1 - d_{NHNFS}(A, B)] \leq 1$, That is $0 \leq EC_{SNFS} \leq 1$.

(2) Let $EC_{SNFS}(A, B) = 1$
$\rightarrow C_{SNFS}(A, B) + 1 - d_{NHNFS}(A, B) = 2$
$\rightarrow C_{SNFS}(A, B) = 2 - 1 + d_{NHNFS}(A, B) = 1 + d_{NHNFS}(A, B)$
According to [15] $0 \leq C_{SNFS}(A, B) \leq 1$. Since $d_{NHNFS}(A, B)$ is the normalized Hausdorff distance between A, B we have $0 \leq d_{NHNFS}(A, B) \leq 1$ Therefore, $C_{SNFS}(A, B) = 0 + 1 = 1$ and $d_{NHNFS}(A, B) = 0$
When $C_{SNFS}(A, B) = 1$
$\rightarrow \mu_A(x_i) = K\mu_B(x_i)$
$\rightarrow T_A(x_i, \mu) = KT_B(x_i, \mu)$
$\rightarrow I_A(x_i, \mu) = KI_B(x_i, \mu)$
$\rightarrow F_A(x_i, \mu) = KF_B(x_i, \mu)$, where K is any constant.
Also $d_{NHNFS}(A, B) = 0 \rightarrow A = B$
In contrast, let $A = B$ then $d_{NHNFS}(A, B) = 0$ and $C_{SNFS}(A, B) = 1$
$\rightarrow EC_{SNFS}(A, B) = 1$

(3) $EC_{SNFS}(A, B) = \frac{1}{2}[C_{SNFS}(A, B) + 1 - d_{NHNFS}(A, B)] = \frac{1}{2}[C_{SNFS}(B, A) + 1 - d_{NHNFS}(B, A)] = EC_{SNFS}(B, A)$ □

6. **Clinical Diagnosis using Similarity-Based Pattern Recognition**

An individual afflicted by an illness will exhibit a range of symptoms, including fever, cough, weariness, sore throat, headache, and sneezing. Furthermore, each viral infection will exhibit various symptoms. Dengue, typhoid, and chikungunya might induce illness, chronic weariness, pharyngitis, coughing, and more symptoms. The symptoms of coronavirus infection encompass fever, cough, nasal congestion, and chronic weariness, among others. The primary symptoms of each disease can be identified by similarity-based pattern recognition involving neutrosophic fuzzy sets, and from the maximum similarity measure or core symptoms, we can ascertain the patient's disease kind. A medical diagnosis is generally established primarily on enduring symptoms, while fleeting symptoms do not yield conclusive insights. The ambiguity

	Temperature	Headache	Sore Throat	Diarrhoea	Shortness of Breath
Dengue Fever	(0.4,0.6,0.2,0.1)	(0.8,0.9,0.1,0.1)	(0.2,0.6.0.3,0.1)	(0.7,0.8,0.0,0.1)	(0.2,0.8,0.3,0.2)
Covid-19	(0.7,0.6,0.2,0.2)	(0.5,0.6,0.2,0.2)	(0.8,0.7.0.1,0.1)	(0.5,0.3,0.4,0.2)	(0.8,0.6,0.3,0.2)
Viral Fever	(0.8,0.7,0.2,0.2)	(0.6,0.6,0.1,0.1)	(0.7,0.5.0.3,0.3)	(0.4,0.2,0.3,0.4)	(0.2,0.3,0.4,0.3)
Rat Fever	(0.8,0.6,0.2,0.2)	(0.7,0.8,0.2,0.1)	(0.2,0.7.0.1,0.2)	(0.3,0.2,0.4,0.5)	(0.2,0.3,0.4,0.4)
Chikungunya	(0.9,0.6,0.5,0.1)	(0.6,0.9,0.1,0.0)	(0.2,0.6.0.1,0.3)	(0.3,0.3,0.5,0.4)	(0.3,0.6,0.2,0.2)

TABLE 1. Ideal values for symptoms of each disease

and uncertainties in the available medical data must be considered when addressing persistent symptoms; however, employing Neutrosophic Fuzzy Sets (NFS) to characterize this data might significantly mitigate these distractions. The similarity metric among neutrosophic fuzzy sets enables the identification of the principal symptoms of each ailment and the deduction of the patient's condition.

For our case study, let us select five patients $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5\}$, each patient having multiple symptoms such as

$\mathbb{S}$={$\mathbb{S}_1$(ProlongedTemperature),$\mathbb{S}_2$(ProlongedHeadache),$\mathbb{S}_3$(Prolonged SoreThroat),$\mathbb{S}_4$(ProlongedDiarrhoea),$\mathbb{S}_5$(Prolonged Shortness of Breath)}.

Now employing the medical data represented by neutrsophic fuzzy sets, we have to deduce the type of disease that affects the person such as

$\mathcal{D}$={$\mathcal{D}_1$(Dengue), $\mathcal{D}_2$(Covid-19), $\mathcal{D}_3$(Viral Fever),$\mathcal{D}_4$(Rat Fever), $\mathcal{D}_5$(Chikungunya)}

Consider the following NFS to be the patient's representation of all symptoms:

$\mathcal{P}_1$={$(\mathbb{S}_1, 0.8, 0.7, 0.2, 0.1)$,

$(\mathbb{S}_2, 0.7, 0.6, 0.2, 0.2)$,$(\mathbb{S}_3, 0.5, 0.6.0.1, 0.1)$,$(\mathbb{S}_4, 0.2, 0.2, 0.0, 0.1)$,$(\mathbb{S}_5, 0.7, 0.3, 0.6, 0.4)$}

We can establish an ideal value for each symptom associated with a given disease when we analyze the medical records of numerous patients who have that disease.**Table 1** provides an ideal value for each symptom associated with each disease.

It is our intention to place pattern$\mathcal{P}_1$ in one of the classes $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4, or \mathcal{D}_5$.Accurate diagnosis $\mathcal{D}_k$ for patient $\mathcal{P}_1$ is derived according to $k = argMax_{1\leq i\leq 5} S.M(\mathcal{P}_1, \mathcal{D}_i)$ where S.M stands for similarity measure. The similarity measures between patient $\mathcal{P}_1$ and disease $\mathcal{D}_i$ are given in **Table 2**.

The maximum similarity measure obtained by employing three different similarity measures according to the given neutrophic fuzzy data is shown in bold in **Table2**. From **Table 2** and **FIGURE-1** we can conclude that patient $\mathcal{P}_1$ is affected by viral fever.

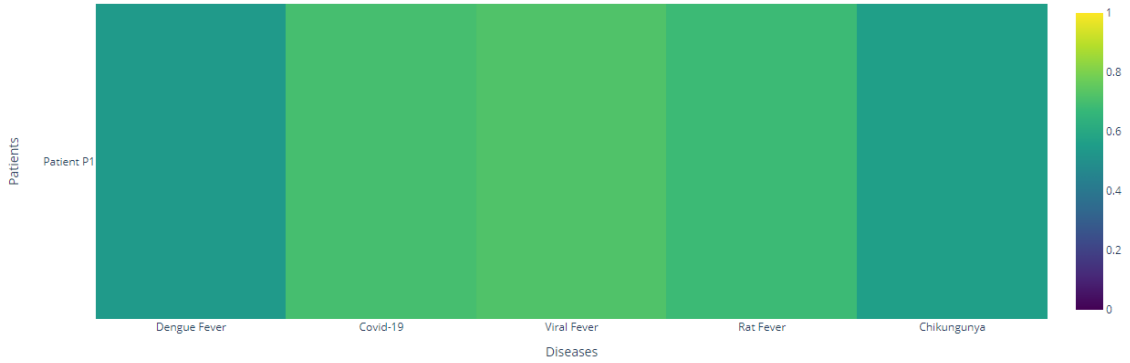


FIGURE 1. Heat map showing patient-disease mapping

	$(\mathcal{P}_1, \mathcal{D}_1)$	$(\mathcal{P}_1, \mathcal{D}_2)$	$(\mathcal{P}_1, \mathcal{D}_3)$	$(\mathcal{P}_1, \mathcal{D}_4)$	$(\mathcal{P}_1, \mathcal{D}_5)$
$S_H(\mathcal{P}_1, \mathcal{D}_i)$	0.58	0.74	**0.76**	0.70	0.64
$S_M(\mathcal{P}_1, \mathcal{D}_i)$	0.56	0.69	**0.70**	0.68	0.58
$EC_{SVNFS}(\mathcal{P}_1, \mathcal{D}_i)$	0.54	0.70	**0.72**	0.68	0.56
$S_E(\mathcal{P}_1, \mathcal{D}_i)$	0.8648	0.9168	**0.9250**	0.9121	0.8833

TABLE 2. Similarity measures between patient $\mathcal{P}_1$ and disease $\mathcal{D}_i$

7. **Comparative Evaluations with Existing Similarity Measures**

The initial similarity measure established for the Neutrosophic Fuzzy Set was the Euclidean similarity measure (S_E), devised by S.Das et al. [4].In this section, we compare and evaluate recent definitions of similarity measures with the traditional Euclidean similarity measure. In **TABLE-2**, we have compiled a summary of various similarity measures between the patient P_1 and each disease D_i. The heat map displayed in **FIGURE-3** will enable us to assess the efficacy of each similarity measure in the given modeling problem.

The recently introduced Hausdorff similarity measure for Neutrosophic Fuzzy Sets (NFS) presents notable benefits in managing the intricacy and multidimensional characteristics of Neutrosophic Fuzzy Sets (NFS), offering a more resilient and all-encompassing evaluation of similarity. However, the Euclidean similarity measure is favored because of its simplicity, computing efficiency, and ability to directly assess similarity between individual points. The selection among the four commonalities is based on the particular demands and limitations of the application. The comparative diagram presented in **FIGURE-2** will provide a comprehensive understanding of the benefits and constraints associated with each similarity.

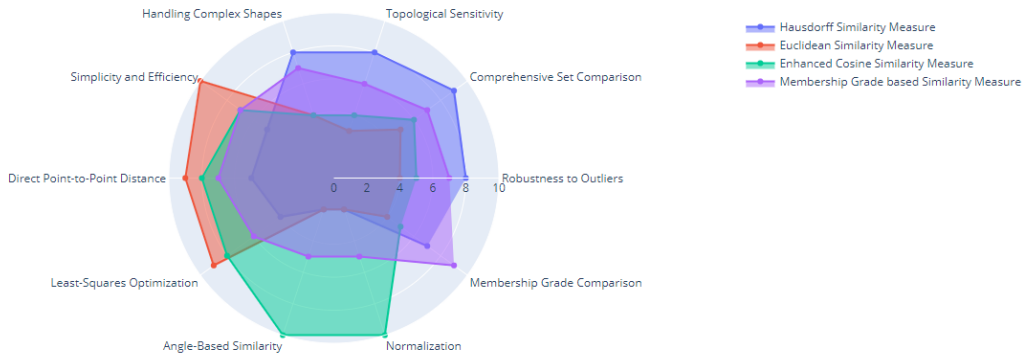


FIGURE 2. Comparison diagram for various similarity measures

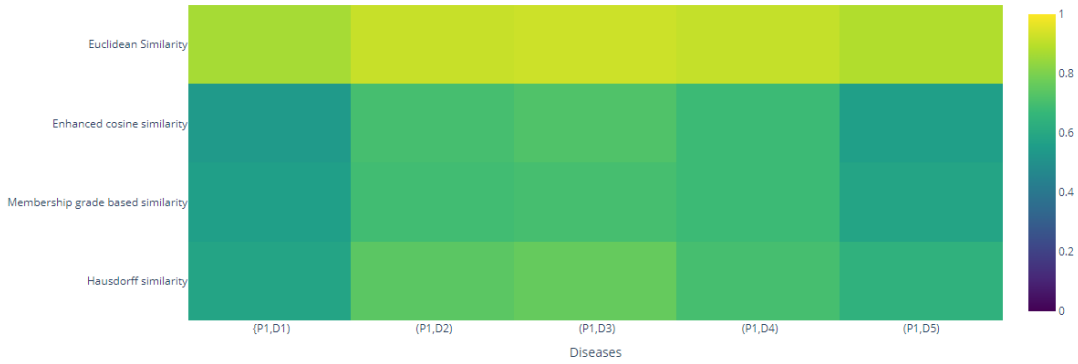


FIGURE 3. Comparison Map

8. **Conclusion**

In Neutrosophic Fuzzy Set theory, the measurement of distance and similarity is a key research area since these concepts are useful tools for handling incomplete and ambiguous information. In the present study, we derive Hausdorff Similarity Measures, Membership-Grade-Based Similarity Measure, and Trigonometric Similarity Measure in a single-valued neutrosophic environment. the use of these similarity measures in pattern recognition represents a significant advancement in the field of medical data analysis.Future work will focus on extending the application of Neutrosophic Fuzzy Sets to more complex medical datasets and exploring their integration with advanced machine learning techniques, such as deep learning, to further enhance the precision and efficiency of symptom extraction and disease characterization. Additionally, incorporating these models into real-time clinical decision support systems can provide substantial benefits to healthcare practitioners by improving diagnostic accuracy and patient outcomes.

References

1. Szmidt, Eulalia and Kacprzyk, Janusz,A note on the Hausdorff distance between Atanassov's intuitionistic fuzzy sets,Notes on Intuitionistic Fuzzy Sets,2009, volume 15, number 1, pp. 1-12.
2. Zadeh, Lotfi A, Fuzzy sets, Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh, World Scientific,1996.pp.394-432.
3. Wang, Haibin and Smarandache, Florentin and Zhang,Yanqing and Sunderraman, Rajshekhar,Single valued neutrosophic sets,Infinite study,2010.
4. Das, Sujit and Roy, Bikash Koli and Kar, Mohuya B and Kar, Samarjit and Pamučar, Dragan, Neutrosophic fuzzy set and its application in decision making, Journal of Ambient Intelligence and Humanized Computing, Springer,2020, volume 11, number 11,pp. 5017–5029.
5. Broumi, Said and Smarandache, Florentin, Several similarity measures of neutrosophic sets, Infinite Study,2013.
6. Zadeh, Lotfi Asker, Fuzzy sets as a basis for a theory of possibility, Fuzzy sets and systems, Elsevier,1978, volume 1, number 1, pp.3-28.
7. Smarandache, Florentin, Neutrosophic set-a generalization of the intuitionistic fuzzy set, International Journal of Pure and Applied Mathematics, Academic Publications,2005, volume 24, number 3, pp. 287.
8. Alblowi, SA and Salama, AA and Eisa, Mohmed, New concepts of neutrosophic sets, Infinite Study, 2014.
9. Peng, Xindong and Dai, Jingguo, A bibliometric analysis of neutrosophic set: two decades review from 1998 to 2017, Artificial Intelligence Review, Springer,2020, volume 53, number 1, pp.199-255.
10. Chaudhuri, Bidyut Baran, and Rosenfeld, Azriel, A modified Hausdorff distance between fuzzy sets, Information Sciences, Elsevier,1999, volume 118, number1-4, pp. 159-171.
11. Zou, Jiawei and Deng, Yong and Hu, Yong and Lin, Ge, Measure distance between neutrosophic sets: An evidential approach, Infinite Study,2018.
12. Candan, K Selçuk and Sapino, Maria Luisa, Data management for multimedia retrieval, Cambridge University Press, 2010.
13. Bhattacharya, A, On a measure of divergence between two multinomial populations, 1946.
14. Ye, Jun, Multicriteria decision making method using the correlation coefficient in a single-valued neutrosophic environment, International Journal of General Systems, Taylor & Francis, volume 42, number 4, pp. 386–394.
15. Ye, Jun, Vector similarity measures of simplified neutrosophic sets and their application in multi-criteria decision-making, Infinite Study, 2014.
16. Mathews, Priya and Sebastian, Lovelymol,Hausdorff distance of neutrosophic fuzzy sets for medical diagnosis,AIP Conference Proceedings,AIP Publishing,volume-2875,number-1,2023.
17. Ye,J. Improved similarity measure between neutrosophic sets and its application in multi-criteria decision-making. Journal of Intelligent & Fuzzy Systems, 38(3), 3293-3303.
18. Peng, J., & Dai, J. (2021). A hybrid similarity measure for neutrosophic sets and its application in decision making. Applied Soft Computing, 101, 107056.
19. Broumi, S., Smarandache, F., & Zavadskas, E. K. (2019). Entropy-based similarity measure for neutrosophic sets and its applications in multi-attribute decision-making. Entropy, 21(2), 117.
20. Yang, X., & Singh, P. Geometric distance-based similarity measure for neutrosophic fuzzy sets in image processing. IEEE Access, 10, 34567-34578.
21. Khan, Q., Ahmad, I., & Ali,Z, Overlap coefficient-based similarity measure for neutrosophic fuzzy sets in medical diagnosis.Journal of Medical Systems, 47(1), 22.
22. Li, F., Zhang, H., & Zhou, X. Integrating similarity measures for neutrosophic fuzzy sets with clustering algorithms for enhanced data analysis. Pattern Recognition Letters, 161, 103-110.

Received: June 29, 2024. Accepted: Oct 12, 2024

NSS
Neutrosophic Sets and Systems, Vol. 75, 2025
University of New Mexico

A Numerical Approach For The Algebra of Two-Fold 2×2 Fuzzy Real Matrices and Anti Fuzzy Matrices

Ahmad A. Abubaker*[1], Mayada Abualhomos[2], Khaled Matarneh[3], Abdallah Al-Husban[4,5]

[1]Faculty of Computer Studies, Arab Open University, Saudi Arabia; a.abubaker@arabou.edu.sa

[2]Applied Science Private University Amman,11931, Jordan; abuhomos@asu.edu.jo

[3]Faculty of Computer Studies, Arab Open University, Saudi Arabia; k.matarneh@arabou.edu.sa

[4]Department of Mathematics, Faculty of Science and Technology, Irbid; National University, P.O. Box: 2600 Irbid, Jordan; dralhosban@inu.edu.jo

[5]Jadara Research Center, Jadara University, Irbid 21110, Jordan.

Abstract:

This paper is dedicated to study for the first time the concept of square 2×2 fuzzy and anti-fuzzy two-fold matrix with real entries, where we present the two-fold algebraic operations between the these matrices and obtain their special properties such as associativity, commutative properties, and the existing of additive of multiplicative inverses. Also, we provide many examples to explain the two-fold algebraic properties and the two-fold algebraic structure.

Keywords: two-fold algebra, two-fold real numbers, two-fold fuzzy matrix, two-fold inverse, anti-fuzzy matrix.

Introduction

The theory of matrices based on mathematical and algebraic logic has been studied by different authors around the world. For example, we can see many important results about the computations of plithogenic matrices in [5].

Also, neutrosophic matrices and their extended types have been defined and handled in [1-3], where many problems such as diagonalization, inverses, and algebraic representations were studied and provided [4, 6].

The two-fold algebra and two-fold algebraic structures began in [7], and then these algebras in fuzzy versions were used in many different contexts, such as mathematical analysis, algebraic elements, and in other types of mathematical structures [8-23].

In this paper we to study for the first time the concept of square 2×2 fuzzy and anti-fuzzy two-fold matrix with real entries, where we present the two-fold algebraic operations between the these matrices and obtain their special properties such as associativity, commutative properties, and the existing of additive of multiplicative inverses.

Two-Fold 2×2 -Fuzzy Real Matrices

Definition:

Let $\mathbb{R}$ be the real field, $\mu: \mathbb{R} \to [0,1]$ such that $\begin{cases}\mu(0) = 0 \\ \mu(1) = 1\end{cases}$ be a fuzzy mapping. Let $\mathbb{R}_F = \{x_{\mu(y)} : x, y \in \mathbb{R}\}$be the corresponding two-fold fuzzy real algebra, then we define:

$A = \begin{pmatrix} x_{1\mu(y_1)} & x_{2\mu(y_2)} \\ x_{3\mu(y_3)} & x_{4\mu(y_4)} \end{pmatrix}$; $x_i, y_i \in \mathbb{R}$.

A is called a 2×2 fuzzy two-fold matrix.

Definition:

Let $a_i = (x_i)_{\mu(y_i)}$ $1 \le i \le 4$, $b_i = (z_i)_{\mu(t_i)}$; $1 \le i \le 4$ and $x_i, y_i, z_i, t_i \in \mathbb{R}$, $a_i, b_i \in \mathbb{R}_F$.

Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$, we define:

$A + B = \begin{pmatrix} a_1 * b_1 & a_2 * b_2 \\ a_3 * b_3 & a_4 * b_4 \end{pmatrix}$; $a_i * b_i = (x_i + z_i)_{\min(\mu(y_i),\mu(t_i))}$

$-A = \begin{pmatrix} -a_1 & -a_2 \\ -a_3 & -a_4 \end{pmatrix}$; $-a_i = (-x_i)_{\mu(-x_i)}$

$A \times B = \begin{pmatrix} (a_1 \circ b_1) * (a_2 \circ b_3) & (a_1 \circ b_2) * (a_2 \circ b_4) \\ (a_3 \circ b_1) * (a_4 \circ b_3) & (a_3 \circ b_2) * (a_4 \circ b_4) \end{pmatrix}$, where $a_i \circ b_j = (x_i z_j)_{\max(\mu(x_i),\mu(t_j))}$

Example:S

Take: $\mu: \mathbb{R} \to [0,1]: \mu(x) = \begin{cases} 0 & ; x = 0 \\ 1 & ; x = 1 \\ \frac{1}{2} & ; x > 1 \\ \frac{1}{3} & ; x < 0 \\ \frac{1}{4} & ; 0 < x < 1 \end{cases}$

And: $a_1 = 3_{\mu(2)} = 3_{\frac{1}{2}}, a_2 = 4_{\mu\left(\frac{1}{3}\right)} = 4_{\frac{1}{4}},$

$$a_3 = 1_{\mu(-5)} = 1_{\frac{1}{3}}, a_4 = 0_{\mu(0)} = 0_0 \ ,$$

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} 3_{\frac{1}{2}} & 4_{\frac{1}{4}} \\ 1_{\frac{1}{3}} & 0_0 \end{pmatrix}.$$

$$b_1 = (-2)_{\mu(1)} = (-2)_1, b_2 = (-1)_{\mu(7)} = (-1)_{\frac{1}{2}}, b_3 = (1)_{\mu(0)} = 1_0 ,$$

$$b_4 = (2)_{\mu\left(\frac{1}{10}\right)} = 2_{\frac{1}{4}} \ , B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} (-2)_1 & (-1)_{\frac{1}{2}} \\ 1_0 & 2_{\frac{1}{4}} \end{pmatrix}.$$

We have:

$$-A = \begin{pmatrix} (-3)_{\mu(-2)} & (-4)_{\mu\left(\frac{-1}{3}\right)} \\ (-1)_{\mu(5)} & (-0)_{\mu(-0)} \end{pmatrix} = \begin{pmatrix} (-3)_{\frac{1}{3}} & (-4)_{\frac{1}{3}} \\ (-1)_{\frac{1}{2}} & 0_0 \end{pmatrix}.$$

$$A + B = \begin{pmatrix} a_1 * b_1 & a_2 * b_2 \\ a_3 * b_3 & a_4 * b_4 \end{pmatrix} = \begin{pmatrix} 1_{\frac{1}{2}} & (3)_{\frac{1}{4}} \\ 2_0 & 2_0 \end{pmatrix}; \begin{cases} a_1 * b_1 = (3-2)_{\frac{1}{2}} = 1_{\frac{1}{2}} \\ a_2 * b_2 = (4-1)_{\frac{1}{4}} = 3_{\frac{1}{4}} \\ a_3 * b_3 = (1+1)_0 = 2_0 \\ a_4 * b_4 = (0+2)_0 = 2_0 \end{cases}$$

$$A \times B = \begin{pmatrix} (a_1 \circ b_1) * (a_2 \circ b_3) & (a_1 \circ b_2) * (a_2 \circ b_4) \\ (a_3 \circ b_1) * (a_4 \circ b_3) & (a_3 \circ b_2) * (a_4 \circ b_4) \end{pmatrix}$$

$$= \begin{pmatrix} (-2)_{\frac{1}{4}} & 5_{\frac{1}{4}} \\ (-2)_0 & (-1)_{\frac{1}{4}} \end{pmatrix}; \begin{cases} a_1 \circ b_1 = (-6)_1, a_2 \circ b_3 = 4_{\frac{1}{4}} \\ (a_1 \circ b_1) * (a_2 \circ b_3) = (-2)_{\frac{1}{4}} \\ a_1 \circ b_2 = (-3)_{\frac{1}{2}}, a_2 \circ b_4 = 8_{\frac{1}{4}} \\ (a_1 \circ b_2) * (a_2 \circ b_4) = 5_{\frac{1}{4}} \end{cases}$$

And: $\begin{cases} a_3 \circ b_1 = (-2)_1, a_4 \circ b_3 = (0)_0 \\ (a_3 \circ b_1) * (a_4 \circ b_3) = (-2)_0 \\ a_3 \circ b_2 = (-1)_{\frac{1}{2}}, a_4 \circ b_4 = 0_{\frac{1}{4}} \\ (a_3 \circ b_2) * (a_4 \circ b_4) = (-1)_{\frac{1}{4}} \end{cases}$

Take $C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} 1_1 & 0_1 \\ 1_0 & 0_0 \end{pmatrix}$, then:

$$B \times C = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \times \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} b_1c_1 + b_2c_3 & b_1c_2 + b_2c_4 \\ b_3c_1 + b_4c_3 & b_3c_2 + b_4c_4 \end{pmatrix}.$$

$b_1c_1 = (-2)_1 \circ (1)_1 = (-2)_1, b_2c_3 = (-1)_{\frac{1}{2}} \circ (1_0) = (-1)_{\frac{1}{2}}$, $b_1c_1 + b_2c_3 = (-2)_1 * (-1)_{\frac{1}{2}} =$

$(-3)_{\frac{1}{2}}$,

$b_1c_2 = (-2)_1 \circ (0)_1 = (0)_1, b_2c_4 = (-1)_{\frac{1}{2}} \circ (0)_0 = (0)_{\frac{1}{2}}$, $b_1c_2 + b_2c_4 = (0)_1 * (0)_{\frac{1}{2}} = (0)_{\frac{1}{2}}$,

$b_3c_1 = (1_0) \circ (1_1) = (1_1), b_4c_3 = (2_{\frac{1}{4}}) \circ (1_0) = (2_{\frac{1}{4}})$, $b_3c_1 + b_4c_3 = (1_1) * (2_{\frac{1}{4}}) = (3_{\frac{1}{4}})$,

$b_3c_2 = (1_0) \circ (0_1) = (0_1), b_4c_4 = (2_{\frac{1}{4}}) \circ (0_0) = (0_{\frac{1}{4}})$, $b_3c_2 + b_4c_4 = (0_1) * (0_{\frac{1}{4}}) = (0_{\frac{1}{4}})$,

Thus $B \times C = \begin{pmatrix} -3_{\frac{1}{2}} & 0_{\frac{1}{2}} \\ 3_{\frac{1}{4}} & 0_{\frac{1}{4}} \end{pmatrix}$.

$$A \times (B \times C) = \begin{pmatrix} 3_{\frac{1}{2}} & 4_{\frac{1}{4}} \\ 1_{\frac{1}{3}} & 0_0 \end{pmatrix} \times \begin{pmatrix} -3_{\frac{1}{2}} & 0_{\frac{1}{2}} \\ 3_{\frac{1}{4}} & 0_{\frac{1}{4}} \end{pmatrix} = \begin{pmatrix} 3_{\frac{1}{4}} & 0_{\frac{1}{4}} \\ (-3)_{\frac{1}{4}} & 0_{\frac{1}{4}} \end{pmatrix},$$

$$\left(3_{\frac{1}{2}}\right) \circ \left(-3_{\frac{1}{2}}\right) * \left(4_{\frac{1}{4}}\right) \circ \left(3_{\frac{1}{4}}\right) = \left(-9_{\frac{1}{2}}\right) * \left(12_{\frac{1}{4}}\right) = 3_{\frac{1}{4}},$$

$$\left(3_{\frac{1}{2}}\right) \circ \left(0_{\frac{1}{2}}\right) * \left(4_{\frac{1}{4}}\right) \circ \left(0_{\frac{1}{4}}\right) = \left(0_{\frac{1}{2}}\right) * \left(0_{\frac{1}{4}}\right) = 0_{\frac{1}{4}},$$

$$\left(1_{\frac{1}{3}}\right) \circ \left(-3_{\frac{1}{2}}\right) * (0_0) \circ \left(3_{\frac{1}{4}}\right) = \left(-3_{\frac{1}{2}}\right) * \left(0_{\frac{1}{4}}\right) = (-3)_{\frac{1}{4}},$$

$$\left(1_{\frac{1}{3}}\right) \circ \left(0_{\frac{1}{2}}\right) * (0_0) \circ \left(0_{\frac{1}{4}}\right) = \left(0_{\frac{1}{2}}\right) * \left(0_{\frac{1}{4}}\right) = 0_{\frac{1}{4}}.$$

$$(A \times B) \times C = \begin{pmatrix} -2_{\frac{1}{4}} & 5_{\frac{1}{4}} \\ -3_1 & -1_{\frac{1}{4}} \end{pmatrix} \begin{pmatrix} 1_1 & 0_1 \\ 1_0 & 0_0 \end{pmatrix} = \begin{pmatrix} 3_{\frac{1}{4}} & 0_{\frac{1}{4}} \\ -3_{\frac{1}{4}} & 0_{\frac{1}{4}} \end{pmatrix},$$

$$\left(-2_{\frac{1}{4}}\right) \circ (1_1) * \left(5_{\frac{1}{4}}\right) \circ (1_0) = (-2_1) * \left(5_{\frac{1}{4}}\right) = 3_{\frac{1}{4}},$$

$$\left(-2_{\frac{1}{4}}\right) \circ (0_1) * \left(5_{\frac{1}{4}}\right) \circ (0_0) = (0_1) * \left(0_{\frac{1}{4}}\right) = 0_{\frac{1}{4}},$$

$$(-2_1) \circ (1_1) * \left(-1_{\frac{1}{4}}\right) \circ (1_0) = (-2_1) * \left(-1_{\frac{1}{4}}\right) = -3_{\frac{1}{4}},$$

$$(-3_1) \circ (0_1) * \left(-1_{\frac{1}{4}}\right) \circ (0_0) = (0_1) * \left(0_{\frac{1}{4}}\right) = 0_{\frac{1}{4}}.$$

Thus $(A \times B) \times C = A \times (B \times C)$.

$$B \times A = \begin{pmatrix} -2_1 & -1_{\frac{1}{2}} \\ 1_0 & 2_{\frac{1}{4}} \end{pmatrix} \times \begin{pmatrix} 3_{\frac{1}{2}} & 4_{\frac{1}{4}} \\ 1_{\frac{1}{3}} & 0_0 \end{pmatrix} = \begin{pmatrix} -7_1 & m_1 \\ m_2 & m_3 \end{pmatrix},$$

$(-2_1) \circ \left(3_{\frac{1}{2}}\right) * \left(-1_{\frac{1}{2}}\right) \circ \left(1_{\frac{1}{3}}\right) = (-6_1) * \left(-1_{\frac{1}{2}}\right) = -7_1$

It is clear that $B \times A \neq A \times B$.

The Algebra Properties of two-fold Matrix Operations:

Property (1):

Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$; $\begin{cases} a_i = (x_i)_{\mu(y_i)} & ; x_i, y_i \in \mathbb{R} \\ b_i = (z_i)_{\mu(t_i)} & ; z_i, t_i \in \mathbb{R} \end{cases}$

We have: $a_i * b_i = (x_i + z_i)_{\min(\mu(y_i),\mu(t_i))} = (z_i + x_i)_{\min(\mu(y_i),\mu(t_i))} = b_i * a_i$, then $A + B = B + A$.

Property (2):

Let $C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$; $c_i = (m_i)_{\mu(n_i)}$, then:

$B + C = \begin{pmatrix} b_1 * c_1 & b_2 * c_2 \\ b_3 * c_3 & b_4 * c_4 \end{pmatrix}, A + (B + C) = \begin{pmatrix} a_1 * b_1 * c_1 & a_2 * b_2 * c_2 \\ a_3 * b_3 * c_3 & a_4 * b_4 * c_4 \end{pmatrix} = (A + B) + C.$

Property (3):

$$A - A = A + (-A) = \begin{pmatrix} (x_1)_{\mu(y_1)} & (x_2)_{\mu(y_2)} \\ (x_3)_{\mu(y_3)} & (x_4)_{\mu(y_4)} \end{pmatrix} + \begin{pmatrix} (-x_1)_{\mu(-y_1)} & (-x_2)_{\mu(-y_2)} \\ (-x_3)_{\mu(-y_3)} & (-x_4)_{\mu(-y_4)} \end{pmatrix} = \begin{pmatrix} 0_{l_1} & 0_{l_2} \\ 0_{l_3} & 0_{l_4} \end{pmatrix} ;$$

$l_i = \min(\mu(y_i), \mu(-y_i))$.

Property (4):

$A \times (B \times C) = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \times \begin{pmatrix} (b_1 \circ c_1) * (b_2 \circ c_3) & (b_1 \circ c_2) * (b_2 \circ c_4) \\ (b_3 \circ c_1) * (b_4 \circ c_3) & (b_3 \circ c_2) * (b_4 \circ c_4) \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \times \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} = \begin{pmatrix} (a_1 \circ d_1) * (a_2 \circ d_3) & (a_1 \circ d_2) * (a_2 \circ d_4) \\ (a_3 \circ d_1) * (a_4 \circ d_3) & (a_3 \circ d_2) * (a_4 \circ d_4) \end{pmatrix}.$

We have:

$a_1 \circ d_1 = a_1 \circ [(b_1 \circ c_1) * (b_2 \circ c_3)] = (a_1 \circ b_1 \circ c_1) * (a_1 \circ b_2 \circ c_3), a_2 \circ d_3 = a_2 \circ [(b_3 \circ c_1) * (b_4 \circ c_3)] = (a_2 \circ b_3 \circ c_1) * (a_2 \circ b_4 \circ c_3),$

Put

$T_1 = (a_1 \circ d_1) * (a_2 \circ d_3) = (a_1 \circ b_1 \circ c_1) * (a_1 \circ b_2 \circ c_3) * (a_2 \circ b_3 \circ c_1) * (a_2 \circ b_4 \circ c_3).$

$a_1 \circ d_2 = a_1 \circ [(b_1 \circ c_2) * (b_2 \circ c_4)] = (a_1 \circ b_1 \circ c_2) * (a_1 \circ b_2 \circ c_4), a_2 \circ d_4 = a_2 \circ [(b_3 \circ c_2) * (b_4 \circ c_4)] = (a_2 \circ b_3 \circ c_2) * (a_2 \circ b_4 \circ c_4),$

$T_2 = (a_1 \circ d_2) * (a_2 \circ d_4) = (a_1 \circ b_1 \circ c_2) * (a_1 \circ b_2 \circ c_4) * (a_2 \circ b_3 \circ c_2) * (a_2 \circ b_4 \circ c_4),$

$a_3 \circ d_1 = a_3 \circ [(b_1 \circ c_1) * (b_2 \circ c_3)] = (a_3 \circ b_1 \circ c_1) * (a_3 \circ b_2 \circ c_3)$, $a_4 \circ d_3 = a_4 \circ [(b_3 \circ c_1) * (b_4 \circ c_3)] = (a_4 \circ b_3 \circ c_1) * (a_4 \circ b_4 \circ c_3)$,

Put

$T_3 = (a_3 \circ d_1) * (a_4 \circ d_3) = (a_3 \circ b_1 \circ c_1) * (a_3 \circ b_2 \circ c_3) * (a_4 \circ b_3 \circ c_1) * (a_4 \circ b_4 \circ c_3)$,

$a_3 \circ d_2 = a_3 \circ [(b_1 \circ c_2) * (b_2 \circ c_4)] = (a_3 \circ b_1 \circ c_2) * (a_3 \circ b_2 \circ c_4)$, $a_4 \circ d_4 = a_4 \circ [(b_3 \circ c_2) * (b_4 \circ c_4)] = (a_4 \circ b_3 \circ c_2) * (a_4 \circ b_4 \circ c_4)$,

$T_4 = (a_3 \circ d_2) * (a_4 \circ d_4) = (a_3 \circ b_1 \circ c_2) * (a_3 \circ b_2 \circ c_4) * (a_4 \circ b_3 \circ c_2) * (a_4 \circ b_4 \circ c_4)$.

Thus $A \times (B \times C) = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$.

On the other hand, we have:

$(A \times B) \times C = \begin{pmatrix} (a_1 \circ b_1) * (a_2 \circ b_3) & (a_1 \circ b_2) * (a_2 \circ b_4) \\ (a_3 \circ b_1) * (a_4 \circ b_3) & (a_3 \circ b_2) * (a_4 \circ b_4) \end{pmatrix} \times \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}$, where

$k_1 = [(a_1 \circ b_1) * (a_2 \circ b_3)] \circ c_1 * [(a_1 \circ b_2) * (a_2 \circ b_4)] \circ c_3 = (a_1 \circ b_1 \circ c_1) * (a_2 \circ b_3 \circ c_1) * (a_1 \circ b_2 \circ c_3) * (a_2 \circ b_4 \circ c_3) = T_1$.

By a similar argument, we can see clearly that:

$k_2 = T_2, k_3 = T_3, k_4 = T_4$, thus $A \times (B \times C) = (A \times B) \times C$.

Property (5):

$A \times (B + C) = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \times \begin{pmatrix} b_1 * c_1 & b_2 * c_2 \\ b_3 * c_3 & b_4 * c_4 \end{pmatrix} = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}$, where:

$F_1 = a_1 \circ (b_1 * c_1) * a_2 \circ (b_3 * c_3) = (a_1 \circ b_1) * (a_1 \circ c_1) * (a_2 \circ b_3) * (a_2 \circ c_3)$,

$F_2 = a_1 \circ (b_2 * c_3) * a_2 \circ (b_4 * c_4) = (a_1 \circ b_2) * (a_1 \circ c_3) * (a_2 \circ b_4) * (a_2 \circ c_4)$,

$F_3 = a_3 \circ (b_1 * c_1) * a_4 \circ (b_3 * c_3) = (a_3 \circ b_1) * (a_3 \circ c_1) * (a_4 \circ b_3) * (a_4 \circ c_3)$,

$F_4 = a_3 \circ (b_2 * c_2) * a_4 \circ (b_4 * c_4) = (a_3 \circ b_2) * (a_3 \circ c_2) * (a_4 \circ b_4) * (a_4 \circ c_4)$,

On the other hand, we have:

$(A \times B) + (A \times C) = \begin{pmatrix} (a_1 \circ b_1) * (a_2 \circ b_3) & (a_1 \circ b_2) * (a_2 \circ b_4) \\ (a_3 \circ b_1) * (a_4 \circ b_3) & (a_3 \circ b_2) * (a_4 \circ b_4) \end{pmatrix} +$

$\begin{pmatrix} (a_1 \circ c_1) * (a_2 \circ c_3) & (a_1 \circ c_2) * (a_2 \circ c_4) \\ (a_3 \circ c_1) * (a_4 \circ c_3) & (a_3 \circ c_2) * (a_4 \circ c_4) \end{pmatrix} = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}$.

Which implies that $A \times (B + C) = (A \times B) + (A \times C)$.

Result:

If TF_μ is the set of all (2×2) two-fold fuzzy matrices with two-fold real entries, then:

1] $(+)$ is a commutative and associative.

2] $(\times)$ is associative.

3] $(\times)$ is a distributive with respect to $(+)$.

Definition:

A (2×2) two-fold zero fuzzy matrix is defined as:

$O = \begin{pmatrix} 0_1 & 0_1 \\ 0_1 & 0_1 \end{pmatrix}$.

Remark:

For any $A \in TF_\mu$, we have $A + O = O + A = A$.

It is clear that $A \times O = O$.

The additive inverse of $A = \begin{pmatrix} (x_1)_{\mu(y_1)} & (x_2)_{\mu(y_2)} \\ (x_3)_{\mu(y_3)} & (x_4)_{\mu(y_4)} \end{pmatrix}$ is not existed.

Definition:

A (2×2) two-fold unitary matrix is:

$I = \begin{pmatrix} 1_0 & 0_1 \\ 0_1 & 1_0 \end{pmatrix}$

Remark:

For $A = \begin{pmatrix} (x_1)_{\mu(y_1)} & (x_2)_{\mu(y_2)} \\ (x_3)_{\mu(y_3)} & (x_4)_{\mu(y_4)} \end{pmatrix}$, we have:

$A \times I = \begin{pmatrix} (x_1)_{\mu(y_1)} & (x_2)_{\mu(y_2)} \\ (x_3)_{\mu(y_3)} & (x_4)_{\mu(y_4)} \end{pmatrix} \times \begin{pmatrix} 1_0 & 0_1 \\ 0_1 & 1_0 \end{pmatrix} = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix}$;

$L_1 = \left((x_1)_{\mu(y_1)} \circ 1_0\right) * \left((x_2)_{\mu(y_2)} \circ (0_1)\right) = (x_1)_{\mu(y_1)} * (0_1) = (x_1)_{\mu(y_1)}$

$L_2 = \left((x_1)_{\mu(y_1)} \circ 0_1\right) * \left((x_2)_{\mu(y_2)} \circ 1_0\right) = (0_1) * (x_2)_{\mu(y_2)} = (x_2)_{\mu(y_2)}$

$L_3 = (x_3)_{\mu(y_3)}$, $L_4 = (x_4)_{\mu(y_4)}$,

thus $A \times I = I \times A = A$.

Finding the multiplicative inverse:

Let $A = \begin{pmatrix} (x_1)_{\mu(y_1)} & (x_2)_{\mu(y_2)} \\ (x_3)_{\mu(y_3)} & (x_4)_{\mu(y_4)} \end{pmatrix} \in TF_\mu$; $x_1x_4 - x_2x_3 \neq 0$

Put $B = \begin{pmatrix} (x_4)_{\mu(z_4)} & (-x_2)_{\mu(z_2)} \\ (-x_3)_{\mu(z_3)} & (x_1)_{\mu(z_1)} \end{pmatrix}$, and compute:

$A \times B = \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix}$ such that:

$\mu_1 = \left[(x_1)_{\mu(y_1)} \circ (x_4)_{\mu(z_4)}\right] * \left[(x_2)_{\mu(y_2)} \circ (-x_3)_{\mu(z_3)}\right] = (x_1x_4 - x_2x_3)_{l_1}$;

$l_1 = \min\left[\max\left(\mu(y_1), \mu(z_4)\right), \max\left(\mu(y_2), \mu(z_3)\right)\right] = 1,$

Thus: we can put: $\mu(z_4) = \mu(z_3) = 1.$

Also, if we put $\mu(z_2) = \mu(z_1) = 1$, then:

$A \times B = \begin{pmatrix} (x_1x_4 - x_2x_3)_1 & 0_1 \\ 0_1 & (x_1x_4 - x_2x_3)_1 \end{pmatrix}.$

Define:

$\det A = (x_1x_4 - x_2x_3)_1$, then:

$\frac{1}{(x_1x_4 - x_2x_3)_1} \times \begin{pmatrix} (x_1x_4 - x_2x_3)_1 & 0_1 \\ 0_1 & (x_1x_4 - x_2x_3)_1 \end{pmatrix} = \begin{pmatrix} 1_1 & 0_1 \\ 0_1 & 1_1 \end{pmatrix} \neq I.$

On the other hand, if we put $l_1 = 0$, then we can not choose $\mu(z_4), \mu(z_3)$to obtain this result, which means that A is not invertible.

Result:

The set of all two-fold fuzzy matrices is not a ring, that is because there are no additive inverses.

Two-Fold 2×2 anti-Fuzzy Real Matrices:

Definition:

Let $\mathbb{R}$ be the real field, $\mu: \mathbb{R} \to [0,1]$ such that $\begin{cases} \mu(0) = 0 \\ \mu(1) = 1 \end{cases}$ be a fuzzy mapping. Let $\mathbb{R}_F = \{x_{\mu(y)} : x, y \in \mathbb{R}\}$be the corresponding two-fold fuzzy real algebra, then we define:

$A = \begin{pmatrix} x_{1\mu(y_1)} & x_{2\mu(y_2)} \\ x_{3\mu(y_3)} & x_{4\mu(y_4)} \end{pmatrix}$; $x_i, y_i \in \mathbb{R}.$

A is called a 2×2 anti-fuzzy two-fold matrix.

Definition:

Let $a_i = (x_i)_{\mu(y_i)} \quad 1 \leq i \leq 4$, $b_i = (z_i)_{\mu(t_i)}$; $1 \leq i \leq 4$ and $x_i, y_i, z_i, t_i \in \mathbb{R}$, $a_i, b_i \in \mathbb{R}_F$.

Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$, we define:

$A + B = \begin{pmatrix} a_1 * b_1 & a_2 * b_2 \\ a_3 * b_3 & a_4 * b_4 \end{pmatrix}$; $a_i * b_i = (x_i + z_i)_{\max(\mu(y_i), \mu(t_i))}$

$-A = \begin{pmatrix} -a_1 & -a_2 \\ -a_3 & -a_4 \end{pmatrix}; -a_i = (-x_i)_{\mu(-x_i)}$

$A \times B = \begin{pmatrix} (a_1 \circ b_1) * (a_2 \circ b_3) & (a_1 \circ b_2) * (a_2 \circ b_4) \\ (a_3 \circ b_1) * (a_4 \circ b_3) & (a_3 \circ b_2) * (a_4 \circ b_4) \end{pmatrix}$, where $a_i \circ b_j = (x_i z_j)_{\min(\mu(x_i), \mu(t_j))}$

Example:

Take: $\mu: \mathbb{R} \to [0,1]: \mu(x) = \begin{cases} 0 & ; x = 0 \\ 1 & ; x = 1 \\ \frac{1}{2} & ; x > 1 \\ \frac{1}{3} & ; x < 0 \\ \frac{1}{4} & ; 0 < x < 1 \end{cases}$

And: $a_1 = 3_{\mu(2)} = 3_{\frac{1}{2}}$, $a_2 = 4_{\mu\left(\frac{1}{3}\right)} = 4_{\frac{1}{4}}$,

$$a_3 = 1_{\mu(-5)} = 1_{\frac{1}{3}}, a_4 = 0_{\mu(0)} = 0_0 \ ,$$

$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} 3_{\frac{1}{2}} & 4_{\frac{1}{4}} \\ 1_{\frac{1}{3}} & 0_0 \end{pmatrix}$.

$$b_1 = (-2)_{\mu(1)} = (-2)_1 , b_2 = (-1)_{\mu(7)} = (-1)_{\frac{1}{2}}, b_3 = (1)_{\mu(0)} = 1_0 ,$$

$b_4 = (2)_{\mu\left(\frac{1}{10}\right)} = 2_{\frac{1}{4}}$, $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} (-2)_1 & (-1)_{\frac{1}{2}} \\ 1_0 & 2_{\frac{1}{4}} \end{pmatrix}$.

We have:

$-A = \begin{pmatrix} (-3)_{\mu(-2)} & (-4)_{\mu\left(\frac{-1}{3}\right)} \\ (-1)_{\mu(5)} & (-0)_{\mu(-0)} \end{pmatrix} = \begin{pmatrix} (-3)_{\frac{1}{3}} & (-4)_{\frac{1}{3}} \\ (-1)_{\frac{1}{2}} & 0_0 \end{pmatrix}$.

$A + B = \begin{pmatrix} a_1 * b_1 & a_2 * b_2 \\ a_3 * b_3 & a_4 * b_4 \end{pmatrix} = \begin{pmatrix} 1_1 & (3)_{\frac{1}{2}} \\ 2_{\frac{1}{3}} & 2_{\frac{1}{4}} \end{pmatrix}$; $\begin{cases} a_1 * b_1 = (3-2)_1 = 1_1 \\ a_2 * b_2 = (4-1)_{\frac{1}{2}} = 3_{\frac{1}{2}} \\ a_3 * b_3 = (1+1)_{\frac{1}{3}} = 2_{\frac{1}{3}} \\ a_4 * b_4 = (0+2)_{\frac{1}{4}} = 2_{\frac{1}{4}} \end{cases}$

The Algebraic Properties of two-fold anti-fuzzy Matrix Operations:

Property (1):

Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$; $\begin{cases} a_i = (x_i)_{\mu(y_i)} & ; x_i, y_i \in \mathbb{R} \\ b_i = (z_i)_{\mu(t_i)} & ; z_i, t_i \in \mathbb{R} \end{cases}$

We have: $a_i * b_i = (x_i + z_i)_{\max(\mu(y_i),\mu(t_i))} = (z_i + x_i)_{\max(\mu(y_i),\mu(t_i))} = b_i * a_i$,

then $A + B = B + A$.

Property (2):

Let $C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$; $c_i = (m_i)_{\mu(n_i)}$, then:

$$B+C=\begin{pmatrix} b_1*c_1 & b_2*c_2 \\ b_3*c_3 & b_4*c_4 \end{pmatrix}, A+(B+C)=\begin{pmatrix} a_1*b_1*c_1 & a_2*b_2*c_2 \\ a_3*b_3*c_3 & a_4*b_4*c_4 \end{pmatrix}=(A+B)+C.$$

Property (3):

$$A-A=A+(-A)=\begin{pmatrix} (x_1)_{\mu(y_1)} & (x_2)_{\mu(y_2)} \\ (x_3)_{\mu(y_3)} & (x_4)_{\mu(y_4)} \end{pmatrix}+\begin{pmatrix} (-x_1)_{\mu(-y_1)} & (-x_2)_{\mu(-y_2)} \\ (-x_3)_{\mu(-y_3)} & (-x_4)_{\mu(-y_4)} \end{pmatrix}=\begin{pmatrix} 0_{l_1} & 0_{l_2} \\ 0_{l_3} & 0_{l_4} \end{pmatrix};$$

$l_i=\max(\mu(y_i),\mu(-y_i))$.

Property (4):

$$A\times(B\times C)=\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\times\begin{pmatrix} (b_1\circ c_1)*(b_2\circ c_3) & (b_1\circ c_2)*(b_2\circ c_4) \\ (b_3\circ c_1)*(b_4\circ c_3) & (b_3\circ c_2)*(b_4\circ c_4) \end{pmatrix}=\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\times$$

$$\begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix}=\begin{pmatrix} (a_1\circ d_1)*(a_2\circ d_3) & (a_1\circ d_2)*(a_2\circ d_4) \\ (a_3\circ d_1)*(a_4\circ d_3) & (a_3\circ d_2)*(a_4\circ d_4) \end{pmatrix}.$$

We have:

$a_1\circ d_1=a_1\circ[(b_1\circ c_1)*(b_2\circ c_3)]=(a_1\circ b_1\circ c_1)*(a_1\circ b_2\circ c_3)$, $a_2\circ d_3=a_2\circ[(b_3\circ c_1)*(b_4\circ c_3)]=(a_2\circ b_3\circ c_1)*(a_2\circ b_4\circ c_3)$,

Put

$T_1=(a_1\circ d_1)*(a_2\circ d_3)=(a_1\circ b_1\circ c_1)*(a_1\circ b_2\circ c_3)*(a_2\circ b_3\circ c_1)*(a_2\circ b_4\circ c_3)$.

$a_1\circ d_2=a_1\circ[(b_1\circ c_2)*(b_2\circ c_4)]=(a_1\circ b_1\circ c_2)*(a_1\circ b_2\circ c_4)$, $a_2\circ d_4=a_2\circ[(b_3\circ c_2)*(b_4\circ c_4)]=(a_2\circ b_3\circ c_2)*(a_2\circ b_4\circ c_4)$,

$T_2=(a_1\circ d_2)*(a_2\circ d_4)=(a_1\circ b_1\circ c_2)*(a_1\circ b_2\circ c_4)*(a_2\circ b_3\circ c_2)*(a_2\circ b_4\circ c_4)$,

$a_3\circ d_1=a_3\circ[(b_1\circ c_1)*(b_2\circ c_3)]=(a_3\circ b_1\circ c_1)*(a_3\circ b_2\circ c_3)$, $a_4\circ d_3=a_4\circ[(b_3\circ c_1)*(b_4\circ c_3)]=(a_4\circ b_3\circ c_1)*(a_4\circ b_4\circ c_3)$,

Put

$T_3=(a_3\circ d_1)*(a_4\circ d_3)=(a_3\circ b_1\circ c_1)*(a_3\circ b_2\circ c_3)*(a_4\circ b_3\circ c_1)*(a_4\circ b_4\circ c_3)$,

$a_3\circ d_2=a_3\circ[(b_1\circ c_2)*(b_2\circ c_4)]=(a_3\circ b_1\circ c_2)*(a_3\circ b_2\circ c_4)$, $a_4\circ d_4=a_4\circ[(b_3\circ c_2)*(b_4\circ c_4)]=(a_4\circ b_3\circ c_2)*(a_4\circ b_4\circ c_4)$,

$T_4=(a_3\circ d_2)*(a_4\circ d_4)=(a_3\circ b_1\circ c_2)*(a_3\circ b_2\circ c_4)*(a_4\circ b_3\circ c_2)*(a_4\circ b_4\circ c_4)$.

Thus $A\times(B\times C)=\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$.

On the other hand, we have:

$(A \times B) \times C = \begin{pmatrix} (a_1 \circ b_1) * (a_2 \circ b_3) & (a_1 \circ b_2) * (a_2 \circ b_4) \\ (a_3 \circ b_1) * (a_4 \circ b_3) & (a_3 \circ b_2) * (a_4 \circ b_4) \end{pmatrix} \times \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}$, where

$k_1 = [(a_1 \circ b_1) * (a_2 \circ b_3)] \circ c_1 * [(a_1 \circ b_2) * (a_2 \circ b_4)] \circ c_3 = (a_1 \circ b_1 \circ c_1) * (a_2 \circ b_3 \circ c_1) *$

$(a_1 \circ b_2 \circ c_3) * (a_2 \circ b_4 \circ c_3) = T_1$.

By a similar argument, we can see clearly that:

$k_2 = T_2, k_3 = T_3, k_4 = T_4$, thus $A \times (B \times C) = (A \times B) \times C$.

Property (5):

$A \times (B + C) = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \times \begin{pmatrix} b_1 * c_1 & b_2 * c_2 \\ b_3 * c_3 & b_4 * c_4 \end{pmatrix} = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}$, where:

$F_1 = a_1 \circ (b_1 * c_1) * a_2 \circ (b_3 * c_3) = (a_1 \circ b_1) * (a_1 \circ c_1) * (a_2 \circ b_3) * (a_2 \circ c_3)$,

$F_2 = a_1 \circ (b_2 * c_3) * a_2 \circ (b_4 * c_4) = (a_1 \circ b_2) * (a_1 \circ c_3) * (a_2 \circ b_4) * (a_2 \circ c_4)$,

$F_3 = a_3 \circ (b_1 * c_1) * a_4 \circ (b_3 * c_3) = (a_3 \circ b_1) * (a_3 \circ c_1) * (a_4 \circ b_3) * (a_4 \circ c_3)$,

$F_4 = a_3 \circ (b_2 * c_2) * a_4 \circ (b_4 * c_4) = (a_3 \circ b_2) * (a_3 \circ c_2) * (a_4 \circ b_4) * (a_4 \circ c_4)$,

On the other hand, we have:

$(A \times B) + (A \times C) = \begin{pmatrix} (a_1 \circ b_1) * (a_2 \circ b_3) & (a_1 \circ b_2) * (a_2 \circ b_4) \\ (a_3 \circ b_1) * (a_4 \circ b_3) & (a_3 \circ b_2) * (a_4 \circ b_4) \end{pmatrix} +$

$\begin{pmatrix} (a_1 \circ c_1) * (a_2 \circ c_3) & (a_1 \circ c_2) * (a_2 \circ c_4) \\ (a_3 \circ c_1) * (a_4 \circ c_3) & (a_3 \circ c_2) * (a_4 \circ c_4) \end{pmatrix} = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}$.

Which implies that $A \times (B + C) = (A \times B) + (A \times C)$.

Result:

If TF_μ is the set of all (2×2) two-fold anti- fuzzy matrices with two-fold real entries, then:

1] $(+)$ is a commutative and associative.

2] $(\times)$ is associative.

3] $(\times)$ is a distributive with respect to $(+)$.

Definition:

A (2×2) two-fold zero anti- fuzzy matrix is defined as:

$O = \begin{pmatrix} 0_0 & 0_0 \\ 0_0 & 0_0 \end{pmatrix}$.

Remark:

For any $A \in TF_\mu$, we have $A + O = O + A = A$.

It is clear that $A \times O = O$.

The additive inverse of $A = \begin{pmatrix} (x_1)_{\mu(y_1)} & (x_2)_{\mu(y_2)} \\ (x_3)_{\mu(y_3)} & (x_4)_{\mu(y_4)} \end{pmatrix}$ is not existed.

Definition:

$A\ (2 \times 2)$ two-fold anti-fuzzy unitary matrix is:

$I = \begin{pmatrix} 1_1 & 0_0 \\ 0_0 & 1_1 \end{pmatrix}$

Remark:

For $A = \begin{pmatrix} (x_1)_{\mu(y_1)} & (x_2)_{\mu(y_2)} \\ (x_3)_{\mu(y_3)} & (x_4)_{\mu(y_4)} \end{pmatrix}$, we have:

$A \times I = \begin{pmatrix} (x_1)_{\mu(y_1)} & (x_2)_{\mu(y_2)} \\ (x_3)_{\mu(y_3)} & (x_4)_{\mu(y_4)} \end{pmatrix} \times \begin{pmatrix} 1_1 & 0_0 \\ 0_0 & 1_1 \end{pmatrix} = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix}$;

$L_1 = \left((x_1)_{\mu(y_1)} \circ 1_1\right) * \left((x_2)_{\mu(y_2)} \circ (0_0)\right) = (x_1)_{\mu(y_1)} * (0_0) = (x_1)_{\mu(y_1)}$

$L_2 = \left((x_1)_{\mu(y_1)} \circ 0_0\right) * \left((x_2)_{\mu(y_2)} \circ 1_1\right) = (0_0) * (x_2)_{\mu(y_2)} = (x_2)_{\mu(y_2)}$

$L_3 = (x_3)_{\mu(y_3)}$, $L_4 = (x_4)_{\mu(y_4)}$,

thus $A \times I = I \times A = A$.

Finding the multiplicative inverse:

Let $A = \begin{pmatrix} (x_1)_{\mu(y_1)} & (x_2)_{\mu(y_2)} \\ (x_3)_{\mu(y_3)} & (x_4)_{\mu(y_4)} \end{pmatrix} \in TF_\mu$; $x_1x_4 - x_2x_3 \neq 0$

Put $B = \begin{pmatrix} (x_4)_{\mu(z_4)} & (-x_2)_{\mu(z_2)} \\ (-x_3)_{\mu(z_3)} & (x_1)_{\mu(z_1)} \end{pmatrix}$, and compute:

$A \times B = \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix}$ such that:

$\mu_1 = \left[(x_1)_{\mu(y_1)} \circ (x_4)_{\mu(z_4)}\right] * \left[(x_2)_{\mu(y_2)} \circ (-x_3)_{\mu(z_3)}\right] = (x_1x_4 - x_2x_3)_{l_1}$;

$l_1 = \max\left[\min\left(\mu(y_1), \mu(z_4)\right), \min\left(\mu(y_2), \mu(z_3)\right)\right] = 1$,

We can not get a similar result by choosing $\mu(z_3), \mu(z_4)$

which means that A is not invertible.

Conclusion

In this paper we have studied for the first time the concept of square 2×2 fuzzy and anti-fuzzy two-fold matrix with real entries, where we presented the two-fold algebraic operations between the these matrices and obtain their special properties such as

associativity, commutative properties, and the existing of additive of multiplicative inverses. Also, we provided many examples to explain the two-fold algebraic properties and the two-fold algebraic structure.

References

1. Abobala, M., On Refined Neutrosophic Matrices and Their Applications In Refined Neutrosophic Algebraic Equations, Journal Of Mathematics, Hindawi, 2021
2. Olgun, N., Hatip, A., Bal, M., and Abobala, M., " A Novel Approach To Necessary and Sufficient Conditions For The Diagonalization of Refined Neutrosophic Matrices", International Journal of neutrosophic Science, Vol. 16, pp. 72-79, 2021.
3. Abobala, M., Hatip, A., and Bal, M., " A Review On Recent Advantages In Algebraic Theory Of Neutrosophic Matrices", International Journal of Neutrosophic Science, Vol.17, 2021.
4. Abobala, M., Ziena, B,M., Doewes, R., and Hussein, Z., "The Representation Of Refined Neutrosophic Matrices By Refined Neutrosophic Linear Functions", International Journal Of Neutrosophic Science, 2022.
5. Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Symbolic 8-Plithogenic Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.
6. Abobala, M., On The Representation of Neutrosophic Matrices by Neutrosophic LinearTransformations, Journal of Mathematics, Hindawi, 2021.
7. Florentine Smarandache. "Neutrosophic Two-Fold Algebra", Plithogenic Logic and Computation, Vol.1, No.1 2024. PP.11-15.
8. Mohammad Abobala. (2023). On a Two-Fold Algebra Based on the Standard Fuzzy
   Number Theoretical System. Journal of Neutrosophic and Fuzzy Systems, 7 (2), 24-29
   (Doi : https://doi.org/10.54216/JNFS.070202).
9. Ahmed Hatip, Necati Olgun. (2023). On the Concepts of Two- Fold Fuzzy Vector Spaces and Algebraic Modules. Journal of Neutrosophic and Fuzzy Systems, 7 (2), 46-52
   (Doi : https://doi.org/10.54216/JNFS.070205).

10. Ramazan Yasar, and Ahmed Hatip, The Structure of The Binary Two Fold Algebra Based On Intuitionistic Fuzzy Groups, Neutrosophic Sets and Systems, Vol. 72, 2024, pp. 134-141, DOI: 10.5281/zenodo.13381912.
11. A. S. Heilat, H. Zureigat, R. Hatamleh, B. Batiha, New Spline Method for Solving Linear Two-Point Boundary Value Problems. uropean Journal of Pure and Applied Mathematics 14 (4), 1283-1294, 2021
12. Al-Husban, A. B. D. U. L. L. A. H., Al-Sharoa, D. O. A. A., Al-Kaseasbeh, M. O. H. A. M. M. A. D., & Mahmood, R. M. S. (2022). Structures of fibers of groups actions on graphs. WSEAS Trans. Math, 7, 650-658.
13. Al-Husban, a. b. d. a. l. l. a. h., al-sharoa, d. o. a. a., saadeh, r., qazza, a., & mahmood, r. (2023). On the fibers of the tree products of groups with amalgamation subgroups. Journal of applied mathematics & informatics, 41(6), 1237-1256.
14. Al-Shbeil, Isra. , Abu, Ibraheem. , Alkhouli, Talat. , Soiman, Ahmed. , Shtayat, Jenan. , Al-Husban, Abdallah. On the Two-Fold Maximal Units in Some Two-Fold Finite Neutrosophic Rings Modulo Integers for 2≤n≤5. International Journal of Neutrosophic Science, vol. , no. , 2025, pp. 155-164. DOI: https://doi.org/10.54216/IJNS.250213
15. Shihadeh, A., Matarneh, K. A. M., Hatamleh, R., Al-Qadri, M. O., & Al-Husban, A. (2024). On The Two-Fold Fuzzy n-Refined Neutrosophic Rings For 2≤ 3. Neutrosophic Sets and Systems, 68, 8-25.
16. Abdallah Shihadeh, Khaled Ahmad Mohammad Matarneh, Raed Hatamleh, Randa Bashir Yousef Hijazeen, Mowafaq Omar Al-Qadri, Abdallah Al-Husban. (2024). An Example of Two-Fold Fuzzy Algebras Based On Neutrosophic Real Numbers, Neutrosophic Sets and Systems, Vol. 67.
17. C. Sivakumar, Mowafaq Omar Al-Qadri, Abdallah shihadeh, Ahmed Atallah Alsaraireh, Abdallah Al-Husban, P. Maragatha Meenakshi, N. Rajesh, M. Palanikumar. (2024). q-rung square root interval-valued neutrosophic sets with respect to aggregated operators using multiple attribute decision making. Journal of , 23 (3), 154-174 (Doi : https://doi.org/10.54216/IJNS.230314)
18. Selvaraj, S., Gharib, G., Al-Husban, A., Al Soudi, M., Kumaran, K. L. M., Palanikumar, M., & Sundareswari, K. New algebraic approach towards interval-valued neutrosophic cubic vague set based on subbisemiring over bisemiring. International Journal of Neutrosophic Science, 23(4), 386-394.
19. Falahah, I. A., Raman, T. T., Al-Husban, A., Alahmade, A., Azhaguvelavan, S., & Palanikumar, M. (2024) Computer purchasing using new type neutrosophic sets

and its extension based on aggregation operators. International Journal of Neutrosophic Science, 23(4), 386-394.

20. A.Abubaker, Ahmad. , M., Wael. , Alrawashdeh, Heba. , Shatarah, Amani. , Mousa, Norah. , Al-Husban, Abdallah. The Mathematical Formulas for Inverting Plithogenic Matrices of Special Orders Between 20 and 24. International Journal of Neutrosophic Science, vol. , no. , 2024, pp. 102-114. DOI: https://doi.org/10.54216/IJNS.240309.
21. Abu, Ibraheem. , Al-Husban, Abdallah. , M., Mutaz. , A., Ahmed. , Shatarah, Amani. , Mousa, Norah. On the Characterization of Some m-Plithogenic Vector Spaces and Their AH-Substructures Under the Condition 6 ≤ dim SPV ≤10. International Journal of Neutrosophic Science, vol.*24* , no. *3*, 2024, pp. 258-267. DOI: https://doi.org/10.54216/IJNS.240322.
22. Isra Al-Shbeil, Wael Mahmoud M. Salameh, Hossam Almahasneh, Khalid Kaabneh, Mutaz Shatnawi, Khaled Al-Husban. (2024). Algorithms for Computing M-Plithogenic Eigen Values and Vectors for Some Special Values of M. International Journal of Neutrosophic Science, 24 (1), 128-135 (Doi : https://doi.org/10.54216/IJNS.240112).
23. Raed Hatamleh, Ayman Hazaymeh.(2024). Finding Minimal Units In Several Two-Fold Fuzzy
Finite Neutrosophic Rings, Neutrosophic Sets and Systems,70,1-16.(DOI:10.5281/zenodo.13160839).

Acknowledgments

The authors extend their appreciation to the Arab Open University for funding this work through AOU research fund no. (AOUKSA-524008).

Received: June 1, 2024. Accepted: Oct 12, 2024

NSS
Neutrosophic Sets and Systems, Vol. 75, 2025
University of New Mexico

On The Continuous and Differentiable Two-Fold Neutrosophic and Fuzzy Real Functions

Raed Hatamleh
Department of Mathematics, Faculty of Science and Information Technology, Jadara University, P.O. Box 733, Irbid 21110, Jordan
Email: raed@jadara.edu.jo

Abstract

This paper is dedicated to defining the concepts of two-fold neutrosophic continuous functions, two-fold fuzzy continuous functions, and two-fold neutrosophic/fuzzy differentiable functions for the first time.

The elementary properties of these novel concepts will be studied and handled through many theorems, as well as many clear examples that clarify the validity of these analytical concepts.

Keywords: two-fold algebra, two-fold neutrosophic function, two-fold fuzzy function, two-fold differentiability.

Introduction

Two-fold neutrosophic algebras are new algebraic structures presented by Smarandache [1] by combining neutrosophic values of truth, falsity, and indeterminacy with classical algebraic sets. These ideas were used by many authors to generalize other famous algebraic structures such as two-fold fuzzy number theoretical systems [2-3], two-fold modules and spaces [4], and two-fold fuzzy rings [5]. Also, they were used in the study of some special two-fold complex functions such as Gamma function [7], and in extending n-refined neutrosophic rings [6].

Neutrosophic real functions and their applications were studied by many authors [8-25] in many different ways.

This has motivated us to introduce the concept of two-fold neutrosophic continuous functions, two-fold fuzzy continuous functions, and two-fold neutrosophic/fuzzy differentiable functions for the first time**.** The elementary properties of these novel concepts will be studied and handled through many theorems, as well as many clear examples that clarify the validity of these analytical concepts.

This study will be very important in the near future to generalize classical real analysis into an extended version based on two-fold algebra of real numbers.

Main Discussion

Definition 1.

Let $i, f, t: \mathbb{R} \to [0,1]$ be three real functions denote to the neutrosophic ordinary real values of truth, indeterminacy and falsity. Let $g: \mathbb{R} \to \mathbb{R}$ be a real function in one variable $g = g(x); x \in \mathbb{R}$. We define the corresponding two-fold neutrosophic real function as follows:

$g_N: R \to R_{(t,i,f)} \ : \ g_N(x) = (g(x))_{(t(x),i(x),f(x))}$

Example 1.

Let

$$i, t, f: \mathbb{R} \to [0,1],$$

such that

$$\begin{cases} i(x) = \min\left(|x|, \dfrac{1}{|x|}\right); x \neq 0 \\ t(x) = \min\left(|x^3|, \dfrac{1}{|x^3|}\right); x \neq 0 \\ f(x) = \min\left(|x^2|, \dfrac{1}{|x^2|}\right); x \neq 0 \end{cases} \text{ and } \begin{cases} i(0) = 1 \\ t(0) = 0 \\ f(0) = \dfrac{1}{2} \end{cases}.$$

Consider

$$g: \mathbb{R} \to \mathbb{R}; g(x) = x^2 + 1,$$

then

$$g_N: \mathbb{R} \to \mathbb{R}_{(t,i,f)},$$

such that:

$$g_N(x) = \begin{cases} (x^2+1)_{(t(x),i(x),f(x))} & ; x \neq 0 \\ (1)_{(0,1,\frac{1}{2})} & ; x = 0 \end{cases}.$$

For example if $x = 2$, then

$$t(x) = \frac{1}{8}, i(x) = \frac{1}{2}, f(x) = \frac{1}{4}, g(x) = 5,$$

$$g_N(x) = (5)_{(\frac{1}{8},\frac{1}{2},\frac{1}{4})}.$$

Definition 2.

Let

$$g_N: \mathbb{R} \to \mathbb{R}_{(t,i,f)},$$

be a two-fold neutrosophic real function, we say that:

1) g_N is fully differential at $x_0 \in \mathbb{R}$ if:

$$\begin{cases} g'(x_0) \text{ is existed} \\ i'(x_0), f'(x_0), t'(x_0) \text{ are existed} , \\ i'(x_0), f'(x_0), t'(x_0) \in [0,1] \end{cases}$$

2) g_N is T-differential (differential with respect to truth component) at $x_0 \in \mathbb{R}$ if:

$$\begin{cases} g'(x_0) \text{ is existed} \\ t'(x_0) \in [0,1] \\ t'(x_0) \text{ is existed} \end{cases}$$

3) g_N is I-differential (differential with respect to the indeterminacy component) at $x_0 \in \mathbb{R}$ if:

$$\begin{cases} g'(x_0) \text{ is existed} \\ i'(x_0) \text{ is existed} , \\ i'(x_0) \in [0,1] \end{cases}$$

4) g_N is F-differential (differential with respect to the falisty component) at $x_0 \in \mathbb{R}$ if:

$$\begin{cases} g'(x_0) \text{ is existed} \\ f'(x_0) \text{ is existed} . \\ f'(x_0) \in [0,1] \end{cases}$$

Example 2.

Take: $f(x) = \min\left(\frac{1}{x^2}, x^2\right)$, $i(x) = \min\left(\frac{1}{x^4}, x^4\right)$, $t(x) = \min\left(|x|, \frac{1}{|x|}\right)$, $t(0) = i(0) = f(0) = \frac{1}{2}$.

$g(x) = x^2 + x + 1$.

Hence,

$$g_N(x): \mathbb{R} \to \mathbb{R}_{(t,i,f)} ,$$

$$g_N(x) = \begin{cases} (x^2 + x + 1)_{(t(x),i(x),f(x))} & ; x \neq 0 \\ (1)_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)} & ; x = 0 \end{cases} .$$

It is easy to see that :

$g'(x_0)$ is existed for all $x_0 \neq 0$,

$t'(x_0), f'(x_0), i'(x_0)$ are existed for all $x_0 \neq 0$,

$t'(x_0), f'(x_0), i'(x_0)$ are not existed for all $x_0 = 0$.

Hence, g_N is not differential for $x_0 = 0$.

For $x_0 = 2$, we have:

$$g'(2) = 5, f(2) = \frac{1}{4}, f'(2) < 0, i(2) = \frac{1}{16}, i'(2) < 0, t(2) = \frac{1}{2}.$$

$$\lim_{x\to 2} \frac{t(x)-t(2)}{x-2} = \lim_{x\to 2} \frac{\frac{1}{|x|}-\frac{1}{2}}{x-2} = \lim_{x\to 2} \frac{\frac{2-|x|}{2|x|}}{x-2} = \lim_{x\to 2} \frac{-1}{2|x|} = -\frac{1}{4} < 0.$$

Hence g_N is not fully differentiable for $x_0 = 2$.

Example 3.

Let $g(x) = x^3 - x - 1$, $i(x) = f(x) = t(x) = \frac{1}{4}$,

then $i'(x) = f'(x) = t'(x) = 0$ for all $x \in \mathbb{R}$.

Hence g_N is fully differentiable for all $x \in \mathbb{R}$.

Remark1.

The set of all points $x \in \mathbb{R}$ for which $g_N(x)$ is fully differentiable is denoted by $FD(g_N)$.

Example 4.

Try to find $FD(g_N)$ for:

$$g(x) = 3x^2 + x - 5\,, i(x) = \begin{cases} \frac{1}{2} & ;x \neq 0 \\ 0 & ;x = 0 \end{cases}, t(x)$$

$$= \begin{cases} \frac{1}{x} & x \geq 1 \\ \frac{-1}{x} & x \leq -1 \\ |x| & -1 < x < 1 \end{cases}, f(x) = \begin{cases} \frac{1}{x^2} & x \geq 1, x \leq -1 \\ x^2 & ;-1 < x < 1 \end{cases},$$

$$g_N(x) = \begin{cases} (3x^2 + x - 5\,)_{(\frac{1}{x}, \frac{1}{2}, \frac{1}{x^2})} & x \geq 1 \\ (3x^2 + x - 5\,)_{(|x|, \frac{1}{2}, x^2)} & -1 < x < 1\,, x \neq 0 \\ (3x^2 + x - 5\,)_{(\frac{-1}{x}, \frac{1}{2}, \frac{1}{x^2})} & x \leq -1 \end{cases}.$$

We remark the following:

The function (g) is differentiable on $\mathbb{R}$, and $g'^{(x)} = 6x + 1$.

The function $i(x)$ is differentiable an $\mathbb{R}^*$ with $i'(x) = 0$ for all $x \neq 0$.

The function $t(x)$ is differentiable on R^* with:

$$t'(x) = \begin{cases} \frac{-1}{x^2} & x \geq 1 \\ \frac{1}{x^2} & x \leq -1 \\ 1 & 0 < x < 1 \\ -1 & -1 < x < 0 \end{cases}.$$

The function $f(x)$ is differentiable an $\mathbb{R}$ with:

$$f'(x) = \begin{cases} \frac{-2}{x^3} & x \geq 1, x \leq -1 \\ 2x & -1 < x < 1 \end{cases}.$$

We can see that:

$$t'(x) \in [0,1] \Leftrightarrow x \in]0,1[\cup]-\infty, -1]\,,$$

$$f'(x) \in [0,1] \Leftrightarrow x \in \left]0, \frac{1}{2}\right[\cup]-\infty, -\sqrt[3]{2}[.$$

$$i'(x) \in [0,1] \Leftrightarrow x \in \mathbb{R}^*.$$

Thus

$$FD(g_N) = \left]0, \frac{1}{2}\right[\cup \left]-\infty, -\sqrt[3]{2}\right[.$$

Definition 3.

Let g_N be a differential two-fold neutrosophic function on $I \subseteq R$, we define

$$g'_N(x) = \left(g'(x)\right)_{\left(t'(x), i'(x), f'(x)\right)}; x \in I.$$

In the previous example, we can see:

$$g'_N(x) = \begin{cases} (6x+1)_{(1,0,2x)} & x \in \left]0, \frac{1}{2}\right[\\ (6x+1)_{\left(\frac{1}{x^2}, 0, \frac{-2}{x^3}\right)} & x \in \left]-\infty, -\sqrt[3]{2}\right[\end{cases}.$$

Definition 4.

Let $g_N, h_N: \mathbb{R} \to \mathbb{R}_{(t,i,f)}$ be two-fold neutrosophic real functions, we define:

1) $(g_N + h_N)(x) = \left(g(x) + h(x)\right)_{(t(x), i(x), f(x))'}$

where

$$\begin{cases} t(x) = \min(t_1(x), t_2(x)) \\ f(x) = \max(f_1(x), f_2(x)) \\ i(x) = \max(i_1(x), i_2(x)) \end{cases},$$

2) $(-g_N)(x) = \left(-g(x)\right)_{(t_1(x), i_1(x), f_1(x))}$,

3) $(g_N . h_N)(x) = \left(g(x).h(x)\right)_{(t(x), i(x), f(x))'}$

where,

$$\begin{cases} t(x) = \max(t_1(x), t_2(x)) \\ f(x) = \min(f_1(x), f_2(x)) \\ i(x) = \min(i_1(x), i_2(x)) \end{cases},$$

4) For $h_N(x) \neq 0, \left(\frac{1}{h_N}\right)(x) = \left(\frac{1}{h(x)}\right)_{(t(x), i(x), f(x))'}$

where,

$$\begin{cases} t(x) = 1 - t_2(x) \\ f(x) = 1 - f_2(x) \\ i(x) = 1 - i_2(x) \end{cases}.$$

Result 1.

From the definition, we get directly:

1) $(g_N - h_N)(x) = \left(g(x) - h(x)\right)_{(t(x), i(x), f(x))'}$

where,

$$\begin{cases} t(x) = \min(t_1(x), t_2(x)) \\ f(x) = \max(f_1(x), f_2(x)) \\ i(x) = \max(i_1(x), i_2(x)) \end{cases},$$

2) $\left(\frac{g_N}{h_N}\right)(x) = \left(\frac{g(x)}{h(x)}\right)_{(t(x),i(x),f(x))'}$

where,

$$\begin{cases} t(x) = \max(t_1(x), 1 - t_2(x)) \\ f(x) = \min(f_1(x), 1 - f_2(x)) \\ i(x) = \min(i_1(x), 1 - i_2(x)) \end{cases},$$

3) $[g_N(x)]^n = ([g(x)]^n)_{(t_1(x),i_1(x),f_1(x))}$.

Theorem 1.

Let $g_N, h_N: \mathbb{R} \to \mathbb{R}_{(t,i,f)}$ be two-fold neutrosophic real functions, then

1) $(g_N + h_N)' = g'_N + h'_N$,

2) $(g_N - h_N)' = g'_N - h'_N$,

3) $(g_N^n)' = n.g_N^{n-1}{}_{(0,1,1)} g'_N$.

Proof.

1) $[(g + h)(x)]' = g'(x) + h'(x), t'(x) = \min\left(t'_1(x), t'_2(x)\right), f'(x) = \max\left(f'_1(x), f'_2(x)\right),\ i'(x) = \max\left(i'_1(x), i'_2(x)\right),$

so that:

$$(g_N + h_N)'(x) = \left(g'(x) + h'(x)\right)_{\left(t'(x),i'(x),f'(x)\right)} = g'_N(x) + h'_N(x).$$

2) it can be proved by a similar argument.

3) $[g^n(x)]' = n.g'(x).g^{n-1}(x)$,and

$$\left(g_N^n(x)\right)' = \left(n.g'(x)g_N^{n-1}(x)\right)_{\left(t'_1(x),i'_1(x),f'_1(x)\right)} = n.\left(g'(x)\right)_{\left(t'_1(x),i'_1(x),f'_1(x)\right)}.\left(g_N^{n-1}(x)\right)_{(0,1,1)}$$

$$= n.g'_N(x).\left(g_N^{n-1}(x)\right)_{(0,1,1)}$$

Remark 2.

If g_N, h_N are fully differentiable on $I \subseteq R$, then

$$\left(g_N, h_N(x)\right)' = (g'h(x), h'g(x))_{(t'(x),i'(x),f'(x))},$$

where,

$$\begin{cases} t'(x) = \max(t'_1(x), t'_2(x)) \\ i'(x) = \min(i'_1(x), i'_2(x)) \\ f'(x) = \min(f'_1(x), f'_2(x)) \end{cases}.$$

Definition 5.

We define

$$\left(\frac{1}{h_N}\right)'(x) = \left(\frac{-h'(x)}{h^2(x)}\right)_{(1-t'_2(x),1-i'_2(x),1-f'_2(x))}.$$

Result 2.

The derivative $\left(\frac{g_N}{h_N}\right)'(x) = \left(\frac{(g'.h-h'g)(x)}{h^2(x)}\right)_{(t(x),i(x),f(x))'}$

where,

$$\begin{cases} t(x) = \max(t_1'(x), 1-t_2'(x)) \\ i(x) = \min(i_1'(x), 1-i_2'(x)) \\ f(x) = \min(f_1'(x), 1-f_2'(x)) \end{cases}.$$

Example 5.

Take: $g(x) = 3x^2 + 1, h(x) = x^3$ as two real functions.

$$t_1(x) = \begin{cases} 0\ ;\ x > 0, x < 0 \\ 1\ ;x = 0 \end{cases} \quad t_2(x) = \begin{cases} 1 & x \geq 0 \\ \frac{1}{2} & ;x < 0 \end{cases},$$

$$i_1(x) = \begin{cases} \frac{-1}{x} & ;x \leq -1 \\ 0\ ;x > -1 \end{cases} \quad i_2(x) = \begin{cases} \frac{1}{3} & ;x \geq 0 \\ \frac{1}{4} & ;x < 0 \end{cases},$$

$$f_1(x) = \{0 \quad ;x \in \mathbb{R} \quad f_2(x) = \begin{cases} \frac{1}{4} & ;x > 1 \\ \frac{1}{3} & ;-1 \leq x \leq 1 \\ \frac{1}{2} & ;x < -1 \end{cases},$$

$g_N, h_N: \mathbb{R} \to \mathbb{R}_{(t,i,f)}$,

such that:

$g_N(x) = (g(x))_{(t_1,i_1,f_1)} = (3x^2+1)_{(t_1(x),i_1(x),f_1(x))}$,

$h_N(x) = (h(x))_{(t_2,i_2,f_2)} = (x^3)_{(t_2(x),i_2(x),f_2(x))}$,

$g(x)$ is differentiable on $\mathbb{R}$, and $g'(x) = 6x$,

$t_1(x)$ is differentiable on $\mathbb{R}^*$, and $t_1'(x) = 0\ ;x \in \mathbb{R}^*$,

$i_1(x)$ is differentiable on $\mathbb{R}|\{-1\}$, and $i_1'(x) = \begin{cases} \frac{1}{x^2} & x < -1 \\ 0 & x > -1 \end{cases}$,

$f_1(x)$ is differentiable on $\mathbb{R}$, and $f_1'(x) = 0$.

So that $g_N(x)$ is differentiable on $\mathbb{R}|\{0,-1\}$, and

$$g_N'(x) = \begin{cases} (6x)_{\left(0,\frac{1}{x^2},0\right)} & x \in]-\infty,-1[\\ (6x)_{(0,0,0)} & x \in]-1,0[\cup]0,\infty[\end{cases}.$$

On the other hand, we have:

$h(x)$ is differentiable on $\mathbb{R}$, and $h'(x) = 3x^2$,

$t_2(x)$ is differentiable on $\mathbb{R}^*$with $t_2'(x) = 0\ ;x \neq 0$,

$i_2(x)$ is differentiable on $\mathbb{R}^*$ with $i_2'(x) = 0\,; x \neq 0$,

$f_2(x)$ is differentiable on $\mathbb{R}|\{-1,1\}$, and $f_2'(x) = 0\,; x \in \mathbb{R}|\{-1,1\}$.

Thus $h_N(x)$ is fully differentiable on $\mathbb{R}|\{0,1,-1\}$, and

$h_N' = (3x^2)_{(0,0,0)}\ ;\ x \in \mathbb{R}|\{0,1,-1\}$,

$(g_N + h_N)'(x) = (6x + 3x^2)_{(0,\frac{1}{x^2},0)}\ ; x \in \mathbb{R}|\{0,1,-1\}$,

$(g_N - h_N)'(x) = (6x - 3x^2)_{(0,\frac{1}{x^2},0)}\ ; x \in \mathbb{R}|\{0,1,-1\}$,

$(g_N.h_N)'(x) = (6x.x^3 + (3x^2+1).3x^2)_{(0,0,0)} = (15x^4 + 3x^2)_{(0,0,0)}$,

$\left(\frac{g_N}{h_N}\right)'(x) = \left(\frac{6x(x^3)-(3x^2)(3x^2+1)}{x^6}\right)_{(1,0,0)} = \left(\frac{-3x^4-3x^2}{x^6}\right)_{(1,0,0)}\ ; x \in \mathbb{R}|\{0,1,-1\}$.

Definition 6.

Let $\mu: \mathbb{R} \to [0,1]$ be a real fuzzy function, and $g: \mathbb{R} \to \mathbb{R}$ be a real function in one variable $g = g(x)$.

We define the corresponding two-fold fuzzy real function $g_\mu(x) = (g(x))_{\mu(x)}\ x \in \mathbb{R}$.

Example 6.

Consider $\mu: \mathbb{R} \to [0,1]$, $g: \mathbb{R} \to \mathbb{R}$ such that:

$$\mu(x) = \begin{cases} \frac{-1}{x} & x \leq -1 \\ x^2 & -1 < x < 0 \\ x & 0 \leq x \leq 1 \\ \frac{1}{x} & x > 1 \end{cases},$$

and $g(x) = 7x^2 + \cos x$, then: $g_\mu: \mathbb{R} \to \mathbb{R}_F$ such that:

$$g_\mu(x) = \begin{cases} (7x^2 + \cos x)_{\frac{-1}{x}} & x \leq -1 \\ (7x^2 + \cos x)_{x^2} & -1 < x < 0 \\ (7x^2 + \cos x)_x & 0 \leq x \leq 1 \\ (7x^2 + \cos x)_{\frac{1}{x}} & x > 1 \end{cases}.$$

For $x = \pi$, we have

$$g_\mu(\pi) = (7\pi^2 - 1)_{\frac{1}{\pi}},$$

For $x = 0$, we have

$$g_\mu(0) = (1)_0.$$

Definition 7.

Let $g_N: \mathbb{R} \to \mathbb{R}_F$ be a two-fold fuzzy real function in one variable, we say that g_N is differentiable at $x_0 \in \mathbb{R}$ if and only if:

$g'(x_0), \mu'(x_0)$ are existed, and $\mu'(x_0) \in [0,1]$.

Example 7.

Take $\mu(x)\colon \mathbb{R} \to [0,1], g(x)\colon \mathbb{R} \to \mathbb{R}$ such that:

$$\mu(x) = \begin{cases} \frac{1}{2} & x \leq 0 \\ \frac{1}{x^2} & x > 0 \end{cases}, g(x) = 1 + x^2 ,$$

$$g_\mu(x)\colon \mathbb{R} \to \mathbb{R}_F\colon g_\mu(x) = \begin{cases} (1+x^2)_{\frac{1}{2}} & ; x \leq 0 \\ (1+x^2)_{\frac{1}{x^2}} & ; x > 0 \end{cases},$$

g_μ is not differentiable on $[0,\infty[$, that is because $g'(0)$ is not existed, $\mu'(x) \notin [0,1] \ ; x > 0$, g_μ is differentiable on $]-\infty, 0[$, that is because $g'(x), \mu'(x)$ are existed for all $x \in]-\infty, 0[$ and $\mu'(x) = 0$.

Thus $g'(x) = \{(2x)_0 \ ; x < 0$.

Definition 8.

Let $(\mu, \alpha)\colon \mathbb{R} \to [0,1], (g,h)\colon \mathbb{R} \to \mathbb{R}, (g_\mu, h_\alpha)\colon \mathbb{R} \to \mathbb{R}_F$ such that: $g_\mu(x) = (g(x))_{\mu(x)}$, $h_\alpha(x) = (h(x))_{\alpha(x)}$, then:

1) $(g_\mu + h_\alpha)(x) = (g(x) + h(x))_{\min(\mu(x),\alpha(x))}$,
2) $(g_\mu - h_\alpha)(x) = (g(x) - h(x))_{\min(\mu(x),\alpha(x))}$,
3) $(g_\mu)^n(x) = (g^n(x))_{\mu(x)}$,
4) $(g_\mu . h_\alpha)(x) = (g(x).h(x))_{\max(\mu(x),\alpha(x))}$,
5) $\left(\frac{1}{g_\mu}\right)(x) = \left(\frac{1}{g_\mu}\right)_{(1-\mu(x))}$,
6) $\left(\frac{h_\alpha}{g_\mu}\right)(x) = \left(\frac{h(x)}{g_\mu}\right)_{\max(\alpha(x),1-\mu(x))}$.

On the other hand, we define:

1) $(g'(x) + h'(x))_{\min(\mu'(x),\alpha'(x))} = (g_\mu + h_\alpha)'$,
2) $(g'(x) - h'(x))_{\min(\mu'(x),\alpha'(x))} = (g_\mu - h_\alpha)'$,
3) $.(g'(x).g^{n-1}(x))_{\mu'(x)} = (g_\mu^n)'$,
4) $(g'(x).h(x) + h'(x).g(x))_{\max(\mu'(x),\alpha'(x))}$,
5) $\left(\frac{-g\prime(x)}{g^2(x)}\right)_{(1-\mu'(x))} = \left(\frac{1}{g_\mu}\right)'$,
6) $\left(\frac{h_\alpha}{g_\mu}\right)' = \left(\frac{h'(x)g(x)-g'(x)h(x)}{g^2(x)}\right)_{\max(\alpha'(x),1-\mu'(x))}$.

Example 8.

Take $(\mu, \alpha)\colon \mathbb{R} \to [0,1], (g,h)\colon \mathbb{R} \to \mathbb{R}, (g_\mu, h_\alpha)\colon \mathbb{R} \to \mathbb{R}_F$,

such that:

$$g(x)=x+1, h(x)=x^2, \mu(x)=\begin{cases}\frac{-1}{x} & x\le -1\\ 1 & x>-1\end{cases}, \alpha(x)=\begin{cases}\frac{1}{3} & x\le -2\\ 0 & -2<x\le 0\\ \frac{1}{2} & x>0\end{cases},$$

$$g_\mu(x)=\begin{cases}(x+1)_{\frac{-1}{x}} & ; x\le -1\\ (x+1)_1 & ; x>-1\end{cases}, h_\alpha(x)=\begin{cases}(x^2)_{\frac{1}{3}} & ; x\le -2\\ (x^2)_0 & ; -2<x\le 0\\ (x^2)_{\frac{1}{2}} & ; x>0\end{cases},$$

$h(x), g(x)$ are differentiable on R, with $g'(x)=1, h'(x)=2x$.

$\alpha(x)$ is differentiable on $\mathbb{R}|\{-2,0\}$with $\alpha'(x)=0$ for all $x\in\mathbb{R}|\{-2,0\}$.

For $\mu(x)$ at $x=-1$, we have:

$$\frac{\mu(x)-\mu(-1)}{x+1}=\frac{\mu(x)-1}{x+1},$$

$$\lim_{x\to -1^+}\frac{\mu(x)-\mu(-1)}{x+1}=\lim_{x\to -1^+}\frac{1-1}{x+1}=0,$$

$$\lim_{x\to -1^-}\frac{\mu(x)-\mu(-1)}{x+1}=\lim_{x\to -1^-}\frac{\frac{-1}{x}-1}{x+1}=\lim_{x\to -1^-}\frac{\frac{-1-x}{x}}{x+1}=\lim_{x\to -1^-}\frac{-1}{x}=1,$$

hence $\mu(x)$ is differentiable at $x=-1$,

$\mu(x)$ is differentiable on $\mathbb{R}|\{-1\}$, and

$$\mu'(x)=\begin{cases}\frac{1}{x^2} & x\le -1\\ 0 & ; x>-1\end{cases},$$

$$(g_\mu+h_\alpha)'(x)=(x+1+x^2)'_{\min(\mu'(x),0)}=(2x+1)_0 \quad ; x\in\mathbb{R}|\{-2,0,-1\},$$

$$(g_\mu-h_\alpha)(x)=(1-2x)_{\min(\mu'(x),0)}=(1-2x)_0 \quad ; x\in\mathbb{R}|\{-2,0,-1\},$$

$$(g_\mu^n)'(x)=n.(x+1)^{n-1}_{\mu'(x)}=\begin{cases}n.(x+1)^{n-1}_{\frac{1}{x^2}} & ; x\le -1\\ n.(x+1)^{n-1}_0 & ; x>-1\end{cases},$$

$$(g_\mu.h_\alpha)'(x)=(3x^2+2x)_{\max(\mu'(x),\alpha'(x))}=\begin{cases}(3x^2+2x)_{\frac{1}{x^2}} & ; x\le -1, x\ne -2\\ (3x^2+2x)_0 & ; x>-1, x\ne 0\end{cases},$$

and so on.

Definition 9.

Let $g_N:\mathbb{R}\to\mathbb{R}_{(t,i,f)}$ be a two-fold neutrosophic real function, we say that:

1) g_N is fully continuous at $x_0\in\mathbb{R}$ if:

$$\begin{cases}g \text{ is continuous at } x_0\\ i,f \text{ are continuous at } x_0\\ t \text{ is continuous at } x_0\end{cases},$$

2) g_N is T- continuous at x_0 (continuous at x_0 with respect to truth component) at $x_0\in\mathbb{R}$ if:

$$\begin{cases}g \text{ is continuous at } x_0\\ t \text{ is continuous at } x_0\end{cases},$$

3) g_N is *I- continuous* (*continuous* with respect to the indeterminacy component) at $x_0 \in \mathbb{R}$ if:

$$\begin{cases} g \text{ is continuous at } x_0 \\ i \text{ is continuous at } x_0 \end{cases},$$

4) g_N is *F*- continuous (continuous with respect to the falisty component) at $x_0 \in \mathbb{R}$ if:

$$\begin{cases} g \text{ is continuous at } x_0 \\ f \text{ is continuous at } x_0 \end{cases}.$$

Definition 10.

Let $g_N: \mathbb{R} \to \mathbb{R}_F$ be a two-fold fuzzy real function in one variable, we say that g_N is *continuous* at $x_0 \in \mathbb{R}$ if and only if:

g, μ are continuous at, $(x_0) \in R$.

Theorem 2.

Let $g_N: \mathbb{R} \to \mathbb{R}_{(t,i,f)}$ be a two-fold neutrosophic real function.If g_N is fully differentiable, then it is fully continuous.

Proof.

Assume that g_N is fully differentiable, then we have:

$g'(x_0)$ is existed
$i'(x_0), f'(x_0), t'(x_0)$ are existed
$i'(x_0), f'(x_0), t'(x_0) \in [0,1]$,

Thus

$$\begin{cases} g \text{ is continuous at } x_0 \\ i, f \text{ are continuous at } x_0 \\ t \text{ is continuous at } x_0 \end{cases}, ,$$

so that it is fully continuous.

Theorem 3.

Let $g_N: \mathbb{R} \to \mathbb{R}_{(t,i,f)}$ be a two-fold neutrosophic real function.If g_N is *T*- differentiable, then it is *T*- continuous.

Proof:

Assume that g_N is T- differentiable, then we have:

$g'(x_0)$ is existed,
$t'(x_0)$ *is* existed,
$t'(x_0) \in [0,1]$.

Thus, g is continuous at x_0, and the function *t* is continuous at x_0 , so that it is *T*- continuous

Theorem 4.

Let $g_N: \mathbb{R} \to \mathbb{R}_{(t,i,f)}$ be a two-fold neutrosophic real function.If g_N is F-differentiable, then it is *F*-continuous.

Proof.

Assume that g_N is F- differentiable, then we have:

$g'(x_0)$ is existed,
$f'(x_0)\ is$ existed,
$f'(x_0) \in [0,1]$.

Thus, g is continuous at x_0, and the function *f* is continuous at x_0 , so that it is *F*- continuous.

Theorem 5.

Let $g_N: \mathbb{R} \to \mathbb{R}_{(t,i,f)}$ be a two-fold neutrosophic real function.If g_N is *I*- differentiable, then it is *I*- continuous.

Proof.

Assume that g_N is I- differentiable, then we have:

$g'(x_0)$ is existed,
$i'(x_0)\ is$ existed,
$i'(x_0) \in [0,1]$.

Thus, g is continuous at x_0, and the function i is continuous at x_0 , so that it is *I*- continuous

Conclusion

In this paper, we defined the concepts of two-fold neutrosophic continuous functions, two-fold fuzzy continuous functions, and two-fold neutrosophic/fuzzy differentiable functions for the first time.

The elementary properties of these novel concepts are studied and handled through many theorems, as well as many clear examples that clarify the validity of these analytical concepts.

References

[1] Florentine Smarandache.(2024). Neutrosophic Two-Fold Algebra, Plithogenic Logic and Computation, 1(1),
11-15.

[2] Mohammad Abobala..(2023) ,On The Foundations of Fuzzy Number Theory and Fuzzy Diophantine
Equations.Galoitica: Journal of Mathematical Structures and Applications, 10(1),17-25.
(Doi **:** https://doi.org/10.54216/GJMSA.0100102).

[3] Mohammad Abobala. (2023). On a Two-Fold Algebra Based on the Standard Fuzzy Number Theoretical
System. Journal of Neutrosophic and Fuzzy Systems, 7 (2), 24-29.
(**Doi :** https://doi.org/10.54216/JNFS.070202).

[4] Ahmed Hatip, Necati Olgun. (2023). On the Concepts of Two- Fold Fuzzy Vector Spaces and Algebraic
Modules. Journal of Neutrosophic and Fuzzy Systems,7(2),46-52 .
(**Doi:** https://doi.org/10.54216/JNFS.070205).

[5] Abdallah Shihadeh, Khaled Ahmad Mohammad Matarneh, Raed Hatamleh, Randa Bashir Yousef Hijazeen, Mowafaq Omar Al-Qadri, Abdallah Al-Husban.(2024). An Example of Two-Fold Fuzzy Algebras Based On Neutrosophic Real Numbers, Neutrosophic Sets and Systems, 67,169-178.

(Doi: 10.5281/zenodo.11151930).

[6] Abdallah Shihadeh, Khaled Ahmad Mohammad Matarneh, Raed Hatamleh, Mowafaq Omar Al-Qadri,

Abdallah Al-Husban, On The Two-Fold Fuzzy n-Refined Neutrosophic Rings For $2 \leq n \leq 3$, Neutrosophic Sets and Systems, Vol. 68, 2024, pp. 8-25. (Doi:10.5281/zenodo.11406449).

[7] Nabil Khuder Salman.(2024). On the Special Gamma Function over the Complex Two-Fold Algebras, Neutrosophic Sets and Systems, 68,26-38.(Doi:10.5281/zenodo.11406461).

[8] Abobala, M., and Zeina, M.B.(2023).A Study of Neutrosophic Real Analysis By Using One Dimensional

Geometric AH-Isometry, Galoitica, Journal of Mathematical Structures and Applications, 3(1),18-24.

[9] Mohammad Bisher Zeina, Mohammad Abobala.(2023). On the Refined Neutrosophic Real Analysis Based on Refined Neutrosophic Algebraic AH-Isometry, Neutrosophic Sets and Systems,47,158-168.

[10] Hatamleh,R.(2024).On the Compactness and Continuity of Uryson's Operator in Orlicz Spaces, International Journal of Neutrosophic Scienc, 24 (3), 233-239.

[11] Barry, W. Al, J. (2024). Computational Intelligence Methodology based on Neutrosophic Set with Multi-

Criteria Decision Making for Evaluating Natural Gas Automobiles. International Journal of Advances in Applied Computational Intelligence, 5(1), 15-28. (**DOI:** https://doi.org/10.54216/IJ\AACI.050102).

[12] Abubaker, Ahmad A, Hatamleh, Raed, Matarneh, Khaled, Al-Husban, Abdallah.(2024). On the Numerical Solutions for Some Neutrosophic Singular Boundary Value Problems by Using (LPM) Polynomials, Inernational Journal of Neutrosophic Science,25(2),197-205. (DOI: https://doi.org/10.54216/IJNS.250217.

[13] Ayman Hazaymeh, Ahmad Qazza, Raed Hatamleh, Mohammad W Alomari, Rania Saadeh.(2023). On Further Refinements of Numerical Radius Inequalities, Axiom-MDPI,12(9),807.

[14] Hatamleh, R., Zolotarev, V. A. (2014). On Two-Dimensional Model Representations of One Class of Commuting Operators. Ukrainian Mathematical Journal, 66(1), 122–144. (Doi: https://doi.org/10.1007/s11253-014-0916-9).

[15] Hatamleh, R., Zolotarev, V. A. (2015). On Model Representations of Non-Selfadjoint Operators with Infinitely Dimensional Imaginary Component, Journal of Mathematical Physics, Analysis, Geometry,

11(2),174–186.(Doi: https://doi.org/10.15407/mag11.02.174).

[16] Heilat, A.S., Batiha,B ,T.Qawasmeh ,Hatamleh R.(2023). Hybrid Cubic B-spline Method for Solving A Class of Singular Boundary Value Problems, European Journal of Pure and Applied Mathematics, 16(2) 751-762. (Doi: https://doi.org/10.29020/nybg.ejpam.v16i2.4725).

[17] C. Sivakumar, Mowafaq Omar Al-Qadri, Abdallah shihadeh, Ahmed Atallah Alsaraireh, Abdallah Al-

Husban, P. Maragatha Meenakshi, N. Rajesh, M. Palanikumar. (2024). q-rung square root interval-valued sets with respect to aggregated operators using multiple attribute decision making. Journal of ,23(3),154-174 (Doi :https://doi.org/10.54216/IJNS.230314).

[18] Selvaraj, S., Gharib, G., Al-Husban, A., Al Soudi, M., Kumaran, K. L. M., Palanikumar, M., &

Sundareswari, K. New algebraic approach towards interval-valued neutrosophic cubic vague set based on
subbisemiring over bisemiring. International Journal of Neutrosophic Science, 23(4), 386-394.
[19] Falahah, I. A., Raman, T. T., Al-Husban, A., Alahmade, A., Azhaguvelavan, S., & Palanikumar, M.
Computer purchasing using new type neutrosophic sets and its extension based on aggregation operators.
International Journal of Neutrosophic Science, 23(4),386-394.
[20] Hatamleh, R. (2003). On the Form of Correlation Function for a Class of Nonstationary Field with a Zero
Spectrum. Rocky Mountain Journal of Mathematics, 33(1)159-173.
(Doi:https://doi.org/10.1216/rmjm/1181069991).
[21] A.Abubaker, Ahmad. , M., Wael. , Alrawashdeh, Heba. , Shatarah, Amani. , Mousa, Norah. , Al-Husban,
Abdallah.(2024). The Mathematical Formulas for Inverting Plithogenic Matrices of Special Orders Between
20 and 24. International Journal of Neutrosophic Science,24(3),102-
114. (Doi: https://doi.org/10.54216/IJNS.240309).
[22] T.Qawasmeh, A. Qazza, R. Hatamleh, M.W. Alomari, R. Saadeh.(2023).Further accurate numerical radius inequalities, Axiom-MDPI, 12 (8), 801.
[23] Batiha, B., Ghanim, F., Alayed, O., Hatamleh, R., Heilat, A. S., Zureigat, H., & Bazighifan, O. (2022). Solving Multispecies Lotka–Volterra Equations by the Daftardar-Gejji and Jafari Method. International Journal of Mathematics and Mathematical Sciences, 2022, 1–7.
https://doi.org/10.1155/2022/1839796
[24] Abu, Ibraheem. , Al-Husban, Abdallah. , M., Mutaz. , A., Ahmed. , Shatarah, Amani. , Mousa, Norah.(2024). On the Characterization of Some m-Plithogenic Vector Spaces and Their AH-Substructures Under the Condition 6 ≤ dim SPV ≤10. International Journal of Neutrosophic Science,24,(3), 258-267. (Doi: https://doi.org/10.54216/IJNS.240322).
[25] Raed Hatamleh, Ayman Hazaymeh.(2024). Finding Minimal Units In Several Two-Fold Fuzzy Finite Neutrosophic Rings, Neutrosophic Sets and Systems,70,1-16.(DOI:10.5281/zenodo.13160839).

Received: June 3, 2024. Accepted: Oct 11, 2024

Neutrosophic Sets and Systems, Vol. 75, 2025
University of New Mexico

Single Valued Neutrosophic Number Ensemble Learning Model for Stability Classification of Open Pit Mine Slopes

Hanzhong Wang, Jun Ye*, Rui Yong*

School of Civil and Environmental Engineering, Ningbo University, Ningbo, Zhejiang 315211, PR China.

* Correspondence: yejun1@nbu.edu.cn; yongrui@nbu.edu.cn

Abstract: China's open pit mining industry faces the dual challenge of increasing production and preventing disasters. In order to ensure the safe exploitation of mineral resources, the stability of slopes must be assessed. In light of the fact that over 1,950 landslide accidents have occurred over the past decade, accounting for 15% of all safety incidents, the evaluation of slope stability has become a critical research focus in the fields of geo-resources and geo-engineering. Traditional slope stability evaluation methods rely on empirical tools and the expertise of professionals to assess slope stability. In contrast, machine learning (ML) methods offer a more comprehensive approach, analyzing the intricate features present in diverse sampling data. As a novel extension of ML, this paper presents a single-valued neutrosophic number-based ensemble learning (SVNN-EL) model. This model employs binary coding groups (1, 0, 0), (0, 1, 0) and (0, 0, 1) to express the learning outcomes of the slope stability, quasi-stability and instability statuses. Subsequently, a similarity measure is employed to determine the classification results of slopes. Finally, the proposed SVNN-EL model is applied to a case study in Yunnan province, China, the proposed model's four performance metrics, namely accuracy, precision, recall, and F1-score (the harmonic mean of precision and recall), are 0.915, 0.894, 0.948, and 0.921, respectively. A comparison with the k-nearest neighbor, support vector machine and random forest methods reveals that the performance metrics of the proposed SVNN-EL model are superior to those of existing methods.

Keywords: single-valued neutrosophic number; ensemble learning; slope stability classification; similarity measure

1. Introduction

As the predominant method of mineral resource development in China, open-pit mining faces two significant challenges: enhancing production and ensuring disaster prevention. It is of the utmost importance to assess the stability of slopes in open-pit mines (OPMs) in order to guarantee the security of the mining process. During the past decade, there have been more than 1,950 landslide accidents in OPMs in China, representing 15% of the total number of safety production accidents [1]. Furthermore, the total number of fatalities resulting from these accidents is the highest among all safety production incidents. The stability of OPM slopes is typically influenced by a range of structural surfaces and features, including those developed at different scales within the slopes themselves. A substantial proportion of slope destabilization incidents is associated with internal slip surfaces and the geometric characteristics of slopes [2]. The evaluation of slope stability in OPMs is of great importance for ensuring the safe and efficient mining of these facilities. Furthermore, it represents a significant research area and a major frontier topic in the field of international geo-resources and geo-engineering.

A slope is a common structure made of solid and rock in the field, and its stability is affected by many factors. As we all know, there is a lot of uncertainty and ambiguity in human nature. Traditional techniques for calculating slope stability factors are common evaluation methods; these methods, including the limit equilibrium method (LEM) [3], the finite element method (FEM) [4] and the discrete element method (DEM) [5], require experienced engineers to spend a significant amount of time to modelling, calculating and testing. In contrast, engineers have developed a number of methods such as rock mass rating (RMR) [6], slope mass rating (SMR) [7], geological strength index (GSI) [8] and Q-classification systems [9, 10] to assess slope stability.

When collecting data on slope features, it is not uncommon to encounter a considerable amount of uncertain, vague, and imprecise information. Consequently, some soft computing methods based on fuzzy theory have been applied to improve the assessment of slope stability. Zhen and Zu [11] first applied the uncertainty of rock mechanical parameters to the Q classification system. Liu and Wang [12] introduced fuzzy mathematics into the neural network (NN) and obtained a comprehensive evaluation for each series of weighted numbers and engineering classification of rocks. A practical and efficient approach using NN and fuzzy clustering systems was applied to the Saeen landslide to estimate the date range of the next probable landslide [13]. An efficiency coefficient method based on fuzzy analytic hierarchy process (FAHP) has been proposed to classify rock slides [14]. Azarafza et al. [15] introduced a fuzzy logical decision making algorithm (DMA) based on block theory to assess the reliability of discontinuous rock slopes under different slip scenarios. Nanehkaran et al. [16] used a fuzzy logic based DMA method and geographical information system (GIS) to evaluate landslide hazard zoning in the Tabriz region. Azarafza et al. [17] used a fuzzy logic DMA to rapidly evaluate block failure instability in discontinuous rock slopes through kinematic analysis.

The application of machine learning (ML) methods has become increasingly common in recent research on slope stability prediction/evaluation. Lin [18] evaluated the effectiveness of four supervised learning methods in slope stability prediction and proposed the gravity search algorithm (GSA), the random forest (RF) algorithm, the support vector machine (SVM) algorithm, and the Naïve Bayesian (NB) algorithm for use in classifier construction. The GSA and RF models demonstrated satisfactory classification results. Ahangari Nanehkaran et al. [19] investigated the efficacy of five machine learning models, namely multilayer perceptron (MLP), SVM, k-nearest neighbor (k-NN), decision tree (DT), and RF, in predicting slope safety factors. The objective of developing these models was to evaluate and optimize them for the calculation of safety factors (FSs) and the selection of slope reinforcement methods. In the ML models, the input parameters included geotechnical indices such as slope height, total slope angle, dry density, cohesion, and internal friction angle, which were used for the estimation of 70 slopes in the South Pars region of Iran. Huang [20] employed a deep learning (DL) algorithm based on a long short-term memory network (LSTM) to predict slope stability. This addressed the limitations of conventional ML models with respect to nonlinear performance, local optimality, and incomplete feature extraction. In his study, RF, a traditional ensemble learning (EL) method, was included in the comparison experiment with more complex deep learning (DL) algorithms, such as a convolutional neural network (CNN). The results demonstrated that RF achieved superior performance on certain programs.

Qin [21] developed the single-valued neutrosophic number (SVNN) based ANFIS to assess slope stability. Subsequently, Qin [22] proposed a Gaussian progress regression (GPR) to predict slope stability in a SVNN scenario, demonstrating its efficacy through the analysis of 167 slope cases. Moreover, a neutrosophic genetic algorithm (NGA) was proposed and used for the clustering of rock discontinuities, demonstrating the capacity to perform optimal clustering analysis of slope sampling data in actual applications.

However, existing ML research relies heavily on the provided FSs as the basis for supervised learning. As a matter of fact, the FSs already indicate the safety information of slope stability, and

there is no need to use ML methods to evaluate slope stability. In order to overcome this problem, this paper proposes a training method for obtaining binary coding-based ML results that reflect the stability, quasi-stability and instability statuses of slopes. Consequently, this study develops a SVNN-based ensemble learning (SVNN-EL) model that utilizes binary coding groups (1, 0, 0), (0, 1, 0), and (0, 0, 1) to express the learning results of the slope stability, quasi-stability and instability statuses. Thereafter, a similarity measure is employed to determine the classification results of slopes. The SVNN-EL model has the potential to enhance its generalizability and applicability in the evaluation and classification process of practical slope stability. Finally, the practical engineering cases of the Lanping OPM are presented to verify the efficiency of the developed model. Based on the accuracy, precision, recall, and F-1 score (the harmonic mean of precision and recall) metrics, the superiority of the proposed SVNN-EL model is demonstrated by a comparison with the existing ML methods. The following section presents a summary of the key findings of this study.

(1) The truth, indeterminacy and falsity degrees in SVNN are capable of comprehensively expressing the hybrid information of uncertainty, inconsistency and incompleteness contained in the raw field slope data. Consequently, the raw data obtained from field investigation and measurement can be transformed into SVNNs by the truth, indeterminacy and falsity membership functions, which are able to capture the comprehensive information in the slope data for ML.

(2) The binary coding groups (1, 0, 0), (0, 1, 0) and (0, 0, 1) are employed to represent the three stability states (stability, quasi-stability, and instability) for slopes and to establish a ML method between the input data (SVNNs) and the output coding groups. Then, the similarity measure value between the ML result and the binary coding groups is used to determine the final classification result for each slope.

(3) A comprehensive comparison of the classification performance of the proposed SVNN-EL model and existing ML methods is performed based on the accuracy, precision, recall, and F-1 score metrics. The results demonstrate that the proposed model outperforms the existing methods in all four-performance metrics.

The remainder of this paper is presented in the following sections. Section 2 provides an overview of the preliminary concepts related to single-valued neutrosophic set (SVNS), SVNN, and EL. Section 3 presents the general framework of a SVNN-EL model to address the slope stability classification problem. In Section 4, the proposed model is applied to the Lanping OPM slopes of Yunnan Province in China as a case study to demonstrate the reasonableness and efficiency of the slope stability classification results in the SVNN scenario. In Section 5, the conclusions derived from the study are presented, accompanied by an analysis of the limitations of the study and suggestions for future research.

2. Preliminaries

2.1 Relative Concepts of Single-Valued Neutrosophic Set (SVNS) and SVNN

From a philosophical perspective, the neutrosophic set (NS) [23] can represent indeterminate and inconsistent information in objects by the use of independent truth, indeterminacy, and falsity membership functions (MFs). As these MFs of NS are situated within a real standard interval [0, 1] or a non-standard interval $]^{-}0, 1^{+}[$, they are difficult to use in the expression and analysis of real-life problems. Consequently, the concept of SVNN was proposed as a subclass of NS based on the standard interval [0, 1] for easy engineering applications [24, 25].

Definition 1 [24, 25]. Let Ψ be a finite set of objects, and let ψ be a common element in Ψ. Then, a SVNS P in Ψ can be defined as

$$P = \left\{ \left\langle \psi, T_p(\psi), U_p(\psi), F_p(\psi) \right\rangle \mid \psi \in \Psi \right\}, \tag{1}$$

where $T_P(\psi)$, $I_P(\psi)$, and $F_P(\psi)$ represents the truth, indeterminacy, and falsity MFs, respectively. For convenience, we use $p_\psi = <T_\psi, U_\psi, F_\psi>$ to represent SVNN as an element in SVNS.

The similarity measure is a critical mathematical tool in clustering analysis and pattern recognition. In this paper, we introduce the cosine similarity measure of SVNNs [26].

Definition 2 [26]. Given two SVNNs $p_{\psi 1}$ = <$T_{\psi 1}$, $U_{\psi 1}$, $F_{\psi 1}$> and $p_{\psi 2}$ = <$T_{\psi 2}$, $U_{\psi 2}$, $F_{\psi 2}$>, the cosine similarity measure between $p_{\psi 1}$ and $p_{\psi 2}$ is represented as

$$S_c(p_{\psi 1}, p_{\psi 2}) = \frac{T_{\psi 1}T_{\psi 2} + U_{\psi 1}U_{\psi 2} + F_{\psi 1}F_{\psi 2}}{\sqrt{T_{\psi 1}^2 + U_{\psi 1}^2 + F_{\psi 1}^2} \cdot \sqrt{T_{\psi 2}^2 + U_{\psi 2}^2 + F_{\psi 2}^2}}, \quad S_c(p_{\psi 1}, p_{\psi 2}) \in [0,1]. \tag{2}$$

2.2 Ensemble Machine Learning Models

2.2.1 Bagging

Bagging, or bootstrap aggregating (BA), is an ensemble learning (EL) technique in ML. Its objective is to reduce the variance of a model by generating multiple independent samples of the training data and training a base model on each sample. The final prediction is then obtained by aggregating the predictions of all the base models. Bagging can be applied to a variety of models, including decision trees, NNs, and SVMs. One of the most popular bagging algorithms is the random forest. RF is a modified version of bagging, in which the classification and regression tree technique is often used as the individual learner. RF uses subsamples of the original data to construct decision trees and randomly selects a subset of variables to determine each split in the tree. RF excludes approximately 30% of training samples from the modeling process due to the use of bootstrapping and random subspace techniques and then it is often used to compute the out-of-bag (OOB) prediction error. The benefits of bagging include improvements in predictive performance, reductions in variance, and increases in model stability. However, bagging can also result in increases in computation time and memory requirements due to the necessity of training and storing multiple models. Overall, bagging is a powerful technique for improving the accuracy and stability of ML models, particularly in situations where the data are noisy or high-dimensional [29, 30]. The diagram of the bagging method is shown in Figure 1.

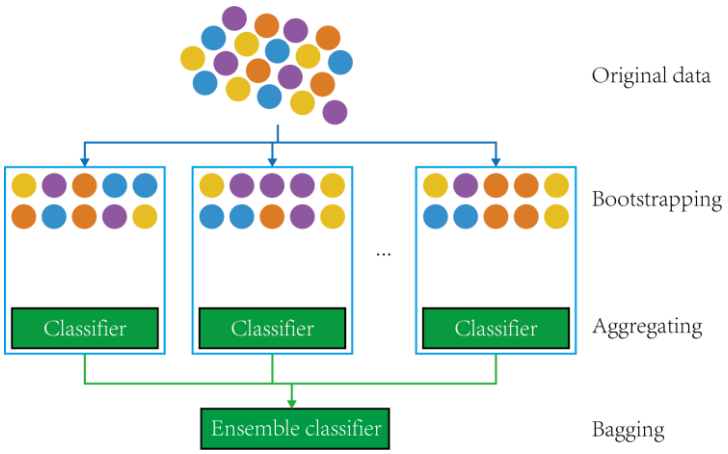


Figure 1. Diagram of the bagging method

2.2.2 Voting

Voting regressor is a EL technique that combines the predictions of multiple base regression models to produce a final prediction. Each base regressor provides its prediction, and the voting regressor aggregates these predictions, often by taking the average, to produce the final output. This approach can be beneficial when different regression models capture different aspects of the underlying data, leading to improved predictive performance compared to using a single model. The

weights of the individual regressors can be uniform or customized based on their performance on the training data. The voting regressor is a versatile tool that can be used to exploit the strengths of various regression algorithms, such as linear regression, decision trees, SVMs and others, to create a robust and accurate predictive model [31].

3. SVNN-EL Model

The section proposes a SVNN-EL model, which is composed of the true bagging method, the indeterminate bagging method, and the false voting method and the cosine similarity measure. Then, its classification process is described as follows.

Step1: Data preparation. Assume that there is a collection of n slopes $D = (d_1, d_2, \ldots, d_n)$, then every slope contains q influence factors $d_i = (a_{i1}, a_{i2}, \ldots, a_{iq})$ $(i = 1, 2, \ldots, n)$ sampled from the field investigation. These slopes imply the three kinds of stability status, which are denoted as three coding groups $y_1 = (y_{t1}, y_{u1}, y_{f1}) = (1, 0, 0)$, $y_2 = (y_{t2}, y_{u2}, y_{f2}) = (0, 1, 0)$, and $y_3 = (y_{t3}, y_{u3}, y_{f3}) = (0, 0, 1)$, corresponding to stable slopes, quasi-stable slopes and unstable slopes.

Step 2: Data neutrosophication. The original data $D = (d_1,d_2, \ldots, d_n)$ are then processed by the truth, indeterminacy and falsity MFs to transform the original data into SVNNs, which are represented as the data sets of the truth, indeterminacy and falsity degrees $D_t = \{d_{t1},d_{t2}, \ldots, d_{tn}, y_{tj}\}$, $D_u = \{d_{u1},d_{u2}, \ldots, d_{un}, y_{ui}\}$, and $D_f = \{d_{f1},d_{f2}, \ldots, d_{fn}, y_{fj}\}$ containing the coding values.

Step3: SVNN-EL model training. In this process, three independent ensemble machine learning methods are established based on the true, false and indeterminate information. Then we use the truth set $D_t = \{d_{t1},d_{t2}, \ldots, d_{tn}, y_{tj}\}$ for the true decision tree-based bagging method, the indeterminacy set $D_u = \{d_{u1},d_{u2}, \ldots, d_{un}, y_{uj}\}$ for the indeterminate decision tree-based bagging method, and the falsity set $D_f = \{d_{f1},d_{f2}, \ldots, d_{fn}, y_{fj}\}$ for the false voting method, where the base algorithms are linear regression (LR), RF and support vector regression (SVR). Their outputs $y_{t,pred}$, $y_{u,pred}$ and $y_{f,pred}$ are constructed as a prediction group $y_{pred} = (y_{t,pred}, y_{u,pred}, y_{f,pred})$.

Step4: Similarity measure. Using Eq. (2), we calculate the similarity measures of $S_C(y_{pred}, y_j)$ $(j = 1, 2, 3)$ for each slop sample.

Step5: Classification result. According to $j^* = \arg\max_{1\le j\le 3}\{S_C(y_{pred}, y_j)\}$, the slope sample with the largest measure value belongs to the corresponding classification y_{j^*}.

In general, the proposed SVNN-EL model is constructed as the flowchart in Figure 2.

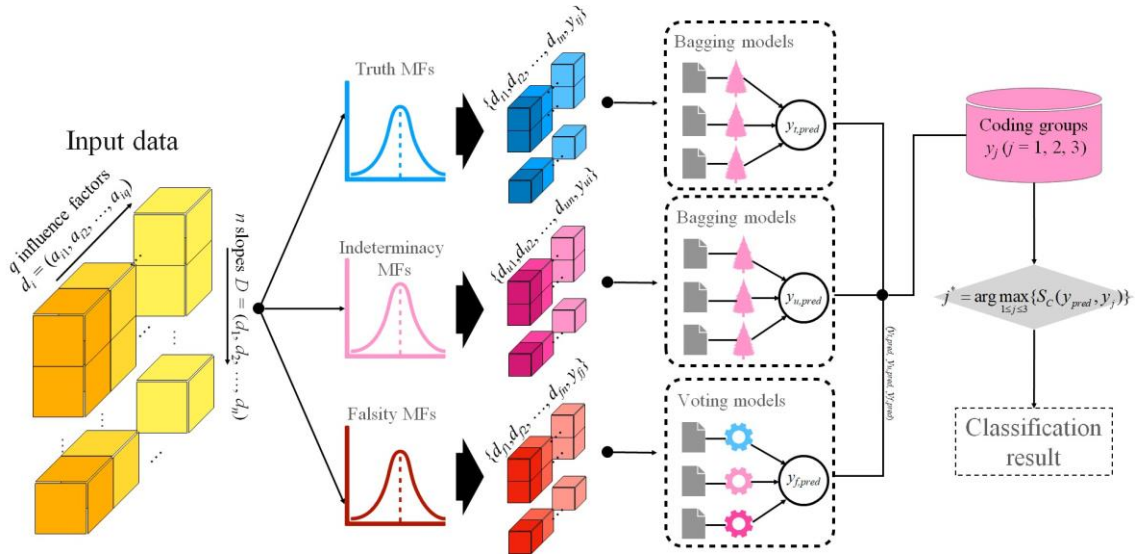


Figure 2. Flowchart of the proposed SVNN-EL model

4. Case Study

4.1 Raw Data Processing of the Lanping OPM Slopes

To validate the effectiveness and suitability of the proposed slope stability classification model, this section applies the proposed SVNN-EL model to classify the stability status of 215 slope samples in Yunnan Province, China. The Lanping OPM is situated in Yunnan Province, China, with geographic coordinates at 99°25′39.47′′E and 26°24′13.30″ N. The highest elevation within the mining area is 2885m, while the lowest elevation is 2475m, resulting in a relative height difference of 410m and characterizing it as a low mountainous terrain. Over years of mining activities, the OPM has taken on an inverted "C" shape; numerous small landslides and failures have occurred within its boundaries. The general conditions and location of this OPM are shown in Figure 3. Hence, it is necessary to provide a classification analysis of slope stability in the Lanping OPM. According to the actual engineering statuses and experts' experience, we consider slope height (a_1), slope angle (a_2), cohesion (a_3), internal friction angle (a_4), and rock density (a_5) as the main factors affecting slope stability. Then, 215 slope samples d_i (i = 1, 2, …, 215) were collected in situ from different areas as a classification case. Figure 3 shows the actual study area of the Lanping OPM.

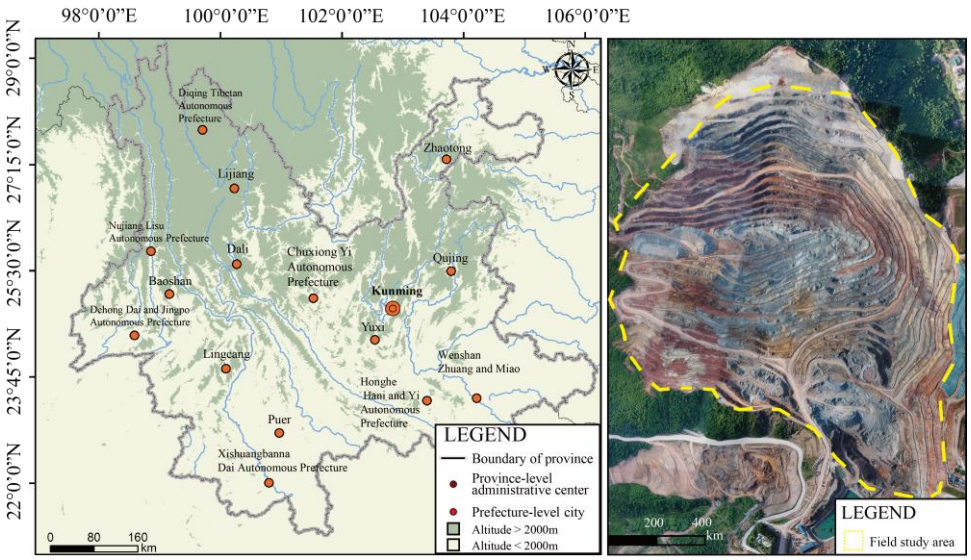


Figure 3. The top view of the actual study area in the Lanping OPM.

Regarding the Lanping OPM, the 215 slopes $D = (d_1, d_2, \ldots, d_{215})$ were collected from the field study area, including the five influence factors: slope angle (a_1), slope height (a_2), cohesion (a_3), friction angle of the rock (a_4) and rock density (a_5). Subsequently, the original data can be divided into the true, indeterminate and false data sets.

First, the original data are neutrosophicated into SVNNs by utilizing truth, indeterminacy and falsity Gaussian membership functions based on the following Gaussian membership function:

$$f(x,\sigma,c) = e^{\frac{-(x-c)^2}{2\sigma^2}}, \tag{3}$$

where σ stands for standard deviation and c stands for mean. The three Gaussian membership functions used in this study are listed in Table 1.

After data neutrosophication, the true, indeterminate and false values in SVNNs and the coding values in y_j (j = 1, 2, 3) are constructed as the true, indeterminate and false data sets $D_t = \{d_{t1}, d_{t2}, \ldots, d_{tn}, y_{tj}\}$, $D_u = \{d_{u1}, d_{u2}, \ldots, d_{un}, y_{uj}\}$, and $D_f = \{d_{f1}, d_{f2}, \ldots, d_{fn}, y_{fj}\}$.

Table 1. Specified truth, indeterminacy, and falsity MFs d_{tk}, d_{uk}, and d_{fk}, (k = 1, 2, 3, 4, 5) for the five affecting factors.

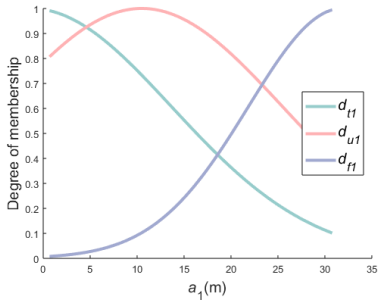
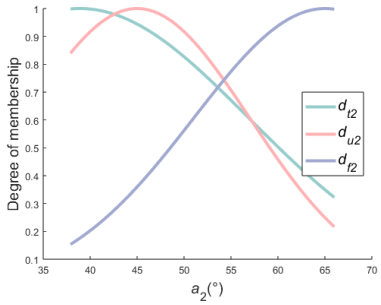
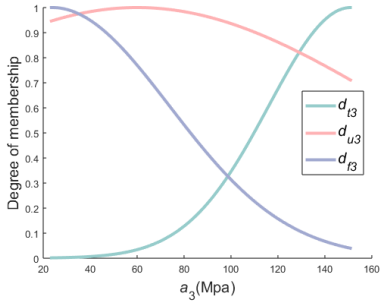
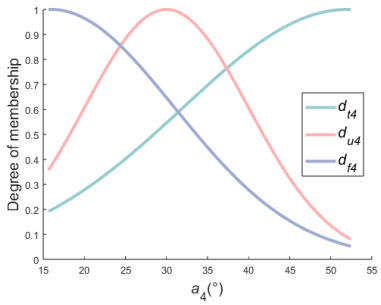
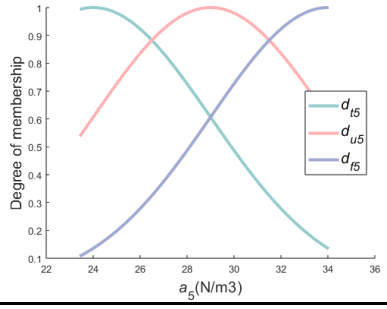
Influence factor	Parameter of Gaussian MFs			Gaussian membership curve
	Truth	Indeterminacy	Falsity	
a_1(m)	σ_{t1}=15.00 c_{t1}=-2.67	σ_{u1}=15.00 c_{u1}=10.50	σ_{f1}=10.00 c_{f1}=31.84	
a_2(°)	σ_{t2}=18.00 c_{t2}=38.90	σ_{u2}=12.00 c_{u2}=45.00	σ_{f2}=14.00 c_{f2}=65.02	
a_3(MPa)	σ_{t3}=35.00 c_{t3}=150.00	σ_{u3}=110.00 c_{u3}=60.00	σ_{f3}=30.00 c_{f3}=23.84	
a_4(°)	σ_{t4}=20.00 c_{t4}=52.00	σ_{u4}=10.00 c_{u4}=30.00	σ_{f4}=15.00 c_{f4}=16.00	
a_5(N/m^3)	σ_{t5}=5.00 c_{t5}=24.00	σ_{u5}=5.00 c_{u5}=29.00	σ_{f5}=5.00 c_{f5}=34.00	

Table 2. Specified parameters of the proposed EL methods

EL method	Parameter	Setting
Bagging method	Base model	Decision tree
	Number of estimators	15
	Max samples	1.0
	Max features	1.0
	Warm start	True
Voting method	Estimator 1	Linear regressor
	Estimator 2	SVM regressor
	Estimator 3	RF regressor
	Weights	None

Since the true, indeterminate, and false EL methods are contained in the proposed SVNN-EL model, their specific parameters are given in Table 2. In order to eliminate the potential influence of randomness in the experimental setup, a fixed random statement (random state = 12) is specified. the true and indeterminate data sets are used for the bagging tree models, comprising multiple decision trees. Then, the false data set is used for the voting tree models, which are also based on a multitude of weak decision trees.

According to the actual slope stability status, the slopes are categorized into the binary coding groups: y_1 = (1, 0, 0), y_2 = (0, 1, 0) and y_3 = (0, 0, 1), which mean stability, quasi-stability, and instability for the slopes. The training and testing data sets are generated using the Python package Scikit-learn, incorporating a constant random statement. The actual stability states and training/testing data sets are presented in Table 3.

4.2 SVNN-EL Classification Results and Discussion

The confusion matrix [32] is a valuable tool to evaluate the performance of a classification model. It provides an overview of the predictions of the model by comparing them with the actual values. The predicted results are listed in the confusion matrix. As shown in Figure 4, the elements c_{ij} in the confusion matrix represent the number of instances, where an observation belonging to the actual class i is predicted to be the class j. The diagonal elements (c_{ii}) represent true positives for each class. Non-diagonal elements (c_{ij}) denote false positives or false negatives.

Actual \ Predicted	Predicted 1	Predicted 2	…	Predicted n
Actual 1	c_{11}	c_{12}	…	c_{1n}
Actual 2	c_{21}	c_{22}	…	c_{2n}
…	…	…	…	…
Actual n	c_{n1}	c_{n2}	…	c_{nn}

Note: Since each class has an individual performance metric, we take their average values for comparative convenience.

Figure 4. Confusion matrix for multi-class classification.

Table 3. Similarity measure values and classification results of the testing data sets.

$y_{t,pred}$	$y_{u,pred}$	$y_{f,pred}$	$S_C(y_{pred}, y_1)$	$S_C(y_{pred}, y_2)$	$S_C(y_{pred}, y_3)$	**Result**	**Real**
0.8651	0.0000	0.0927	0.9943	0.0000	0.1066	(1,0,0)	(1,0,0)
0.8379	0.0000	0.0354	0.9991	0.0000	0.0422	(1,0,0)	(1,0,0)
0.3481	0.0914	0.6213	0.4847	0.1273	0.8653	(0,0,1)	(0,0,1)
0.3536	0.0017	0.7857	0.4104	0.0019	0.9119	(0,0,1)	(0,0,1)
0.7556	0.0671	0.0232	0.9956	0.0885	0.0305	(1,0,0)	(1,0,0)
0.3495	0.0081	0.3335	0.7233	0.0167	0.6903	(1,0,0)	(1,0,0)
0.2701	0.5086	0.0000	0.4690	0.8832	0.0000	(0,1,0)	(0,1,0)
0.3645	0.0036	0.7966	0.4161	0.0041	0.9093	(0,0,1)	(0,0,1)
0.3844	0.0236	0.7503	0.4558	0.0280	0.8896	(0,0,1)	(0,0,1)
0.4894	0.4629	0.0817	0.7212	0.6822	0.1204	(1,0,0)	(1,0,0)
0.3401	0.0300	0.7504	0.4125	0.0364	0.9102	(0,0,1)	(0,0,1)
0.4590	0.4610	0.0000	0.7056	0.7087	0.0000	(0,1,0)	(0,1,0)
0.9046	0.0000	0.0419	0.9989	0.0000	0.0462	(1,0,0)	(1,0,0)
0.4987	0.4829	0.0588	0.7158	0.6932	0.0844	(1,0,0)	(1,0,0)
0.3978	0.0671	0.6506	0.5196	0.0877	0.8499	(0,0,1)	(0,0,1)
0.3227	0.4837	0.0746	0.5504	0.8251	0.1272	(0,1,0)	(0,1,0)
0.3757	0.0359	0.2882	0.7912	0.0755	0.6069	(1,0,0)	(1,0,0)
0.4334	0.4749	0.0000	0.6741	0.7386	0.0000	(0,1,0)	(0,1,0)
0.4425	0.0295	0.8035	0.4822	0.0321	0.8755	(0,0,1)	(0,0,1)
0.6833	0.0833	0.0000	0.9926	0.1211	0.0000	(1,0,0)	(1,0,0)
0.5423	0.4888	0.0000	0.7428	0.6695	0.0000	(1,0,0)	(0,1,0)
0.3294	0.0711	0.0000	0.9775	0.2111	0.0000	(0,0,1)	(0,0,1)
0.6667	0.0833	0.0000	0.9923	0.1240	0.0000	(1,0,0)	(1,0,0)
0.3378	0.0000	0.6204	0.4782	0.0000	0.8783	(0,0,1)	(0,0,1)
0.4657	0.0295	0.8642	0.4742	0.0300	0.8799	(0,0,1)	(0,0,1)
0.7119	0.0594	0.0626	0.9927	0.0829	0.0873	(1,0,0)	(1,0,0)
0.3201	0.5027	0.0000	0.5371	0.8435	0.0000	(0,1,0)	(0,1,0)
0.3308	0.0100	0.7187	0.4181	0.0126	0.9083	(0,0,1)	(0,0,1)
0.4219	0.1081	0.4364	0.6844	0.1753	0.7077	(0,0,1)	(1,0,0)
0.4068	0.0300	0.2971	0.8061	0.0595	0.5888	(1,0,0)	(1,0,0)
0.7530	0.0278	0.0505	0.9971	0.0368	0.0669	(1,0,0)	(1,0,0)
0.7814	0.0019	0.1013	0.9917	0.0024	0.1285	(1,0,0)	(1,0,0)
0.4511	0.1278	0.7096	0.5303	0.1502	0.8344	(0,0,1)	(1,0,0)
0.5401	0.0300	0.5007	0.7328	0.0407	0.6792	(1,0,0)	(0,0,1)
0.7556	0.0278	0.0000	0.9993	0.0367	0.0000	(1,0,0)	(1,0,0)
0.3747	0.0914	0.4748	0.6125	0.1494	0.7762	(0,0,1)	(0,0,1)
0.7333	0.1042	0.0000	0.9901	0.1406	0.0000	(1,0,0)	(1,0,0)
0.4536	0.0017	0.8119	0.4877	0.0018	0.8730	(0,0,1)	(0,0,1)
0.8493	0.0019	0.0714	0.9965	0.0022	0.0838	(1,0,0)	(1,0,0)

0.3324	0.0771	0.7875	0.3873	0.0898	0.9176	(0,0,1)	(0,0,1)
0.3349	0.1748	0.6848	0.4282	0.2234	0.8756	(0,0,1)	(0,0,1)
0.3452	0.0850	0.5786	0.5084	0.1252	0.8520	(0,0,1)	(0,0,1)
0.3012	0.4996	0.1254	0.5047	0.8373	0.2102	(0,1,0)	(0,1,0)
0.3844	0.0236	0.7095	0.4761	0.0292	0.8789	(0,0,1)	(0,0,1)
0.3082	0.0833	0.6719	0.4143	0.1120	0.9032	(0,0,1)	(0,0,1)
0.5415	0.0220	0.2306	0.9194	0.0373	0.3915	(1,0,0)	(1,0,0)
0.3479	0.5281	0.0232	0.5497	0.8346	0.0367	(0,1,0)	(0,1,0)

Table 4. Classification results for the proposed SVNN-EL model.

	Stable	**Quasi-stable**	**Unstable**
Actual	19	1	1
Actual	0	7	0
Actual	2	0	17

All the results of the testing data are presented in Table 3. Here, the similarity measure values between the EL results $y_{t,pred}$, $y_{u,pred}$, $y_{f,pred}$ and the three coding groups (1, 0, 0), (0, 1, 0) and (0, 0, 1) are obtained by Eq. (2). Subsequently, each slope is classified in terms of the classification method of Step 5. The total classification results of the 47 testing slopes are presented in Table 4. The classification results can be visualized in a three-dimensional scatter plot (Figure 5), which is enabled to show a more detailed representation of the distance between the classified points and the target coding groups.

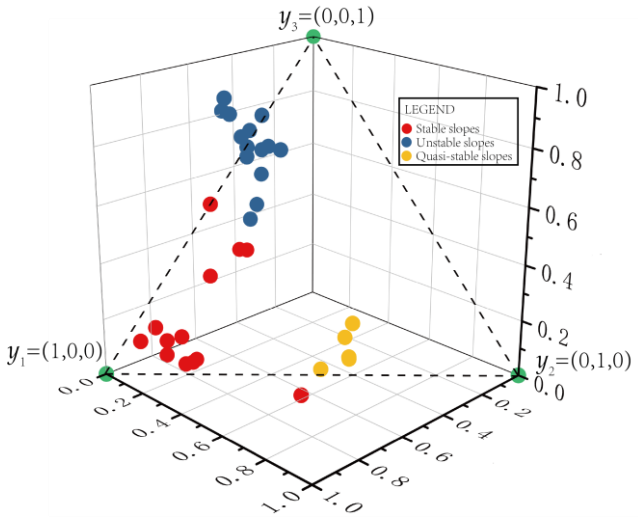


Figure 5. Classification result of the 47 testing slopes.

To show the effectiveness of the proposed SVNN-EL model, the four-performance metrics of the Acc, precision, recall and F1-score are considered below.

Accuracy (Acc) measures how well the model predicts both positive and negative classes by

$$Acc = \frac{\sum_{i=0}^{n-1} C_{ii}}{\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} C_{ij}}. \tag{4}$$

Precision quantifies how many of the positively classified instances are actually correct by

$$Precision_i = \frac{C_{ii}}{\sum_{j=0}^{n-1} C_{ji}}. \tag{5}$$

Recall or sensitivity estimates how well the model identifies all real positives by

$$Recall_i = \frac{C_{ii}}{\sum_{j=0}^{n-1} C_{ij}}. \tag{6}$$

F1-score combines precision and recall into one metric using their harmonic mean value by

$$F1-score_i = \frac{2Precision \cdot Recall}{Precision + Recall}. \tag{7}$$

By Eqs. (4)–(7), the performance metrics of the Acc, precision, recall and F1-score are 0.915, 0.894, 0.948, 0.921, respectively. To show the effectiveness of the proposed SVNN-EL model, we chose the classical ML methods of KNN, SVM, RF for comparison, the specific experiment settings are shown in the Table 2. The training and testing data sets also keep the same with the proposed SVNN-EL model. The comparison results are shown in Figure 6.

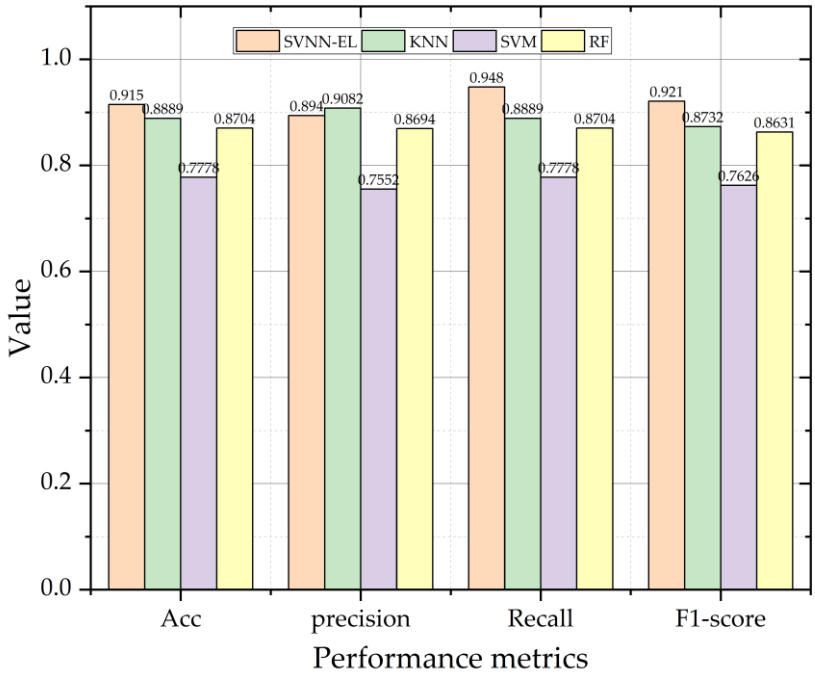


Figure 6. Performance comparison of KNN, SVM, RF and the proposed method.

As illustrated in Figure 6, the proposed SVNN-EL model exhibits the optimal accuracy, recall, and F1-score. Our findings indicate that the proposed SVNN-EL model outperforms the traditional

methods in scenarios where the features are not readily discernible. Additionally, the proposed model offers the flexibility to be adapted to specific engineering conditions, a distinctive superiority over the traditional ML methods.

5. Conclusions

Based on the hybrid form of SVNN and the EL method, the SVNN-EL model was proposed as a method of classifying the stability of OPM slopes. Subsequently, the efficacy of the proposed model was validated by using a dataset comprising 215 slope samples sourced from Yunnan province, China. Then, the proposed model can reach better classification precision of the slopes. The comparative outcomes demonstrated that the proposed SVNN-EL model was superior to the existing ML methods of KNN, SVM, RF in the performance metrics of the Acc, precision, recall and F1-score. This approach offered a valuable way for mining engineers. In general, the proposed SVNN-EL model includes the following main highlights.

(1) The truth, indeterminacy and falsity degrees in SVNN can comprehensively express the hybrid information of uncertainty, imprecision and incompleteness in the raw field slope data, which provide the necessary information for the proposed SVNN-EL model.

(2) The coding groups were used to represent the stability status of each slope. Then a relationship between the coding groups and the slope stability status can be established in the ML process. Furthermore, a classification method based on the similarity measure was developed to determine the final classification result.

(3) The proposed SVNN-EL model demonstrated better performance in terms of the Acc, precision, recall and F-1 score metrics, along with the metric values of 0.915, 0.894, 0.948 and 0.921, respectively. The comparative results demonstrated that the proposed model exhibits superiority over the existing ML methods of KNN, SVM, RF in all four-performance metrics.

It should be noted that the proposed model has a limitation in that it exclusively considers the fundamental engineering conditions and plane failure modes of OPM slopes. In the future, the proposed model will be extended to encompass more complex engineering conditions, such as those associated with earthquakes and blasting loads. Furthermore, it will be applied to slopes where circular and wedge-shaped failure modes may occur. The novel technique presented in this study has the potential to be extended to other applications, including rock burst risk assessment, galaxy identification, and medical diagnosis in the presence of uncertain or inconsistent data.

Funding: This research was financially supported by the National Natural Science Foundation of China (Grant No. 42177117).

Conflicts of Interest: The authors declare that they have no conflicts of interest.

References

1. Zhai, M.; Hu, R.; Wang, Y. et al. Mineral resource science in China: review and perspective. *Geography and Sustainability* **2021**, *2*, 107–114.
2. Pandit, B.; Tiwari, G.; Latha, G.M.; Sivakumar Babu, G. Stability analysis of a large gold mine open-pit slope using advanced probabilistic method. *Rock Mechanics and Rock Engineering* **2018**, *51*, 2153–2174.
3. Morgenstern, N.; Price, V.E. The analysis of the stability of general slip surfaces. *Geotechnique* **1965**, *15*, 79–93.
4. Liu, C.; Kelly, R.G. A review of the application of finite element method (FEM) to localized corrosion modeling. *Corrosion* **2019**, *75*, 1285–1299.

5. Horabik, J.; Molenda, M. Parameters and contact models for dem simulations of agricultural granular materials: a review. *Biosystems engineering* **2016**, *147*, 206–225.
6. Bieniawski, Z.T. *Engineering rock mass classifications: A complete manual for engineers and geologists in mining, civil, and petroleum engineering*. John Wiley & Sons, 1989.
7. Romana, M. New adjustment ratings for application of Bieniawski classification to slopes. In Proceedings of the international symposium on role of rock mechanics, Zacatecas, Mexico, **1985**, pp. 49–53.
8. Hoek, E.; Marinos, P.; Benissi, M. Applicability of the geological strength index (GSI) classification for very weak and sheared rock masses. the case of the Athens Schist formation. *Bulletin of Engineering Geology and the Environment* **1998**, *57*, 151–160.
9. Barton, N.; Lien, R.; Lunde, J. Engineering classification of rock masses for the design of tunnel support. *Rock mechanics* **1974**, *6*, 189–236.
10. Barton, N.; Bar, N. Introducing the Q-slope method and its intended use within civil and mining engineering projects. *ISRM EUROCK*, **2015**, ISRM.
11. Zhen-yu, T.; Zu-zeng, P. Application of fuzzy mathematics to the engineering classification of rocks. *Chinese Journal of Geotechnical Engineering* **1981**, *3*, 36–45.
12. Liu, P.; Wang, Y. Multiple attribute decision-making method based on single-valued neutrosophic normalized weighted Bonferroni mean. *Neural Computing and Applications* **2014**, *25*, 2001–2010.
13. Alimohammadlou, Y.; Najafi, A.; Gokceoglu, C. Estimation of rainfall-induced landslides using ANN and fuzzy clustering methods: a case study in Saeen slope. Azerbaijan Province, Iran. *Catena* **2014**, *120*, 149–162.
14. Shengwu, Q.; Rongguo, C.; Guangjie, L.I.; Zhongjun, M.A.; Jiangfeng, L.V. Stability classification of rock slope based on FAHP efficiency coefficient method. *Journal of Catastrophology* **2016**, *31*(4), 27–32.
15. Azarafza, M.; Akgün, H.; Feizi-Derakhshi, M.-R.; Azarafza, M.; Rahnamarad, J.; Derakhshani, R. Discontinuous rock slope stability analysis under blocky structural sliding by fuzzy key-block analysis method. *Heliyon* **2020**, *6*, e03907.
16. Nanehkaran, Y.A.; Mao, Y.; Azarafza, M.; Kockar, M.K.; Zhu, H.-H. Fuzzy-based multiple decision method for landslide susceptibility and hazard assessment: a case study of Tabriz, Iran. *Geomechanics and Engineering* **2021**, *24*, 407–418.
17. Azarafza, M.; Asghari-Kaljahi, E.; Ghazifard, A.; Akgün, H. Application of fuzzy expert decision-making system for rock slope block-toppling modeling and assessment: a case study. *Modeling Earth Systems and Environment*. **2021**, *7*, 159–168.
18. Lin, Y.; Zhou, K.; Li, J. Prediction of slope stability using four supervised learning methods. *IEEE Access* **2018**, *6*, 31169–31179.
19. Ahangari Nanehkaran, Y.; Pusatli, T.; Chengyong, J.; Chen, J.; Cemiloglu, A.; Azarafza, M.; Derakhshani, R. Application of machine learning techniques for the estimation of the safety factor in slope stability analysis. *Water* **2022**, *14*, 3743.
20. Huang, F.; Xiong, H.; Chen, S.; Lv, Z.; Huang, J.; Chang, Z.; Catani, F. Slope stability prediction based on a long short-term memory neural network: comparisons with convolutional neural networks, support vector machines and random forest models. *International Journal of Coal Science & Technology* **2023**, *10*, 18.
21. Qin, J.; Du, S.; Ye, J.; Yong, R. SVNN-ANFIS approach for stability evaluation of open-pit mine slopes. *Expert Systems with Applications* **2022**, *198*, 116816.

22. Qin, J.; Ye, J.; Sun, X.; Yong, R.; Du, S. A single-valued neutrosophic gaussian process regression approach for stability prediction of open-pit mine slopes. *Applied Intelligence* **2023**, *53*, 13206–13223.
23. Smarandache, F. Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis. **1998**.
24. Wang, H.; Smarandache, F.; Zhang, Y.; Sunderraman, R. Single valued neutrosophic sets. *Multispace and Multistructure* **2010**, 4, 410-413.
25. Ye, J. A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets. *Journal of Intelligent & Fuzzy Systems* **2014**, *26*, 2459–2466.
26. Ye, J. Vector similarity measures of simplified neutrosophic sets and their application in multicriteria decision making. *International Journal of Fuzzy Systems* **2014**, *16*(2), *204–211*.
27. Kaifeng, Z.; Jun, Y.E.; Bo, L.I. Application of clustering analysis based on similarity measure in slope stability classification. *Journal of Shaoxing University (Natural Science)* **2018,** *38*(8), 65–71.
28. Ye, J.; Zhang, Z.; Yong, R.; Du, S. MAGDM model using the exponential similarity measure of neutrosophic confidence cubic sets in a single-valued neutrosophic multivalued circumstance. *Journal of Mathematics* **2023**, *2023*, Article ID 5150472.
29. Breiman, L. Bagging predictors. *Machine Learning* **1996**, *24*, 123–140.
30. Kavitha, B.; Karthikeyan, S.; Maybell, P.S. An ensemble design of intrusion detection system for handling uncertainty using neutrosophic logic classifier. *Knowledge-Based Systems* **2012**, *28*, 88–96.
31. Leon, F.; Floria, S.-A.; Bădică, C. Evaluating the effect of voting methods on ensemble-based classification. *In Proceedings of the 2017 IEEE International Conference on Innovations in Intelligent Systems and Applications (INISTA)*. Gdynia, Poland, July 2017, pp. 1–6.
32. Rani, N.; Gupta, D.K.; Singh, S. Multi-class classification of breast cancer abnormality using transfer learning. *Multimedia Tools and Applications* **2024,** *83*, 75085–75100.

Received: June 4, 2024. Accepted: Oct 12, 2024

Neutrosophic Sets and Systems, Vol. 75, 2025
University of New Mexico

Identification of influential factors affecting student performance in semester examinations in the educational institution using score topological indices in Single Valued Neutrosophic Graphs

S. Narasimman[1], M. Shanmugapriya[2], R. Sundareswaran[3], Laxmi Rathour[4,*], Lakshmi Narayan Mishra[5], Vinita Dewangan[6], Vishnu Narayan Mishra[7,*]

[1]Controller of Examination, Sri Sivasubramaniya Nadar College of Engineering, Chennai - 603 110, India.
[2,3]Department of Mathematics, Sri Sivasubramaniya Nadar College of Engineering, Chennai - 603 110, India.
[4]Department of Mathematics, National Institute of Technology, Chaltlang, Aizawl 796 012, Mizoram, India.
[5]Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore 632 014, Tamil Nadu, India.
[6]Department of Mathematics, Mats School of Engineering and Information Technology MATS University, Raipur, Chhattisgarh, India.
[7*]Department of Mathematics, Faculty of Science, Indira Gandhi National Tribal University, Lalpur, Amarkantak, Anuppur, Madhya Pradesh 484 887, India.
Email: *narasimmans@ssn.edu.in*, *shanmugapriyam@ssn.edu.in*, *laxmirathour817@gmail.com*, *lakshminarayanmishra04@gmail.com*, *vinitadewangan12@gmail.com.*

*Corresponding author details
E-mail: *laxmirathour817@gmail.com*, *vishnunarayanmishra@gmail.com*, *vnm@igntu.ac.in*

Abstract: Education not only measures the progress of an individual, but it also contributes to the development of a community and a nation. To understand the factors that affect the academic performance of students in institutions of higher education, there are many factors within the institution and outside the institution that affect the academic performance of students. Includes affecting student academic performance are class size, parent-related factors, academic performance, the contribution of institutional factors to student academic performance, and the impact of poverty on students. The interplay of personal attributes, learning habits, prior academic preparation, and the school environment is essential to understanding and supporting student academic achievement. Each of these factors contributes uniquely to academic performance, and their relative importance may vary according to each student's circumstances. All these factors include uncertainty in nature. Neutrosophic graphs provide a powerful mathematical framework for dealing with uncertainty and complexity in various mathematical and real-world problems. This manuscript comprises the study of some innovative results based on the score topological indices in SVN_{graphs}. Finally, we identify the influential factors in the successful conduct of semester examinations using these new parameters.

Keywords: *Score Function, SVN_{graph}, Score Topological Indices*

Abbreviation	Description
F_{set}	Fuzzy set
$F_{relation}$	Fuzzy relation
F_{graph}	Fuzzy graph
IF_{set}	Intuitionistic Fuzzy set
CF_{set}	Classical fuzzy set
N_{set}	Neutrosophic set

SVN_{set}	Single Valued Neutrosophic set
N_{graph}	Neutrosophic graph
$F_{numbers}$	Fuzzy number
IF_{number}	Intuitionistic Fuzzy number
$TriIF_{numbers}$	Triangular Intuitionistic Fuzzy number
SVN_{values}	Single Valued Neutrosophic values
U_{set}	Universe set
FV_{set}	Fuzzy vertex set
FI_{set}	Fuzzy index set
NV_{set}	Neutrosophic vertex set
NI_{set}	Neutrosophic index set
$T_{mem}, I_{mem}, F_{mem}$	Truth membership, Indeterminacy membership, Falsity membership
$SVNV_{set}$	Single Value Neutrosophic vertex set
$SVNI_{set}$	Single Value Neutrosophic index set
N_{cycle}	Neutrosophic cycle
SD_{vertex}	Score Degree of vertex
$SVN_{subgraph}$	Single-valued neutrosophic sub-graph

1. Introduction

In 1965, Zadeh [1] introduced the concept of the degree of membership/truth (T) and defined the F_{set}. Kaufmann [2] later extended this by applying the concept of fuzziness to graph theory using $F_{relation}$. Rosenfeld [3] introduced several concepts related to F_{graph}, including bridges, cycles, paths, trees, and connectedness, and explored some properties ofF_{graph}. An IF_{set}Considers both the membership grade and the non-membership grade for any entity, with the requirement that their sum does not exceed one. Atanassov [4–5] introduced theIF_{set} as a modified version of theCF_{set}.

In 1995, Smarandache [7] introduced the concept of indeterminacy/neutrality (I) as an independent component and defined theN_{set}, which consists of three components: Truth (T), Indeterminacy (I), and Falsity (F). N_{set} are characterized by three functions: $T_{mem}, I_{mem}, and\ \ F_{mem}$ where their values are real standard or non-standard subsets of the unit interval]−0, 1+[. The SVN_{set} [8], which takes values from the subset of [0, 1], is a specific instance of aN_{set} and is particularly useful for addressing real-world problems, especially in decision support systems.

Topological indices are molecular descriptors to study the properties of a chemical compound in chemical graph theory. These numerical quantities of a chemical graph describe its topology. A molecular graph can be represented using an atom as a vertex and a bond between two atoms represents an edge. There are many indices available in the literature such as the Wiener index [13], Zagreb index [22-23], Randic index [16], Atom-Bond-Connectivity index [17], Harmonic index[18], sum-connectivity and general sum-connectivity indices [19-20], Schultz index or degree distance index [21] and Gutman index [22].

InF_{graphs}, the study of these indices has been much studied in recent years by many researchers. Shriram Kalathian et. al [23] introduced some topological indices in F_{graphs}. Masoud Ghods and Zahra Rostami studied the first and second Zagreb indices, the Harmonic index, the Randic index, and the Connectivity index in N_{graphs} [24]. The score and accuracy of uncertainty graphs made it easy to rank them. In this work, we introduce a new kind of topological indices based on score function and prove a few theorems related to these indices. Ranking of IF_{number} Was done by Mitchell, et. al and Nayagam et. al. [26 -29]. Also, Sahin, R. [30] analyzed a multi-criteria decision-making method in a neutrosophic environment, utilizing score and accuracy functions. This approach aims to

enhance decision-making processes by effectively addressing the uncertainty and indeterminacy inherent inN_{set}.

Ranking$F_{numbers}$Was first introduced by Jain [27]. Some of the researchers proposed ranking methods based on triangular and trapezoidal membership functions. Later, a few of them also introduced ranking $F_{numbers}$ Based on the centroid point. Yager discussed centroid-based ranking. $F_{numbers}$Initially. Mitchell discussed a new ranking method for $IF_{numbers}$ [28]. Gomthi Nayagam et.al introduced the concept of ranking the $IF_{numbers}$ and Modified ranking of $IF_{numbers}$ [29,30]. A ranking Method of $TriIF_{numbers}$ and its applications to decision-making were studied by Li [31]. In 2016, Sahin [32] introduced score and accuracy functions within a neutrosophic environment to solve multi-criteria decision-making problems.

Grzegorzewski [33] treated $IF_{numbers}$As two families of metrics and developed a ranking method for these numbers. In 2016, Nancy and Harish Garg [34] introduced an improved score function for ranking N_{set}, providing a more refined approach for evaluating and comparing alternatives in a neutrosophic decision-making environment. In this work, we introduce some of the topological indices based on the following score function which is useful for ranking. SVN_{set}.

$\tilde{S} = \frac{\tilde{T} + \tilde{I} + 1 - \tilde{F}}{3}$, $\tilde{S} \in [0,1]$(1).

Education is a process through which a person can transmit his findings and experiences for survival and development over the generations. Education aims to strengthen the individual's as well as society's culture. Examination is a very interesting and important tool to examine a student's achievement. Evaluating a student's performance through examination leads to both government and non-government organizations' support and treatment for improving the quality of education for the benefit of social changes and carrying out education sector reforms. The performance of a student in examinations is influenced by a variety of factors, and understanding these factors is crucial for improving educational outcomes. Saima Rasul and Qadir Bukhsh designed to measure the factors affecting students' performance in examinations at the university level [35]. Kassu Mehari Beyene Jemal Ayalew Yimam discussed the Multilevel Analysis for Identifying Factors Influencing Academic Achievement of Students in Higher Education Institutions in Wollo University [36]. Yousuf Nasser Said Al Husaini andNur Syufiza Ahmad Shukor [37] analyzed a comprehensive review of the factors affecting student academic performance based on low entry grades, family background, food and accommodation facility, gender, past assessment grade, and students' learning activity. Kawtar Tani et.al. [38] evaluated the factors contributing to the poor academic performance of students in higher education and the effect of factors relating to family obligations, work, social commitments, and financial concerns were analyzed. They showed a relationship between the academic performance of students and their attendance. They also concluded that the effect of absenteeism is a major one for determining student performance in most educational institutions and encouraged students to attend classes to achieve better grades in their future courses. Wei Liu and Lei Zhang [39] showed the evaluation of student performance and identified the key influencing factors based on the Entropy-Weighted TOPSIS Model.

In all the above studies, the amount of uncertainty was not an important factor in finding the key factors of a student's performance in the end-semester examination. This motivates us to study the performance of a student using a new mathematical tool namely score-based topological indices using SVN_{values}. In this study, we aim to analyze the impact of different factors that affect student performance at the higher education level based on the factors psychological, socioeconomic, and physical influences, changes in question paper patterns, guidance adequacy, exam-related anxieties, internal environment, question paper difficulty, and evaluator impact. A survey of over 500 students was conducted to gather data. The neutrosophic theory was employed to handle uncertainties in student satisfaction levels.

In sectionδ.2, all the basic definitions are presented. Section δ.3 introduces the concept of score-based topological indices, along with illustrations to demonstrate their application.

2. Preliminaries

Definition 1: [1] Let X be theU_{set}. A $F_{set}\tilde{A}$ on X is defined as $\tilde{A} = \{\langle x, T_{\tilde{A}}(x)\rangle : x \in X\}$, where $T_{\tilde{A}}(x): X \to {}^{-}]0,1[^{+}$ is said to be the T_{mem}function, which represents the degree of confidence.

Definition 2:[3]Let $G = (V_G, E_G)$ be a simple graph, where V_G, E_G be the set of vertices and edges, respectively. Then, a $F_{graph}G$ is denoted by $\tilde{G} = (V_G,\ \tilde{\mu}, \tilde{\rho})$, where $\tilde{\mu} = T_{\tilde{\mu}}$is the FV_{set} on $V_G$and $\tilde{\rho} = T_{\tilde{\rho}}$ is the FI_{set} on $E_G \subseteq V_G \times V_G$ *where* $(T_{\tilde{\mu}}: V_G \to [0,1]: T_{\tilde{\rho}}: V_G \times V_G \to [0,1])$, and is defined as$T_{\tilde{\rho}}(u_i, v_j) \le T_{\tilde{\mu}}(u_i) \wedge T_{\tilde{\mu}}(v_j)\ , (u_i, v_j) \in E_G(i, j = 1,2,3 \dots n)$.

Definition 3: [4,5] A $F_{set}\tilde{A}$ in the universal discourse X. An $IF_{set}\ \tilde{A}$ on X is defined as $\tilde{A} = \{\langle x, T_{\tilde{A}}(x),\ F_{\tilde{A}}(x)\rangle : x \in X\}$, where $T_{\tilde{A}}(x): X \to {}^{-}]0,1[^{+}$ is said to be the certainty membership function, which specifies the degree of confidence, $I_{\tilde{A}}(x): X \to {}^{-}]0,1[^{+}$is said to be the uncertainty membership, which represents the degree of indistinctness, respectively of the element $x \in X\ in\ \tilde{A}$, such that $0 \le T_{\tilde{A}}(x) + F_{\tilde{A}}(x) \le 1$.

Definition 4:[5] An IF_{graph} of G is of the form $\tilde{G} = (V_G,\ \tilde{\mu}, \tilde{\rho})$, where $\tilde{\mu} = (T_{\tilde{\mu}}, F_{\tilde{\mu}})$ is aNV_{set} on V_G and $\tilde{\rho} = (T_{\tilde{\rho}}, F_{\tilde{\rho}})$ is the NI_{set} on $E_G \subseteq V_G \times V_G$. $V_G = \{v_1, v_2, v_3, \dots v_n\}$ such that $(T_{\tilde{\mu}}: V_G \to [0,1],\ F_{\tilde{\mu}}: V_G \to [0,1]$, denote the degree of certainty membership, and degree of indistinctness membership, respectively of the element $v_i \in V_G$and $, 0 \le T_{\tilde{\mu}}(v_i) + F_{\tilde{\mu}}(v_i) \le 1\ for\ every\ v_i \in V_G. E_G \subseteq V_G \times V_G where\ T_{\tilde{\rho}}: V_G \times V_G \to [0,1], F_{\tilde{\rho}}: V_G \times V_G \to [0,1]$are such that $T_{\tilde{\rho}}(v_i, v_j) \le [\ T_{\tilde{\mu}}(v_i) \wedge T_{\tilde{\mu}}(v_j)]$, $F_{\tilde{\rho}}(v_i, v_j) \le [\ F_{\tilde{\mu}}(v_i) \vee F_{\tilde{\mu}}(v_j)]$ and $0 \le T_{\tilde{\rho}}(v_i, v_j) + F_{\tilde{\rho}}(v_i, v_j) \le 1\ for\ every\ (v_i, v_j) \in E_G(i, j = 1,2,3 \dots n)$.

Definition 5: [7] Let X be a U_{set}. A $N_{set}\tilde{A}$ on X is defined as $\tilde{A} = \{\langle x, T_{\tilde{A}}(x), I_{\tilde{A}}(x), F_{\tilde{A}}(x)\rangle : x \in X\}$, where $T_{\tilde{A}}(x): X \to {}^{-}]0,1[^{+}$ is said to be the T_{mem}function, which represents the degree of confidence, $I_{\tilde{A}}(x): X \to {}^{-}]0,1[^{+}$is said to be the I_{mem}, which represents the degree of uncertainty, and $F_{\tilde{A}}(x): X \to {}^{-}]0,1[^{+}$ is said to be the F_{mem}, which represents the degree of skepticism, respectively of the element $x \in X\ in\ \tilde{A}$, such that ⬚$^{-}0 \le T_{\tilde{A}}(x) + I_{\tilde{A}}(x) + F_{\tilde{A}}(x) \le 3^{+}, \forall x \in X$.

Definition 6: N_{graph} of G is of the form $\tilde{G} = (V_G,\ \tilde{\mu}, \tilde{\rho})$, where $\tilde{\mu} = (T_{\tilde{\mu}}, I_{\tilde{\mu}}, F_{\tilde{\mu}})$ is a NV_{set}on V_G and $\tilde{\rho} = (T_{\tilde{\rho}}, I_{\tilde{\rho}}, F_{\tilde{\rho}})$ is the NI_{set} on $E_G \subseteq V_G \times V_G$.

i. $V_G = \{v_1, v_2, v_3, \dots v_n\}$ such that $(T_{\tilde{\mu}}: V_G \to [0,1], I_{\tilde{\mu}}: V_G \to [0,1], F_{\tilde{\mu}}: V_G \to [0,1]$, denote the degree of $T_{mem}, I_{mem}, F_{mem}$, respectively of the element $v_i \in V_G$and , ⬚$^{-}0 \le T_{\tilde{\mu}}(v_i) + I_{\tilde{\mu}}(v_i) + F_{\tilde{\mu}}(v_i) \le 3^{+}\ for\ every\ v_i \in V_G$.
ii. $E_G \subseteq V_G \times V_G where\ T_{\tilde{\rho}}: V_G \times V_G \to [0,1], I_{\tilde{\rho}}: V_G \times V_G \to [0,1], F_{\tilde{\rho}}: V_G \times V_G \to [0,1]$ are such that $T_{\tilde{\rho}}(v_i, v_j) \le [\ T_{\tilde{\mu}}(v_i) \wedge T_{\tilde{\mu}}(v_j)]$, $I_{\tilde{\rho}}(v_i, v_j) \le [\ I_{\tilde{\mu}}(v_i) \wedge I_{\tilde{\mu}}(v_j)]$, $F_{\tilde{\rho}}(v_i, v_j) \le [\ F_{\tilde{\mu}}(v_i) \wedge F_{\tilde{\mu}}(v_j)]$ and ⬚$^{-}0 \le T_{\tilde{\rho}}(v_i, v_j) + I_{\tilde{\rho}}(v_i, v_j) + F_{\tilde{\rho}}(v_i, v_j) \le 3^{+}\ for\ every\ (v_i, v_j) \in E_G(i, j = 1,2,3 \dots n)$.

Definition 7: Let X be a universe set. A$SVN_{set}\tilde{S}$ on X is defined as $\tilde{S} = \{\langle x, T_{\tilde{S}}(x), I_{\tilde{S}}(x), F_{\tilde{S}}(x)\rangle : x \in X\}$, where $T_{\tilde{S}}(x): X \to {}^{-}]0,1[^{+}$ is said to be the T_{mem} function, $I_{\tilde{S}}(x): X \to {}^{-}]0,1[^{+}$is said to be the I_{mem}, and $F_{\tilde{S}}(x): X \to {}^{-}]0,1[^{+}$ is said to be the F_{mem}, respectively of the element $x \in X\ on\ \tilde{S}$, such that ⬚$^{-}0 \le T_{\tilde{S}}(x) + I_{\tilde{S}}(x) + F_{\tilde{S}}(x) \le 3^{+}, \forall x \in X$.

Definition 8: A $SVN_{graph}\tilde{S}$ of G is of the form $\tilde{G} = (V_G,\ \tilde{\mu}, \tilde{\rho})$, where $\tilde{\mu} = (T_{\tilde{\mu}}, I_{\tilde{\mu}}, F_{\tilde{\mu}})$ is a$SVNV_{set}$on V_G

and $\tilde{\rho} = (T_{\tilde{\rho}}, I_{\tilde{\rho}}, F_{\tilde{\rho}})$ is the $SVNI_{set}$ on $E_G \subseteq V_G \times V_G$.

i. $V_G = \{v_1, v_2, v_3, \dots v_n\}$ such that $T_{\tilde{\mu}}: V_G \to [0,1], I_{\tilde{\mu}}: V_G \to [0,1], F_{\tilde{\mu}}: V_G \to [0,1]$, denote the degree of $T_{mem}, I_{mem}, F_{mem}$, respectively of the element $v_i \in V_G$and , $0 \leq T_{\tilde{\mu}}(v_i) + I_{\tilde{\mu}}(v_i) + F_{\tilde{\mu}}(v_i) \leq 3^+ \; for \; every \; v_i \in V_G$.

ii. $E_G \subseteq V_G \times V_G where \; T_{\tilde{\rho}}: V_G \times V_G \to [0,1], I_{\tilde{\rho}}: V_G \times V_G \to [0,1], F_{\tilde{\rho}}: V_G \times V_G \to [0,1]$ are such that $T_{\tilde{\rho}}(v_i, v_j) \leq [\, T_{\tilde{\mu}}(v_i) \wedge T_{\tilde{\mu}}(v_j)], I_{\tilde{\rho}}(v_i, v_j) \leq [\, I_{\tilde{\mu}}(v_i) \vee I_{\tilde{\mu}}(v_j)], F_{\tilde{\rho}}(v_i, v_j) \geq [\, F_{\tilde{\mu}}(v_i) \vee F_{\tilde{\mu}}(v_j)]$and $0 \leq T_{\tilde{\rho}}(v_i, v_j) + I_{\tilde{\rho}}(v_i, v_j) + F_{\tilde{\rho}}(v_i, v_j) \leq 3^+ \; for \; every \; (v_i, v_j) \in E_G (i, j = 1,2,3 \dots n). T_{\tilde{\rho}}(v_i, v_j) \leq min[\, T_{\tilde{\mu}}(v_i), T_{\tilde{\mu}}(v_j)], \quad I_{\tilde{\rho}}(v_i, v_j) \geq max[\, I_{\tilde{\mu}}(v_i), I_{\tilde{\mu}}(v_j), \quad F_{\tilde{\rho}}(v_i, v_j) \geq max[\, F_{\tilde{\mu}}(v_i) F_{\tilde{\mu}}(v_j)]$,

Definition 9: Let $\tilde{G} = (\tilde{\mu}, \tilde{\rho})$be a SVN_{graph} and $\acute{P}$ is a path in $\tilde{G}$. $\acute{P}$ is a collection of vertices, $v_0, v_1, v_2, \cdots, v_n$ such that $\left[T_{\tilde{\mu}}(v_{i-1}, v_i), I_{\tilde{\mu}}(v_{i-1}, v_i), F_{\tilde{\mu}}(v_{i-1}, v_i)\right] > 0$ for $0 \leq i \leq n$. $\acute{C}$ is a N_{cycle} if $v_0 = v_n$ and $n \geq 3$.

3. Topological indices of SVN_{graph}

Definition 10: Let $\tilde{G} = (\tilde{\mu}, \tilde{\rho})$ be a SVN_{graph} and $v \in V_G$. Then the$SD_{vertex}v$ is defined as $d_s(v) = \sum_{u_i \epsilon N(v)} \tilde{S}(u_i, v)$, where $N(v)$ represents the number of neighbors of v.

Definition 11: Let $\tilde{G} = (\tilde{\mu}, \tilde{\rho})$ be a SVN_{graph} and $v \in V_G$. Then the k- $SD_{vertex}v$ is defined as $d_s^k(v) = \sum_{u_i \epsilon N(v)} \tilde{S}(u_i, v)^k$, where $N(v)$ represents the number of vertices which are adjacent to v.

3.1 Zagreb Index

Definition 12: Let $\tilde{G}: (\tilde{\mu}, \tilde{\rho})$ be a $SVN_{graph}$with a nonempty vertex set. The first kind of Zagreb index $ZI(\tilde{G})$of a SVN_{graph} is defined as

$$ZI(\tilde{G}) = \sum_{u_i \epsilon \sigma^*} \tilde{S}(u_i) d_s^2(u_i), u_i \in V_G.$$

Definition 13: The second kind of Zagreb index $ZI^*(\tilde{G})$of a SVN_{graph} is defined as

$$ZI^*(\tilde{G}) = \sum_{(u_i, u_j) \epsilon \mu} \tilde{S}(u_i) d_s(u_i) \tilde{S}(u_j) d_s(u_j), \qquad \forall i \neq j.$$

Example 14: Let $\tilde{G}$ be the SVN_{graph} as shown in Fig. 1 such that $V_{\tilde{G}} = \{v_1, v_2, v_3, v_4\}$.

The score function for all vertices is

$\tilde{S}(v_1) = \frac{0.5+0.1+1-0.4}{3} = \frac{1.2}{3} = 0.4$, $\tilde{S}(v_2) = \frac{0.6+0.3+1-0.2}{3} = \frac{1.7}{3} = 0.567$, $\tilde{S}(v_3) = \frac{0.2+0.3+1-0.4}{3} = \frac{1.1}{3} = 0.367$, $\tilde{S}(v_4) = \frac{0.4+0.2+1-0.5}{3} = \frac{1.1}{3} = 0.367$, $\tilde{S}(v_1, v_2) = \frac{0.5+0.4+1-0.5}{3} = \frac{1.4}{3} = 0.467$; $\tilde{S}(v_2, v_3) = \frac{0.2+0.3+1-0.4}{3} = \frac{1.1}{3} = 0.367$, $\tilde{S}(v_3, v_4) = \frac{0.2+0.4+1-0.5}{3} = \frac{1.1}{3} = 0.367$, $\tilde{S}(v_4, v_1) = \frac{0.4+0.3+1-0.6}{3} = \frac{1.1}{3} = 0.367$.

Now, we have

$$d_s(v_1) = \tilde{S}(v_1, v_2) + \tilde{S}(v_4, v_1) = 0.467 + 0.367 = 0.834$$
$$d_s(v_2) = \tilde{S}(v_1, v_2) + \tilde{S}(v_2, v_3) = 0.467 + 0.367 = 0.834$$
$$d_s(v_3) = \tilde{S}(v_2, v_3) + \tilde{S}(v_3, v_4) = 0.367 + 0.367 = 0.734$$
$$d_s(v_4) = \tilde{S}(v_3, v_4) + \tilde{S}(v_4, v_1) = 0.367 + 0.367 = 0.734$$

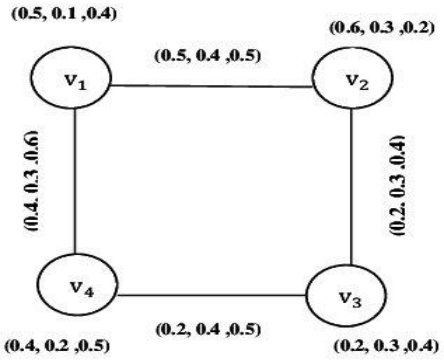


Fig.1. SVN_{graph} *with* $PI(\tilde{G}) = 0.956$, $ZI(\tilde{G}) = .0606, ZI^*(\tilde{G}) = 0.224$

From definitions [12, 13], we have

$$ZI(\tilde{G}) = 0.4(0.467^2 + 0.367^2) + 0.567(0.467^2 + 0.367^2) + 0.367(0.367^2 + 0.367^2) + 0.367(0.367^2 + 0.367^2)$$

$$= 0.4(0.3523) + 0.567(0.3523) + 0.367(0.1347) + 0.367(0.1347) = 0.0606.$$

$$ZI^*(\tilde{G}) = \sum_{(v_i, v_j)\epsilon\mu} \tilde{S}(v_i)d_s(v_i)\tilde{S}(v_j)d_s(v_j), \forall\, i \neq j \text{ and } v_i, v_j \in V_{\tilde{G}}, i, j = \{1, 2, 3, 4\}$$

$$ZI^*(\tilde{G}) = \{0.1578 + 0.1274 + 0.0726 + 0.0899\} = 0.4477.$$

Observation 15: Let $\tilde{G}$ is the SVN_{graph} and $\tilde{H}$ is the $SVN_{subgraph}\tilde{G}$ such that $\tilde{H} = \tilde{G} - u$ then $ZI(\tilde{H})$< ZI $(\tilde{G})$ and $ZI^*(\tilde{H})$<$ZI^*(\tilde{G})$.

Definition 16: A $SVN_{graph}\tilde{G} = (\tilde{\mu}, \tilde{\rho})$ is said to be a score regular SVN_{graph} if $d_s(v) = k$, for all vertices $v \ in\ V_G$.

Observation 17: Let $\tilde{G}$: $(\tilde{\mu}, \tilde{\rho})$be the score regular N_{graph}. Then, $ZI^*(\tilde{G}) = k^2 d_s(v)$.

Theorem 18: Suppose $\tilde{G}$: $(\tilde{\mu}, \tilde{\rho})$ be a SVN_{graph} with $\tilde{n}$ vertices and $\tilde{m}$ Edges. Then $ZI^*(\tilde{G}) \leq \left(\frac{16}{81}\right)\tilde{\Delta}^2$.

Proof. As $d_s(v) \leq \frac{2}{3}$and $\tilde{S}(v) \leq \frac{2}{3}$,. $\tilde{S}(u_i)d_s(u_i)\tilde{S}(u_j)d_s(u_j) \leq \left(\frac{16}{81}\right)\tilde{\Delta}^2$where $\tilde{\Delta}$ represents the maximum degree of $\tilde{G}$.

Theorem 19: Let $\tilde{P}$ be a path with n vertices. Then $ZI^*(\tilde{P}) \leq \frac{64}{81}(n-2)$.

Proof. Let $(v_1, \cdots, v_n)$ be a vertex set of path $\tilde{P}$. Then $d_s(v_1) = \tilde{S}(u_i, v_1) = \tilde{S}(v_1, v_2) \leq \frac{2}{3} d_s(v_n) = \sum_{u_i \epsilon N(v_n)} \tilde{S}(u_i, v_n) = \tilde{S}(v_n, v_{n-1}) \leq \frac{2}{3}$, and $d_s(v_i) = \sum_{u_i \epsilon N(v_i)} \tilde{S}(u_i, v_i) \leq \frac{4}{3}$, for $i = 2, 3, \cdots, n-1$ and $\tilde{S}(v_i) \leq \frac{2}{3}, \forall i$.

Therefore, $ZI^*(\tilde{P}) = \sum_{(u_i, u_j)\epsilon\mu} \tilde{S}(u_i)d_s(u_i)\tilde{S}(u_j)d_s(u_j)$

$$= \left[\tilde{S}(v_1)d_s(v_1)\tilde{S}(v_2)d_s(v_2) + \tilde{S}(v_n)d_s(v_n)\tilde{S}(v_{n-1})d_s(v_n) + \sum_{i=2}^{n-2} \tilde{S}(v_i)d_s(v_i)\tilde{S}(v_{i-1})d_s(v_{i-1})\right]$$

$$\leq \left[\frac{32}{81} + \frac{32}{81} + \frac{64}{81}(n-3)\right] = \frac{64}{81}(n-2).$$

Example 19:

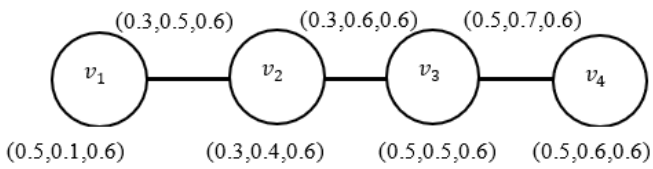


Fig.2. *SVN_{graph} $\breve{P}$ on 4 vertices with $ZI^*(\tilde{G}) = 0$.*

The score function for all vertices is

$\tilde{S}(v_1) = \frac{0.5+0.1+1-0.6}{3} = 0.33, \tilde{S}(v_2) = \frac{0.3+0.4+1-0.6}{3} = 0.37, \tilde{S}(v_3) = \frac{0.5+0.5+1-0.6}{3} = 0.47;$ $\tilde{S}(v_4) = \frac{0.5+0.6+1-0.6}{3} = \frac{1.5}{3} = 0.5, \tilde{S}(v_1, v_2) = \frac{0.3+0.5+1-0.6}{3} = \frac{1.2}{3} = 0.4, \tilde{S}(v_2, v_3) = \frac{0.3+0.6+1-0.6}{3} = \frac{1.3}{3} = 0.43, \tilde{S}(v_3, v_4) = \frac{0.5+0.7+1-0.6}{3} = \frac{1.6}{3} = 0.53.$

Now, we have $d_s(v_1) = \tilde{S}(v_1, v_2) = 0.4$; $d_s(v_2) = \tilde{S}(v_1, v_2) + \tilde{S}(v_2, v_3) = 0.4 + 0.43 = 0.83$; $d_s(v_3) = \tilde{S}(v_2, v_3) + \tilde{S}(v_3, v_4) = 0.43 + 0.53 = 0.96$; $d_s(v_4) = \tilde{S}(v_3, v_4) = 0.53$

From definitions [13], we have

$$ZI^*(\tilde{G}) = \tilde{S}(v_1)d_s(v_1)\tilde{S}(v_2)d_s(v_2) + \tilde{S}(v_2)d_s(v_2)\tilde{S}(v_3)d_s(v_3) + \tilde{S}(v_3)d_s(v_3)\tilde{S}(v_4)d_s(v_4)$$
$$= .33 \times .4 \times .37 \times .83 + .37 \times .83 \times .96 \times .43 + .47 \times .43 \times .5 \times .53 = 0.279 .$$

Theorem 20: Let $\breve{C}$ is a cycle with n vertices. Then $ZI^*(\breve{C}) \leq \frac{64}{81}(n-1)$.

Proof. Let $(v_1, \cdots, v_n)$ be a vertex set of cycle $\breve{C}$. Then $d_s(v_1) = \sum_{u_i \in N(v_1)} \tilde{S}(u_i, v_n) = \tilde{S}(v_1, v_n) \leq \frac{2}{3}$ and $d_s(v_i) = \sum_{u_i \in N(v_i)} \tilde{S}(u_i, v_i) = \tilde{S}(v_{i-1}, v_i) \leq \frac{4}{9}$, for $i = 2, 3, \cdots, n-1$ and $\tilde{S}(v_i) \leq \frac{2}{3}, \forall i$.

Therefore,

$$ZI^*(\breve{C}) = \sum_{(u_i, u_j) \epsilon \mu} \tilde{S}(u_i)d_s(u_i)\tilde{S}(u_j)d_s(u_j)$$
$$= \left[\tilde{S}(v_1)d_s(v_1)\tilde{S}(v_2)d_s(v_2) + \tilde{S}(v_1)d_s(v_n)\tilde{S}(v_n)d_s(v_n) + \sum_{i=2}^{n-2} \tilde{S}(v_i)d_s(v_{i+1})\tilde{S}(u_{i+1})d_s(v_{i+1})\right]$$
$$\leq \left[\frac{64}{81} + \frac{64}{81} + \frac{64}{81}(n-3)\right] = \frac{64}{81}(n-1).$$

Example 20:

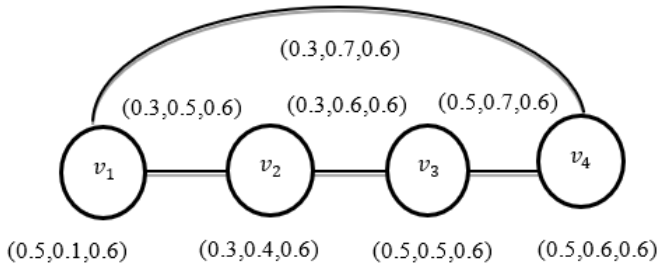


Fig.3. *$SVN_{graph}\breve{C}$ on 4 vertices with $ZI^*(\breve{C}) = 0$.*

The score function for all vertices is

$\tilde{S}(v_1) = \frac{0.5+0.1+1-0.6}{3} = 0.33, \tilde{S}(v_2) = \frac{0.3+0.4+1-0.6}{3} = 0.37, \tilde{S}(v_3) = \frac{0.5+0.5+1-0.6}{3} = 0.47;$ $\tilde{S}(v_4) = \frac{0.5+0.6+1-0.6}{3} = \frac{1.5}{3} = 0.5, \tilde{S}(v_1, v_2) = \frac{0.3+0.5+1-0.6}{3} = \frac{1.2}{3} = 0.4, \tilde{S}(v_2, v_3) = \frac{0.3+0.6+1-0.6}{3} = \frac{1.3}{3} = 0.43, \tilde{S}(v_3, v_4) = \frac{0.5+0.7+1-0.6}{3} = \frac{1.6}{3} = 0.53, \tilde{S}(v_1, v_4) = \frac{0.3+0.7+1-0.6}{3} = \frac{1.4}{3} = 0.47.$

Now, we have $d_s(v_1) = \tilde{S}(v_1, v_2) + \tilde{S}(v_1, v_4) = 0.4 + .47 = .87$; $d_s(v_2) = \tilde{S}(v_1, v_2) + \tilde{S}(v_2, v_3) = 0.4 + 0.43 = 0.83$; $d_s(v_3) = \tilde{S}(v_2, v_3) + \tilde{S}(v_3, v_4) = 0.43 + 0.53 = 0.96$; $d_s(v_4) = \tilde{S}(v_3, v_4) + \tilde{S}(v_1, v_4) = 0.53 + .47 = 1$.

From definitions [13], we have

$$ZI^*(\breve{C}) = \tilde{S}(v_1)d_s(v_1)\tilde{S}(v_2)d_s(v_2) + \tilde{S}(v_2)d_s(v_2)\tilde{S}(v_3)d_s(v_3) + \tilde{S}(v_3)d_s(v_3)\tilde{S}(v_4)d_s(v_4) + \tilde{S}(v_4)d_s(v_4)\tilde{S}(v_1)d_s(v_1)$$
$$= .33 \times .4 \times .37 \times .83 + .37 \times .83 \times .96 \times .13 + .13 \times .5 \times .53 \times .96 + .5 \times .33 \times 1 \times .87$$
$$= .255.$$

Theorem 20: Let $\breve{S}$ be a star with n vertices. Then $ZI^*(\breve{S}) \leq \frac{8}{9}(n-1)$.

Proof. Let $(u, v_1, \cdots, v_{n-1})$ be a vertex set of $\check{S}$. $d_s(u) = \sum_{u_i \epsilon N(u)} \tilde{S}(u_i, u) \leq \frac{2}{3}(n-1)$ and $d_s(v_i) = \sum_{u_i \epsilon N(v_i)} \tilde{S}(u_i, v_i) \leq \frac{2}{3}$, for $i = 2, 3, \cdots, n-1$ and $\tilde{S}(v_i) \leq \frac{2}{3}, \forall i$.
Therefore,

$$ZI^*(\check{C}) = \sum_{(u_i,u_j)\epsilon\mu} \tilde{S}(u_i)d_s(u_i)\tilde{S}(u_j)d_s(u_j) = \left[\tilde{S}(u)\sum_{i=2}^{n-2} d_s(u)\tilde{S}(v_i)d_s(v_i)\right]$$

$$\leq \frac{2}{3}\left[\frac{4}{3}(n-1)\right] = \frac{8}{9}(n-1).$$

3.2 *Connectivity Index*

Definition 18: Let $\tilde{G}: (\tilde{\mu}, \tilde{\rho})$ be a SVN_{graph}.Then the strength of score connectedness $SCONN_{\tilde{G}}(u_i, v_j)$is between two vertices u_i and $v_j$is defined as $SCONN_{\tilde{G}}(u_i, v_j) = \tilde{S}(u_i, v_j)$.

Definition 19: Let $\tilde{G}: (\tilde{\mu}, \tilde{\rho})$ be a SVN_{graph}. The Connectivity index $CI(\tilde{G})$of a SVN_{graph} is defined as $CI(\tilde{G}) = \sum_{(u_i,v_j)\epsilon\sigma^*} \tilde{S}(u_i)\tilde{S}(v_j)SCONN_{\tilde{G}}(u_i, v_j), \forall i \neq j \; and \; (v_i, v_j) \in E_G$.

Example 20: Let $\tilde{G}: (V_G, \tilde{\mu}, \tilde{\rho})$ be a SVN_{graph} in Fig.4 with $\sigma^* = [a, b, c, d]$.
The score function for all vertices is given by,

$$\tilde{S}(a) = \frac{\tilde{T}_a + \tilde{I}_a + 1 - \tilde{F}_a}{3} = 0.33; \; \tilde{S}(b) = \frac{\tilde{T}_b + \tilde{I}_b + 1 - \tilde{F}_b}{3} = 0.367;$$

$$\tilde{S}(c) = \frac{\tilde{T}_c + \tilde{I}_c + 1 - \tilde{F}_c}{3} = 0.43; \; \tilde{S}(d) = \frac{\tilde{T}_d + \tilde{I}_d + 1 - \tilde{F}_d}{3} = 0.5.$$

and the edges the score function is given by,

$$\tilde{S}(a,b) = \frac{\tilde{T}_{ab} + \tilde{I}_{ab} + 1 - \tilde{F}_{ab}}{3} = 0.3; \; \tilde{S}(b,d) = \frac{\tilde{T}_{bd} + \tilde{I}_{bd} + 1 - \tilde{F}_{bd}}{3} = 0.33;$$

$$\tilde{S}(d,c) = \frac{\tilde{T}_{dc} + \tilde{I}_{dc} + 1 - \tilde{F}_{dc}}{3} = 0.3; \; \tilde{S}(c,a) = \frac{\tilde{T}_{ca} + \tilde{I}_{ca} + 1 - \tilde{F}_{ca}}{3} = 0.3.$$

Using Eq.(1), the strength of connectedness ($CONN_{\tilde{G}}(u_i, v_j)$ can be determined to be
$SCONN_{\tilde{G}}(a,b) = 0.3; \; SCONN_{\tilde{G}}(a,c) = 0.3; \; SCONN_{\tilde{G}}(a,d) = 0.3;$
$SCONN_{\tilde{G}}(b,c) = 0.3; \; SCONN_{\tilde{G}}(b,d) = 0.33; SCONN_{\tilde{G}}(c,d) = 0.3.$
After computing the connectedness between all pairs of vertices, we found that,
$CI(\tilde{G}) = 0.590$.

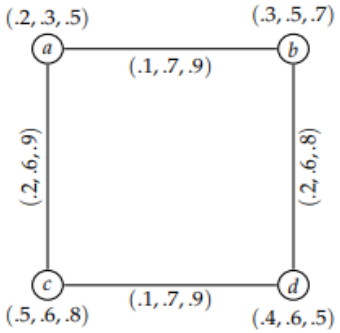


Fig.4. A $SVN_{graph}$with $CI(\tilde{G}) = 0.590, WI(\tilde{G}) = 0.795, HWI(\tilde{G}) = 0.427$.

3.3 *Wiener Index*

Definition 21: The Wiener index $WI(\tilde{G})$of a SVN_{graph} is defined as

$$WI(\tilde{G}) = \sum_{(u_i,v_j)\epsilon\sigma^*} \tilde{S}(u_i)\tilde{S}(v_j)d_s(u_i, v_j), \forall i \neq j \; and \; (v_i, v_j) \in E_G$$

where $d_s(u_i, v_j)$ is the sum of minimum average score functions from u_i *to* v_j.

Example 22: Consider the $SVN_{graph}\tilde{G}: (V_G, \tilde{\mu}, \tilde{\rho})$ *with* $V_G = \{a, b, c, d\}, \tilde{S}(a, b) = 0.3, \tilde{S}(b, d) = 0.33, \tilde{S}(d, c) = 0.3, \tilde{S}(c, a)0.3$ (Fig.4).

To find the $d_s(u_i, v_j)$for two vertices $u_i$and v_j Such that the distance between the vertices has the minimum value. Therefore, $d_s(a, b) = 0.3;\ d_s(a, c) = 0.3;$

$d_s(b, c) = \tilde{S}(b, a) + \tilde{S}(a, c) = 0.3 + 0.3 = 0.6;\ d_s(b, d) = 0.33;\ d_s(c, d) = 0.3$

Hence,

$$WI(\tilde{G}) = \sum_{(u_i, v_j)\epsilon\sigma^*} \tilde{S}(u)\tilde{S}(v)d_s(u, v) =$$
$$= \tilde{S}(a)\tilde{S}(b)d_s(a, b) + \tilde{S}(a)\tilde{S}(c)d_s(a, c) + \tilde{S}(a)\tilde{S}(d)d_s(a, d) + \tilde{S}(b)\tilde{S}(a)d_s(b, a)$$
$$+ \tilde{S}(b)\tilde{S}(c)d_s(b, c) + \tilde{S}(b)\tilde{S}(d)d_s(b, d) + \tilde{S}(c)\tilde{S}(a)d_s(c, a) + \tilde{S}(c)\tilde{S}(b)d_s(c, b)$$
$$+ \tilde{S}(c)\tilde{S}(d)d_s(c, d) + \tilde{S}(d)\tilde{S}(a)d_s(d, a) + \tilde{S}(d)\tilde{S}(b)d_s(d, b) + \tilde{S}(d)\tilde{S}(c)d_s(d, c)$$
$$= 0.795.$$

3.4 Modified Wiener Index

Definition 23: The Modified Wiener index $MWI(\tilde{G})$of a SVN_{graph} is defined as

$$MWI(\tilde{G}) = \frac{1}{2} \sum_{(u_i, v_j)\epsilon\sigma^*} \tilde{S}(u_i)\tilde{S}(v_j)d_s(u_i, v_j), \forall i \neq j \ and \ (v_i, v_j) \in E_G.$$

Example 24: From Fig.4

$$MWI(\tilde{G}) = \frac{1}{2} \sum_{(u_i, v_j)\epsilon\sigma^*} \tilde{S}(u_i)\tilde{S}(v_j)d_s(u_i, v_j) = \frac{1}{2} \times WI(\tilde{G}) = 0.398.$$

3.5 Hyper Wiener Index

Definition 25: The Hyper Wiener index $HWI(\tilde{G})$of a SVN_{graph} is defined as

$$HWI(\tilde{G}) = \frac{1}{2} \sum_{(u_i, v_j)\epsilon\sigma^*} \left(\tilde{S}(u_i)\tilde{S}(v_j)d_s(u_i, v_j) + \left(\tilde{S}(u_i)\tilde{S}(v_j)d_s(u_i, v_j)\right)^2\right), \forall i \neq j \ and \ (v_i, v_j) \in E_G.$$

Example 26: From Fig.4

$$HWI(\tilde{G}) = \frac{1}{2}.2\big[\big((0.33 \times 0.367 \times 0.3) + (0.33 \times 0.367 \times 0.3)^2\big)$$
$$+ \big((0.33 \times 0.43 \times 0.3) + (0.33 \times 0.43 \times 0.3)^2\big)$$
$$+ \big((0.33 \times 0.5 \times 0.6) + (0.33 \times 0.5 \times 0.6)^2\big)$$
$$+ \big((0.367 \times 0.43 \times 0.6) + (0.367 \times 0.43 \times 0.6)^2\big)$$
$$+ \big((0.367 \times 0.5 \times 0.33) + (0.367 \times 0.5 \times 0.33)^2\big)$$
$$+ \big((0.43 \times 0.5 \times 0.3) + (0.43 \times 0.5 \times 0.3)^2\big)\big] = 0.427.$$

3.6 Schultz Index

Definition 27: The Schultz index $SCI(\tilde{G})$of a SVN_{graph} is defined as Schultz index given by

$$SCI(\tilde{G}) = \sum_{u(u_i, v_j)\epsilon\sigma^*} \tilde{S}(u_i)\tilde{S}(v_j)d_s(u_i, v_j)\big[d_s(u_i) + d_s(v_j)\big], \forall i \neq j \ and \ (v_i, v_j) \in E_G.$$

Example 28: Consider the $SVN_{graph}\tilde{G}: (V_{\tilde{G}}, \tilde{\mu}, \tilde{\rho})$ as shown in Figure 5, with the vertex set $V_{\tilde{G}} = \{v_1, v_2, v_3, v_4\}$ such that $(\tilde{T}, \tilde{I}, \tilde{F})(v_1) = (0.5, 0.1, 0.4),\ (\tilde{T}, \tilde{I}, \tilde{F})(v_2) = (0.6, 0.3, 0.2), (\tilde{T}, \tilde{I}, \tilde{F})(v_3) = (0.2, 0.3, 0.4), and\ (\tilde{T}, \tilde{I}, \tilde{F})(v_4) = (0.4, 0.2, 0.5)$ The edge set contains $(\tilde{T}, \tilde{I}, \tilde{F})(v_1, v_2) = (0.5, 0.4, 0.5), (\tilde{T}, \tilde{I}, \tilde{F})(v_2, v_3) = (0.2, 0.3, 0.4), (\tilde{T}, \tilde{I}, \tilde{F})(v_3, v_4) = (0.2, 0.4, 0.4), and\ (\tilde{T}, \tilde{I}, \tilde{F})(v_4, v_1) = (0.4, 0.3, 0.6).$

We have,

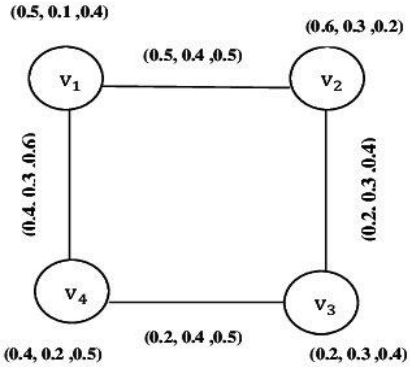


Fig.5.$SVN_{graph}$with $SCI(\tilde{G}) = 4.491$, $GI(\tilde{G}) = 1.764$, $HI(\tilde{G}) = 3.051$.

The score function for all vertices is

$\tilde{S}(v_1) = \frac{0.5+0.1+1-0.4}{3} = \frac{1.2}{3} = 0.4;\quad \tilde{S}(v_2) = \frac{0.6+0.3+1-0.2}{3} = \frac{1.7}{3} = 0.567, \tilde{S}(v_3) = \frac{0.2+0.3+1-0.4}{3} = \frac{1.1}{3} = 0.367;$
$\tilde{S}(v_4) = \frac{0.4+0.2+1-0.5}{3} = \frac{1.1}{3} = 0.367, \tilde{S}(v_1, v_2) = \frac{0.5+0.4+1-0.5}{3} = \frac{1.4}{3} = 0.467; \tilde{S}(v_2, v_3) = \frac{0.2+0.3+1-0.4}{3} = \frac{1.1}{3} = 0.367, \tilde{S}(v_3, v_4) = \frac{0.2+0.4+1-0.5}{3} = \frac{1.1}{3} = 0.367; \tilde{S}(v_4, v_1) = \frac{0.4+0.3+1-0.6}{3} = \frac{1.1}{3} = 0.367.$

now, we have,

$d_s(v_1) = \tilde{S}(v_1, v_2) + \tilde{S}(v_4, v_1) = 0.467 + 0.367 = 0.834;$
$d_s(v_2) = \tilde{S}(v_1, v_2) + \tilde{S}(v_2, v_3) = 0.467 + 0.367 = 0.834;$
$d_s(v_3) = \tilde{S}(v_2, v_3) + \tilde{S}(v_3, v_4) = 0.367 + 0.367 = 0.734;$
$d_s(v_4) = \tilde{S}(v_3, v_4) + \tilde{S}(v_4, v_1) = 0.367 + 0.367 = 0.734;$

and $d_s(v_1, v_2) = 1, d_s(v_1, v_3) = 2, d_s(v_1, v_4) = 1, d_s(v_2, v_3) = 1, d_s(v_2, v_4) = 2, d_s(v_3, v_4) = 1.$

$$\begin{aligned} SCI(\tilde{G}) = {} & 0.4 \times 0.567 \times 1 \times (0.834 + 0.834) + 0.4 \times 0.367 \times 2 \times (0.834 + 0.734) \\ & + 0.4 \times 0.367 \times 1(0.834 + 0.734) + 0.4 \times 0.567 \times 1 \times (0.834 + 0.834) \\ & + 0.567 \times 0.367 \times 1 \times (0.834 + 0.734) + 0.567 \times 0.367 \times 2 \times (0.834 + 0.734) \\ & + 0.4 \times 0.367 \times 2 \times (0.834 + 0.734) + 0.567 \times 0.367 \times 1 \times (0.834 + 0.734) + 0.367 \\ & * 0.367 \times 1 \times (0.734 + 0.734) + 0.4 \times 0.367 \times 1 \times (0.834 + 0.734) \\ & + 0.567 \times 0.367 \times 2 \times (0.834 + 0.734) + 0.367 \times 0.367 \times 1 \times (0.734 + 0.734) \end{aligned}$$

$SCI(\tilde{G}) = 2 \times [0.3783024 + 0.4603648 + 0.2301824 + 0.32628355 + 0.6525671 + 0.19772345]$
$SCI(\tilde{G}) = 2 \times 2.2454237 = 4.491.$

3.7 Gutman Index

Definition 29: The Gutman index $GI(\tilde{G})$of a SVN_{graph} is defined as

$GI(\tilde{G}) = \sum_{(u_i, v_j) \epsilon \sigma^*} \tilde{S}(u_i)\tilde{S}(v_j) d_s(u_i, v_j) d_s(u_i) d_s(v_j), \forall i \neq j \; and \; (v_i, v_j) \in E_G.$

Example 30: Consider the graph in Figure 5,

$$GI(\tilde{G}) = \sum_{u,v \epsilon \sigma^*} \tilde{S}(u_i)\tilde{S}(v_j) d_s(u_i, v_j) d_s(u_i) d_s(v_j),$$

$$\begin{aligned} = {} & 0.4 \times 0.567 \times 1 \times 0.834 \times 0.834 + 0.4 \times 0.367 \times 2 \times 0.834 \times 0.734 \\ & + 0.4 \times 0.367 \times 1 \times 0.834 \times 0.734 + 0.4 \times 0.567 \times 1 \times 0.834 \times 0.834 \\ & + 0.567 \times 0.367 \times 1 \times 0.834 \times 0.734 + 0.567 \times 0.367 \times 2 \times 0.834 \times 0.734 \\ & + 0.4 \times 0.367 \times 2 \times 0.834 \times 0.734 + 0.567 \times 0.367 \times 1 \times 0.834 \times 0.734 \\ & + 0.367 \times 0.367 \times 1 \times 0.734 \times 0.734 + 0.4 \times 0.367 \times 1 \times 0.834 \times 0.734 \\ & + 0.567 \times 0.367 \times 2 \times 0.834 \times 0.734 + 0.367 \times 0.367 \times 1 \times 0.734 \times 0.734 \end{aligned}$$

$= 2 \times [0.1577521 + 0.179729 + 0.0898645 + 0.12738293 + 0.25476586 + 0.07256451 = 1.764.$

3.8 Planarity Index

Definition 31: The Planarity index $PI(\tilde{G})$of a SVN_{graph} is defined as

$$PI(\tilde{G}) = min\left\{\frac{1}{2}\sum_{\tilde{G}-xy} \tilde{S}(u_i)\tilde{S}(v_j)d_s(u_i, v_j)\right\}.$$

Example 32: For the SVN_{graph} in Fig. 6, $\tilde{S}(v_1) = 0.6$, $\tilde{S}(v_2) = 0.5$, $\tilde{S}(v_3) = 0.433, \tilde{S}(v_4) = 0.567$; $\tilde{S}(v_1, v_2) = 0.533, \tilde{S}(v_2, v_3) = 0.467, \tilde{S}(v_3, v_4) = 0.433, \tilde{S}(v_4, v_1) = 0.6, \tilde{S}(v_1, v_3) = .5, \tilde{S}(v_2, v_4) = 0.5; d_s(v_1, v_2) = 0.533, d_s(v_1, v_3) = 1, d_s(v_1, v_4) = 0.6, d_s(v_2, v_3) = 0.467, d_s(v_2, v_4) = 0.5, d_s(v_3, v_4) = 0.433$ ($when\ v_1v_3 is\ deleted$); $d_s(v_1, v_2) = 0.533, d_s(v_1, v_3) = 0.5$, $d_s(v_1, v_4) = 0.6, d_s(v_2, v_3) = 0.467, d_s(v_2, v_4) = 0.9$, $d_s(v_3, v_4) = 0.433$ ($when\ v_2v_4\ is\ deleted$)and$PI(\tilde{G}) = min\{0.9729, 0.95648\} = 0.956$.

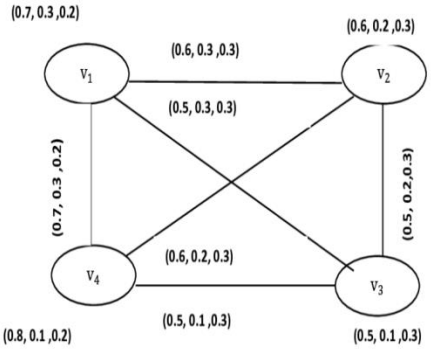


Fig.6.SVN_{graph} with $PI(\tilde{G}) = 0.956$, $RI(\tilde{G}) = 6.184$.

3.9 Randic Index

Definition 33: Let $\tilde{G}$: $(\tilde{\mu}, \tilde{\rho})$ be a SVN_{graph}.The Randic index $RI(\tilde{G})$of a SVN_{graph} is defined as

$$RI(\tilde{G}) = \frac{1}{2}\sum_{(u_i,u_j)\epsilon\mu} \left[\tilde{S}(u_i)\tilde{S}(u_j)d_s(u_i)d_s(u_j)\right]^{\frac{-1}{2}} \ , \forall i \neq j.$$

Example 34: Consider the SVNG as shown in Figure 6, by simple calculations, we get

$$RI(\tilde{G}) = \frac{1}{2}\Big\{[\tilde{S}(v_1)\tilde{S}(v_2)d_s(v_1)d_s(v_2)]^{\frac{-1}{2}} + [\tilde{S}(v_2)\tilde{S}(v_3)d_s(v_2)d_s(v_3)]^{\frac{-1}{2}} + [\tilde{S}(v_3)\tilde{S}(v_4)d_s(v_3)d_s(v_4)]^{\frac{-1}{2}} + [\tilde{S}(v_4)\tilde{S}(v_1)d_s(v_4)d_s(v_1)]^{\frac{-1}{2}}\Big\}$$

$$= \frac{1}{2}\{2.51774893 + 2.80184681 + 3.7122555 + 3.33584545\} = 6.184.$$

3.10Harmonic Index

Definition 35: Let $\tilde{G}$: $(\tilde{\mu}, \tilde{\rho})$ be a SVN_{graph}.The Harmonic index $HI(\tilde{G})$of a SVN_{graph} is defined as

$$HI(\tilde{G}) = \frac{1}{2}\sum_{(u_i,u_j)\epsilon\mu} \left[\tilde{S}(u_i)d_s(u_i)+\tilde{S}(u_j)d_s(u_j)\right]^{-1} \ , \forall i \neq j.$$

Example 36: Consider the SVN_{graph} as shown in Figure 6.

$$HI(\tilde{G}) = \frac{1}{2}\Big\{\left[\tilde{S}(v_1)d_s(v_1) + \tilde{S}(v_2)d_s(v_2)\right]^{-1} + \left[\tilde{S}(v_2)d_s(v_2) + \tilde{S}(v_3)d_s(v_3)\right]^{-1} + \left[\tilde{S}(v_3)d_s(v_3) + \tilde{S}(v_4)d_s(v_4)\right]^{-1} + \left[\tilde{S}(v_4)d_s(v_4) + \tilde{S}(v_1)d_s(v_1)\right]^{-1}\Big\}$$

$$= \frac{1}{2}(1.23995943 + 1.34724408 + 1.85612782 + 1.6584353) = 3.051.$$

4 Identification of influential factors affecting students' performance in semester examinations in the educational institution using score topological indices

The conduct of a semester examination in an education institute depends on many factors, out of which few factors affect the students' performance either directly or indirectly in the examination. These factors are classified into the following four categories.

Category 1: Examination hall related issues: (Temperature, Light disturbing sounds both inside and

outside the examination hall) Environment of exam Hall (H_1), Attitude of the invigilator (H_2), Seating Plan (H_3).

Category 2: Question paper-related issues: Change in question pattern (Q_1), Similarities in question paper (Q_2), Ambiguity in questions of question paper (Q_3), Mood of paper corrector (Q_4), Difficulty levels in question paper (Q_5), Extra and detailed study for examination (Q_6), Selection of questions in the choice-based questions (Q_7).

Category 3: Personal issues of a student: Shortage of attendance (P_1), Domestic problems (P_2), financial condition (P_3), Attention and interest of parents (P_4).

Category 4: Exam Preparation related issues: Preparation without determination of objectives (E_1), Overconfidence (E_2), Methods for approaching the questions (E_3), Handwriting (E_4), Examination fever (E_5).

The questionnaire was prepared based on the above main four factors and used as a tool for this research. The questionnaire was circulated among more than 50 engineering colleges in the city, around 250 students, and 50 controllers of examination in various institutions and universities in and around Chennai city. The main objective of the study is to determine influential factors in each of the above-mentioned four classes that affect the student's performance in the examination. Based on the collected questionnaire data and discussion with the CoEs from different institutions in the city, we draw the relation SVN_{graph} of the most influential factors affecting students' performance in semester examinations with neutrosophic values, represented in Figures 8-11. Also, vertex and edge-score values, and the score degree of the vertex are calculated for different neutrosophic values, illustrated in Tables 1-4.

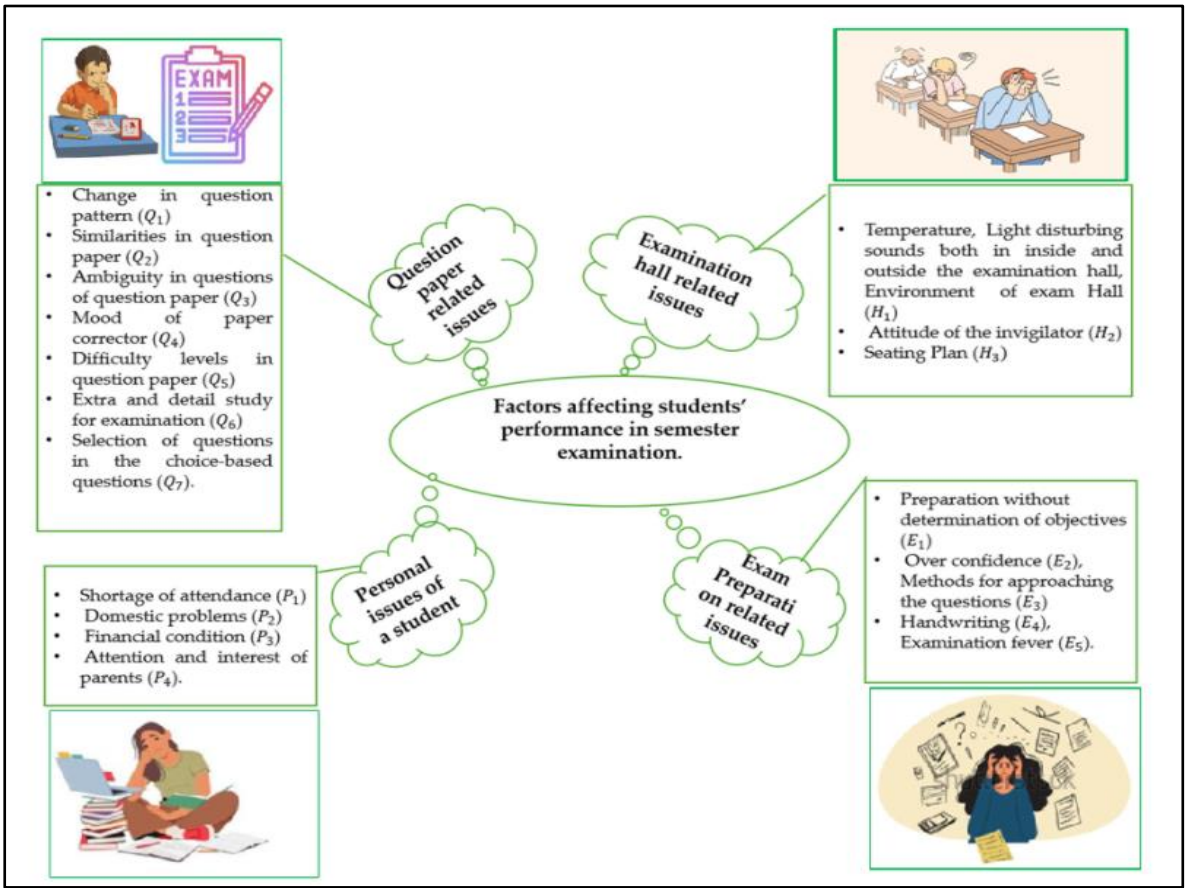

EXAM
Change in question pattern (Q_1)
Similarities in question paper (Q_2)
Ambiguity in questions of question paper (Q_3)
Mood of paper corrector (Q_4)
Difficulty levels in question paper (Q_5)
Extra and detail study for examination (Q_6)
Selection of questions in the choice-based questions (Q_7).
Question paper related issues
Examination hall related issues
Temperature, Light disturbing sounds both in inside and outside the examination hall, Environment of exam Hall (H_1)
Attitude of the invigilator (H_2)
Seating Plan (H_3)
Factors affecting students' performance in semester examination.
Shortage of attendance (P_1)
Domestic problems (P_2)
Financial condition (P_3)
Attention and interest of parents (P_4).
Personal issues of a student
Exam Preparation related issues
Preparation without determination of objectives (E_1)
Over confidence (E_2), Methods for approaching the questions (E_3)
Handwriting (E_4), Examination fever (E_5).

Fig.7. Influential factors affecting student performance

4.1 Analysis of Category 1

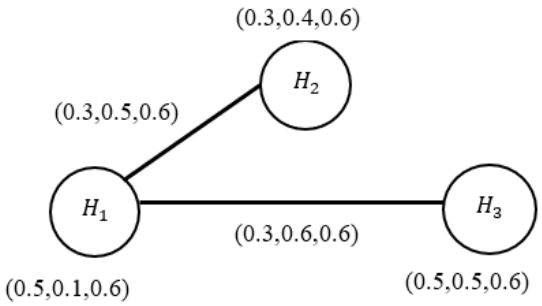


Fig.8.SVN_{graph} −Examination Hall related issues

Table 1. Calculated values of vertex and edge --score, and score degree of vertex for category 1

Vertex-score values	Edge-score values	Score degree of the vertex
$\tilde{S}(H_1) = \frac{0.5+0.1+1-0.6}{3} = 0.33$	$\tilde{S}(H_1,H_2) = \frac{0.3+0.5+1-0.6}{3} = 0.27$ $\tilde{S}(H_1,H_3) = \frac{0.3+0.6+1-0.6}{3} = 0.43$	$d_s(H_1) = \tilde{S}(H_1,H_2) + \tilde{S}(H_1,H_3)$ $= 0.27 + 0.43 = 0.70$
$\tilde{S}(H_2) = \frac{0.3+0.4+1-0.6}{3} = 0.37$		$d_s(H_2) = \tilde{S}(H_1,H_2) = 0.27$
$\tilde{S}(H_3) = \frac{0.5+0.5+1-0.6}{3} = 0.47$		$d_s(H_3) = \tilde{S}(H_1,H_3) = 0.43$

The Zagreb's indices of $\tilde{G}$ On first and second kinds are $ZI(\tilde{G}) = \tilde{S}(H_i)d_s^2(H_i), H_i \in V_G \; where \; i = \{1,2,3\}$

$ZI(\tilde{G}) = 0.33(0.27^2 + 0.43^2) + 0.37(0.27^2) + 0.47(0.43^2)$

$= 0.33(0.2578) + 0.37(0.0729) + 0.47(0.0729)$

$= 0.0367.$

4.2 Analysis of Category 2

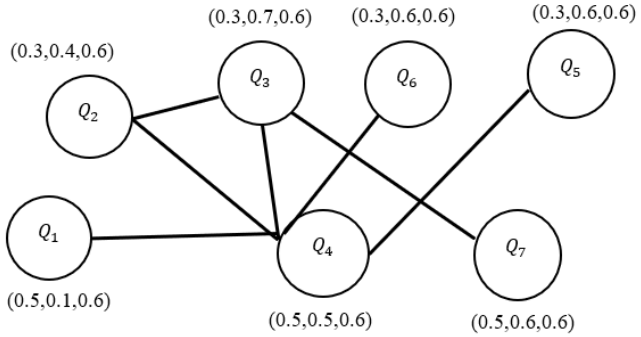


Fig.9.SVN_{graph} −Question paper-related issues

Table 2. Calculated values of vertex and edge-score, and score degree of the vertex for category 2

Vertex-score values	Edge-score values	Score degree of the vertex
$\tilde{S}(Q_1) = \frac{0.5+0.1+1-0.4}{3} = 0.4$	$\tilde{S}(Q_1,Q_4) = \frac{0.5+0.5+1-0.6}{3} = 0.4$ $\tilde{S}(Q_2,Q_3) = \frac{0.3+0.7+1-0.6}{3} = 0.47$ $\tilde{S}(Q_2,Q_4) = \frac{0.3+0.7+1-0.6}{3} = 0.47$ $\tilde{S}(Q_3,Q_4) = \frac{0.3+0.7+1-0.6}{3} = 0.47$ $\tilde{S}(Q_3,Q_7) = \frac{0.3+0.7+1-0.6}{3} = 0.47$	$d_s(Q_1) = \tilde{S}(Q_1,Q_4) = 0.4$
$\tilde{S}(Q_2) = \frac{0.3+0.4+1-0.6}{3} = 0.37$		$d_s(Q_2) = \tilde{S}(Q_2,Q_3) + \tilde{S}(Q_2,Q_4)$ $= 0.47 + 0.47 = 0.94$
$\tilde{S}(Q_3) = \frac{0.5+0.7+1-0.6}{3} = 0.533$		$d_s(Q_3) = \tilde{S}(Q_3,Q_4) + \tilde{S}(Q_3,Q_7)$ $= 0.47 + 0.47 = 1.03$
$\tilde{S}(Q_4) = \frac{0.5+0.6+1-0.6}{3} = 0.5$		$d_s(Q_4) = \tilde{S}(Q_1,Q_4) + \tilde{S}(Q_2,Q_4)$ $+ \tilde{S}(Q_3,Q_4)$ $+ \tilde{S}(Q_4,Q_5)$ $+ \tilde{S}(Q_4,Q_6)$

	$\tilde{S}(Q_4,Q_5)=\frac{0.3+0.7+1-0.6}{3}=0.47$	$=0.4+0.47+0.47+0.47+0.47$
$\tilde{S}(Q_5)=\frac{0.5+0.1+1-0.4}{3}=0.4$		$d_s(Q_5)=\tilde{S}\left(Q_4,Q_5\right)=0.47$
$\tilde{S}(Q_6)=\frac{0.5+0.1+1-0.4}{3}=0.4$	$\tilde{S}(Q_4,Q_6)=\frac{0.3+0.7+1-0.6}{3}=0.47$	$d_s(Q_6)=\tilde{S}\left(Q_4,Q_6\right)=0.47$
$\tilde{S}(Q_7)=\frac{0.5+0.1+1-0.4}{3}=0.4$		$d_s(Q_7)=\tilde{S}\left(Q_3,Q_7\right)=0.47$

The Zagreb's indices of $\tilde{G}$ On first and second kinds are $ZI(\tilde{G})=\sum_{Q_i\epsilon\sigma^*}\tilde{S}(Q_i)d_s^2(Q_i), Q_i\in V_G \; where \; i=\{1,2,3,4,5,6,7\}$.

$ZI(\tilde{G})=0.4(0.4^2)+0.37(0.47^2+0.47^2)+0.533(0.47^2+0.47^2)$
$+0.5(0.4^2+0.47^2+0.47^2+0.47^2+0.47^2)+0.4(0.47^2)+0.4(0.47^2)+0.4(0.47^2)$
$=0.4(0.16)+0.37(0.2697)+0.533(0.2697)+0.5(1.0436)+3\,X\,0.4(0.2209)$
$=0.3274.$

4.3 Analysis of Category 3

The first and second kinds of Zagreb's indices for Fig.10 is

$ZI(\tilde{G})=0.4(0.4^2+0.47^2)+0.37(0.4^2+0.47^2+0.5^2)+0.533(0.53^2+0.47^2+0.47^2)$
$+0.5(0.5^2+0.53^2)$
$=0.4(0.3809)+0.567(0.6309)+0.367(0.7227)+0.367(0.5309)$
$=0.5098.$

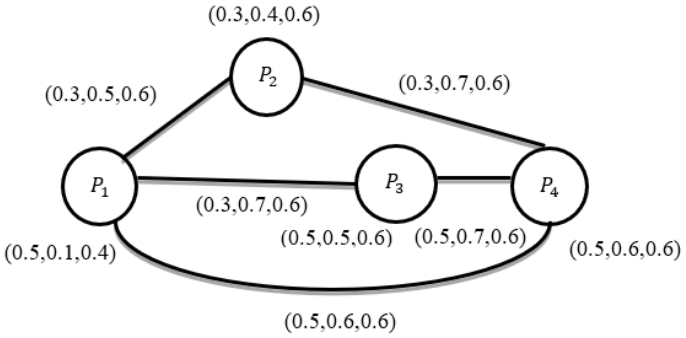


Fig.10. SVN_{graph} −Personal issues of a student

Table 3. Calculated values of vertex and edge-score, and score degree of vertex for category 3

Vertex-score values	Edge-score values	Score degree of the vertex
$\tilde{S}(P_1)=\frac{0.5+0.1+1-0.4}{3}=0.4$	$\tilde{S}(P_1,P_2)=\frac{0.3+0.5+1-0.6}{3}=0.4$	$d_s(P_1)=\tilde{S}\left(P_1,P_2\right)+\tilde{S}\left(P_1,P_3\right)=0.4+0.47=0.87$
$\tilde{S}(P_2)=\frac{0.3+0.4+1-0.6}{3}=0.37$	$\tilde{S}(P_1,P_3)=\frac{0.3+0.7+1-0.6}{3}=0.47$	$d_s(P_2)=\tilde{S}\left(P_1,P_2\right)+\tilde{S}\left(P_2,P_3\right)+\tilde{S}\left(P_2,P_4\right)=0.4+0.47+.5=1.37$
$\tilde{S}(P_3)=\frac{0.5+0.7+1-0.6}{3}=0.533$	$\tilde{S}(P_2,P_3)=\frac{0.3+0.7+1-0.6}{3}=0.47$	$d_s(P_3)=\tilde{S}\left(P_3,P_4\right)+\tilde{S}\left(P_2,P_3\right)+\tilde{S}\left(P_1,P_3\right)=0.53+0.47+0.47=1.47$
$\tilde{S}(P_4)=\frac{0.5+0.6+1-0.6}{3}=0.5$	$\tilde{S}(P_2,P_4)=\frac{0.5+0.6+1-0.6}{3}=0.5$	$d_s(P_4)=\tilde{S}\left(P_3,P_4\right)+\tilde{S}\left(P_2,P_4\right)=0.5+0.53=1.03$
	$\tilde{S}(P_3,P_4)=\frac{0.5+0.7+1-0.6}{3}=0.53$	

4.4 Analysis of Category 4

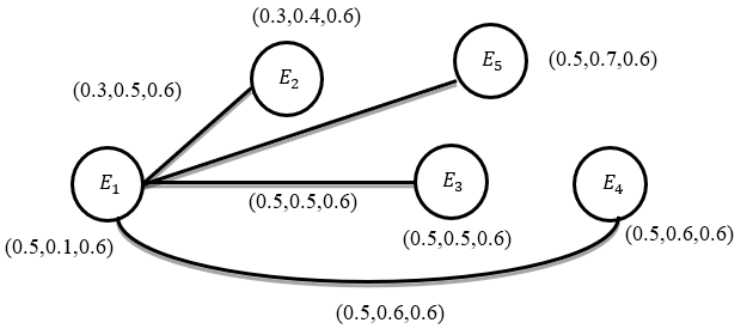


Fig.11.SVN_{graph} −Examination Hall related issues

Table 4. Calculated values of vertex and edge-score, and score degree of vertex for category 4

Vertex-score values	Edge-score values	Score degree of the vertex
$\tilde{S}(E_1) = \frac{0.5+0.1+1-0.6}{3} = 0.33$	$\tilde{S}(E_1, E_2) = \frac{0.3+0.5+1-0.6}{3} = 0.4$ $\tilde{S}(E_1, E_5) = \frac{0.3+0.7+1-0.6}{3} = 0.47$ $\tilde{S}(E_1, E_3) = \frac{0.5+0.5+1-0.6}{3} = 0.47$ $\tilde{S}(E_1, E_4) = \frac{0.5+0.6+1-0.6}{3} = 0.5$	$d_s(E_1) = \tilde{S}(E_1, E_2) + \tilde{S}(E_1, E_3) + \tilde{S}(E_1, E_4) + \tilde{S}(E_1, E_5) = 0.4 + 0.47 + 0.47 + .5 = 1.84$
$\tilde{S}(E_2) = \frac{0.3+0.4+1-0.6}{3} = 0.37$		$d_s(E_2) = \tilde{S}(E_1, E_2) = 0.4$
$\tilde{S}(E_3) = \frac{0.5+0.5+1-0.6}{3} = 0.47$		$d_s(E_3) = \tilde{S}(E_1, E_3) = 0.47$
$\tilde{S}(E_4) = \frac{0.5+0.6+1-0.6}{3} = 0.5$		$d_s(E_4) = \tilde{S}(E_1, E_4) = 0.5$
$\tilde{S}(E_5) = \frac{0.5+0.7+1-0.6}{3} = 0.53$		$d_s(E_5) = \tilde{S}(E_1, E_5) = .47$

The Zagreb's indices of Figure 11 are calculated as follows:

$ZI(\tilde{G}) = 0.4(0.4^2 + 0.47^2 + 0.47^2 + 0.5^2) + 0.37(0.4^2) + 0.47(0.47^2) + 0.5(0.47^2) + .53(.47^2) +$

$= 0.4(0.8518) + 0.37(0.16) + 0.47(0.2209) + 0.5(0.2209) + 0.53(0.2209)$

$= 0.1478.$

5 Conclusion

We concluded that the personal issues of a student play an important role in their performance in the examination even though the change in question paper pattern or other hall environment issues or evaluation process is based on the values of the first Zagreb index of all the above four categories. The other factors affect the student's performance to a negligible level compared with individual student's issues.

References:

1. Zadeh, L.A. Fuzzy sets. Inf. Control, 8, 338–353, 1965.
2. Kaufmann, A, Introduction to the Theory of Fuzzy Subsets; Academic Press: Cambridge, MA, USA, 2.
3. Rosenfeld, A, Fuzzy graphs. In Fuzzy Sets and Their Applications to Cognitive and Decision Processes; Elsevier: Amsterdam, The Netherlands, 77–95,1975.
4. Atanassov, K.T. Intuitionistic fuzzy sets. Fuzzy Sets Syst., 20, 87–96,1986.
5. Atanassov, K.T. Intuitionistic fuzzy sets. In Intuitionistic Fuzzy Sets; Springer: Berlin, Germany, 1–137,1999.
6. K. Atanassov and A. Shannon,On a generalisation of Intuitionistic fuzzy graph,Notes on Intuitionistic fuzzy Sets, 8, 73-78, 2002.

7. Smarandache, F. A Unifying Field in Logic: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability and Statistics; Infinite Study; Modern Science Publisher: New York, NY, USA, 2005.
8. F. Smarandache," Refined Literal Indeterminacy and the Multiplication Law of Sub-Indeterminacies," Neutrosophic Sets and Systems,9, 58 - 63, 2015.
9. F. Smarandache, "Types of Neutrosophic Graphs and neutrosophic Algebraic Structures together with their Applications in Technology," seminar, Universitatea Transilvania din Brasov, Facultatea de Design de Produs si Mediu, Brasov, Romania. 06 June 2015.
10. S. Broumi, M. Talea, A. Bakali, F. Smarandache, "Single Valued Neutrosophic Graphs," Journal of New Theory, N 10, 86-101, 2016.
11. Broumi, S., Talea, M., Smarandache, F. and Bakali, A. Single Valued Neutrosophic Graphs: Degree, Order and Size, IEEE International Conference on Fuzzy Systems, 11, 2016,84-102.
12. Mitchell, H. B., Ranking intuitionistic fuzzy numbers, Int. J. Uncertainty, Fuzziness Knowledge Based Syst., 12(3), 377–386, 2007.
13. Eliasi M, Raeisi G and Taeri B, Wiener index of some graph operations Discrete. Appl. Math. 160 1333–44,2012.
14. Ghorbani M and Hosseinzadeh M A, A new version of Zagreb indices Filomat, 26, 93–100, 2012.
15. Idrees N, Saif M J, Rauf A and Mustafa S, First and second Zagreb eccentricity indices of thorny graphs Symmetry (Basel), 9, 1–9,2017.
16. Li X and Shi Y, A survey on the Randic index MATCH Commun. Math. Comput. Chem, 59, 127– 56,2008.
17. Xing R, Zhou B and Dong F, On atom-bond connectivity index of connected graphs, Discrete Appl. Math. ,2011.
18. Zhong L, The harmonic index for graphs Appl. Math. Lett., 25 561–6,2012.
19. Lučić B, Trinajstić N and Zhou B 2009 Comparison between the sum-connectivity index and productconnectivity index for benzenoid hydrocarbons Chem. Phys. Lett. 475 146–8
20. Zhou B and Trinajstić N 2010 On general sum-connectivity index J. Math. Chem. 47 210–8
21. HUA H, Wiener and Schultz molecular topological indices of graphs with specified cut edges MATCH Commun. Math. Comput. Chem. 61,643, 2009.
22. Andova V, Dimitrov D, Fink J and Škrekovski R, Bounds on Gutman index MatchCommunications Math. Comput. Chem. 67, 515, 2012.
23. Shriram Kalathian, Sujatha Ramalingam, Sundareswaran Raman and Narasimman Srinivasan, Some topological indices in fuzzy graphs, Journal of Intelligent & Fuzzy Systems, 39(5), 6033-6046, 2020.
24. Masoud Ghods, Zahra Rostami, Introduction to Topological Indices in Neutrosophic Graphs, Neutrosophic Sets and Systems, 35, 2020.
25. P.T. Chang, E.S. Lee, Fuzzy decision networks and deconvolution Comput. Math. Appl., 37, 53-63, 1999.
26. H. Lee-Kwang, J.-H. Lee, A method for ranking fuzzy numbers and its application to decision-making
27. IEEE Trans. Fuzzy Sets Syst., 7, 677-685, 1999.
28. R. Jain, Decision-making in the presence of fuzzy variables IEEE Trans. Syst. Man Cybern., 6 , 698-703, 1976,
29. Mitchell, H.B., Ranking - Intuitionistic Fuzzy numbers. Int. J. Uncert. Fuzz. Knowl. Based Syst. 12, 377–386, 2004.
30. Nayagam, V. L. G., Venkateshwari, G., and Sivaraman, G., Ranking of intuitionistic fuzzy numbers, In Proc. of the IEEE Int. Conf. on Fuzzy Systems (FUZZ-IEEE'08), 1971–1974, 2008.
31. Nayagam, V.L.G., Venkateshwari, G., Sivaraman, G., Modified ranking of intuitionistic fuzzy numbers. Notes on Intuitionistic Fuzzy Sets 17, 5–22, 2011.

32. Li, D.F., Nan J.X., Zhang M.J., A ranking Method of triangular intuitionistic fuzzy numbers and applications to decision making, international Journal of Computational Intelligence Systems 3, 522–530, 2010.
33. Sahin, R., Multi-criteria neutrosophic decision making method based on score and accuracy functions under neutrosophic environment, International Journal for Uncertainty Quantification, 6 (5), 377–385, 2016.
34. Przemysław Grzegorzewski, Distances between intuitionistic fuzzy sets and/or interval-valued fuzzy sets based on the Hausdorff metric, Fuzzy Sets and Systems, 148(2), 319-328, 2004.
35. Nancy and Harish Garg, an improved score function for ranking neutrosophic sets and its application to decision-making process,International Journal for Uncertainty Quantification, 6 (5), 377–385, 2016.
36. Saima Rasul, Qadir Bukhsh, A study of factors affecting students' performance in examination at university level, Procedia Social and Behavioral Sciences, 15, 2042–2047, 2011.
37. Kassu Mehari Beyene Jemal Ayalew Yimam,Multilevel Analysis for Identifying Factors Influencing Academic Achievement of Students in Higher Education Institution: The Case of Wollo University, Journal of Education and Practice, 7(13), 2016.
38. Yousuf Nasser Said Al Husaini,Nur Syufiza Ahmad Shukor, Factors Affecting Students' Academic Performance: A review,Published/ publié in Res Militaris (resmilitaris.net), 12(6), Winter 2022.
39. Kawtar Tani, Elizabeth Dalzell, Nathan Ehambaranathan, Sheela Murugasu and Anne Steele,Evaluation of factors affecting students' performance in tertiary education,Journal of Pedagogical Research ,3 (2), 2019.
40. Wei Liu and Lei Zhang, Performance Evaluation and Identification of Key Influencing Factors for Student Achievement Based on the Entropy-Weighted TOPSIS Model, Computational Intelligence and Neuroscience Volume 2023, Article ID 1253824, 12 pages.

Received: June 5, 2024. Accepted: Oct 13, 2024

NSS
Neutrosophic Sets and Systems, Vol. 75, 2025
University of New Mexico

Enhancing Multiple-Attribute Decision-Making Framework for Talent Cultivation Quality Evaluation Through Using Weighted Dual Hamy Mean and Prioritized Aggregation Operators

Bing Sun[1], Dongyan Gao[1], Shujuan Yin[2]*

[1]Anhui Wenda University of Information Engineering, Hefei, 231201, Anhui, China (Bing Sun, E-mail: wendaxhf@163.com ; Dongyan Gao, E-mail: herosb812@126.com)

[2]Baotou Teachers' College, Inner Mongolia University of Science & Technology, Baotou, 014030, China

*Corresponding author, E-mail: yueying_82@163.com

Abstract: With the rapid development of the economy and the transformation and upgrading of industrial structure, universities have increased their focus on theoretical knowledge and application, actively providing practical platforms for students, enriching the connection between theoretical knowledge and practice, deepening the reform of quality education, constructing a modern education system, and ensuring that students can adapt their professional skills to the requirements of industry technology in the field of computer information. Cultivating high-quality computer information talents can not only promote better employment for students, but also promote the development of computer information education in universities. The talent cultivation quality evaluation in practical aspects of computer information major is looked as the multiple-attribute decision-making (MADM) issue. In this paper, we extended the weighted dual Hamy mean (WDHM) operator and prioritized aggregation (PA) operator to 2-tuple linguistic neutrosophic sets (2TLNSs) to propose the 2-tuple linguistic neutrosophic prioritized WDHM (2TLNPWDHM) operator. Finally, a numerical example for talent cultivation quality evaluation in practical aspects of computer information major is employed to verify the 2TLNPWDHM operator. The main contributions of this study are summarized: (1) the 2TLNPWDHM operator is built; (2) the 2TLNPWDHM operator is designed to cope with the MADM with 2TLNNs; (3) a numerical example for talent cultivation quality evaluation in practical aspects of computer information major is supplied to proof the designed method; (4) some comparative studies are utilized to verify the rationality of the 2TLNPDHM.

Keywords: Multiple-attribute decision-making (MADM); 2TLNSs; 2TLNPWDHM; talent cultivation quality evaluation; computer information major

1. Introduction

Focusing on national strategies such as Made in China 2025 and Artificial Intelligence 2.0, as well as the industrial transformation and upgrading of the old industrial base in Northeast China, as well as the demand for high-quality skilled talents in emerging industries such as the Internet of Things, big data, and cloud computing, we will promote the deep integration of emerging technologies and computer information science and technology professional knowledge, abilities, and qualities. Starting from existing professional advantages, we will integrate resources, thoroughly analyze the gap between the existing talent training programs in the field of Computer Information Science and technology and the requirements of new engineering construction, develop talent training programs that are suitable for the transformation and upgrading of traditional industries and the cultivation and development of emerging industries, clarify training objectives, scientifically design curriculum systems, increase the proportion of practical courses, improve innovation and entrepreneurship levels, and actively adapt to the requirements of new engineering construction[1, 2]. Cultivate composite high-quality new engineering talents with a solid theoretical foundation, strong practical innovation ability, and international competitiveness. Professional construction should be integrated with industry, and talent cultivation should be combined with job positions, so that computer information technology can better serve other majors. Adhere to the educational philosophy of "student-centered, outcome oriented, and continuous improvement", fully implement the fundamental task of "cultivating virtue and nurturing talents", adhere to the principle of "people-oriented", promote the "four returns", and form a "highly unified foundation and major, organic integration of theory and practice, deep integration of science and technology, close cooperation between universities and enterprises, and complementary advantages between domestic and foreign"[3, 4]. The excellent talent training model and practical talent training system for computer information

science and technology promote the integration of industry and education, joint construction and sharing, and international cooperation, achieving deep integration of educational elements, and actively promoting the construction of new engineering disciplines in computer information science and technology. Construct a talent training program for computer information science and technology, continuously strengthen the construction of professional connotation, integrate ideological education, vocational quality education, and innovation and entrepreneurship education into the talent training process, scientifically construct a curriculum teaching system, and through interdisciplinary cooperation, personalized mentor system training, and school enterprise collaborative education practice mechanism innovation, build a talent training system that integrates disciplines and collaborates on education[5, 6]. Emphasize the connection between profession and industry, and face the needs of the industry; Emphasis should be placed on interdisciplinary studies, with a focus on the intersection of computer information science and technology with computer science, big data, communication, artificial intelligence, biomedical and other disciplines, and the cultivation of versatile talents should be strengthened; Emphasize the cultivation of students' innovation and entrepreneurship abilities, cultivate high-quality specialized talents with innovative abilities and awareness, entrepreneurial spirit, and the courage to engage in practice; Emphasize student-centered, personalized curriculum system with multiple modules, and implement a mentor system and small class teaching system; Implementing the fundamental task of cultivating virtue and talent, strengthening the construction of ideological and political education in the curriculum, and emphasizing the cultivation of patriotism; Based on the characteristics of interdisciplinary integration, intensive knowledge of new technologies, and strong engineering applications in the field of computer information science and technology, the positioning of cultivating innovative talents with a wide range of perspectives is clarified, and a talent training plan that applies what is learned is formed, highlighting the applicability and practicality of the discipline, and further improving the quality of talent training. Develop a student-centered and results oriented professional training program, and promote interdisciplinary training models.

Integrating multiple fields such as computers, communication, and computer, highlighting the penetration and integration of science and engineering[7]. Thick foundation and wide caliber, serving the main battlefield of national economic information technology; Emphasize cross cutting and strong innovation, facing major strategic needs in the field of national computer information; To cultivate scientists, engineers, and industry elites with a solid knowledge foundation and professional skills in computer information science and technology, comprehensive development in morality, intelligence, physical fitness, aesthetics, and labor, a strong sense of patriotism and social responsibility, and excellent comprehensive qualities[8]. Highlighting the education and teaching philosophy of "student-centered", closely following the development trend of computer technology, with the goal of serving the Internet of Things, big data, cloud computing, artificial intelligence and other directions, adhering to the principles of "emphasizing foundation, strengthening practice, focusing on innovation, and showcasing individuality", and closely surrounding the "gender equality" standard, constructing a curriculum system based on the basic idea of "basic platform module+characteristic innovation module+independent personalized module"[9, 10]. Increase the proportion of practical courses, strengthen the cultivation of students' practical skills, and enable talent cultivation to adapt to diverse social needs. Strengthen the construction of "golden courses", form a shared and integrated curriculum construction system, and continue to play the exemplary and leading role of high-quality courses in curriculum ideological and political education, blended teaching, and research feedback teaching[11]. Open a batch of new courses that reflect the latest developments in industry and technology, and enhance the support of the courses for professional training goals and graduation requirements[12, 13]. Strengthen the construction of practical teaching centers, invest sufficient funds to improve the hardware conditions of practical teaching; Closely cooperate with theoretical teaching and set up supporting experimental courses for important foundational and professional courses; Offering large-scale comprehensive experimental courses that are relatively independent and involve several theoretical courses[14]; Building a universal platform for basic experiments for lower grade undergraduate students, a

professional experimental platform for higher grade undergraduate students, and a high-end platform for comprehensive experiments for undergraduate and graduate students; Organize students to participate in the college's series of scientific research lectures and academic seminars, expanding their scientific and technological perspectives; Encourage students to enter research teams in advance and actively participate in research training; Encourage teachers to transform research content into innovative training projects and graduation project topics, open to students, stimulate their research enthusiasm, and enhance their research practical abilities[15-17]; Strengthen extracurricular practical training, establish open laboratories, add "open innovation experimental projects" and "college student innovation and entrepreneurship training projects", actively organize students to participate in various levels of "science and technology competitions", replace practice with competitions, apply what is learned, and improve students' innovation awareness and hands-on practical ability at multiple levels and aspects. By participating in practical activities such as scientific research projects, various subject competitions, and college student innovation and entrepreneurship training programs, we can quickly enhance students' innovation and practical abilities, cultivate high-quality innovation and entrepreneurship talents for the computer information science and technology industry, and truly achieve a training model that combines theory and practice[16].

Decision making is one of the most common activities in daily life [18-21]. The key to information driven MADM lies in the management of decision information, including the expression, transformation and integration of information[22-24]. When using MADM theory to solve decision-making issues, it is first necessary to portray the decision information through an appropriate way [25-29]. Then, Zadeh [30] devised the fuzzy sets(FSs) to portray uncertain information. Smarandache [31] devised the neutrosophic sets (NSs). The SVNSs [32] and INSs [33] were devised for uncertain MADM. Wang et al. [34] came up with the 2TLNSs which the portrayed information is portrayed through 2TLs[35-37]. The 2TLNSs

[34] integrate the advantages of the single-values neutrosophic sets (SVNSs) [38] and 2-tuple linguistic information representation model[35-37]. Wang et al. [39] came up with CODAS model to solve the MAGDM under 2TLNSs. Wu et al. [40] came up with the Hamy mean information models for 2TLNNs.

The talent cultivation quality evaluation in practical aspects of computer information major is looked as the MADM) issue. The PA operator [41] could prioritization relationship among the attributes and WDHM operator [40] could consider the interrelationship of the futilized arguments. The 2TLNSs [34] integrate the advantages of the single-values neutrosophic sets (SVNSs) [38] and 2-tuple linguistic model[35-37], which can effectively portray and describe uncertain information during the talent cultivation quality evaluation in practical aspects of computer information major. In order to take the full use of these two kinds of operators, in this paper, we combine the WDHM operator and PA operator with 2TLNSs to propose the 2TLNPWDHM operator. The 2TLNPWDHM could consider the prioritization among the attributes and it could also consider the relationships between the attributes and could effectively avoid the problem of information loss in the process of talent cultivation quality evaluation in practical aspects of computer information major. Then, the 2TLNPWDHM operator is utilized to handle MADM with 2TLNS. Finally, taking the talent cultivation quality evaluation in practical aspects of computer information major as an example, the proposed method is verified. The main aim and research motivation of this study are summarized as (1) the WDHM operator and PA operator are utilized to derive the 2TLNPWDHM operator under 2TLNSs; (2) the 2TLNPWDHM operator is developed for

the MADM under 2TLNSs, (3) a numerical study is provided for talent cultivation quality evaluation in practical aspects of computer information major to demonstrate the 2TLNPWDHM, and (4) some comparative studies are provided to justify the 2TLNPWDHM operator.

Then, the research structure is outlined. In Sect. 2, the concept of 2TLNSs is introduced. In Sect. 3, the 2TLNPWDHM operator is designed for MADM. In Sect. 4, a numerical example is utilized to solve the talent cultivation quality evaluation in practical aspects of computer information major. Sect. 5 concludes this study.

2. Preliminaries

Wang et al. [34] constructed the 2TLNSs.

2.1. 2TLSs

Definition 1[35]. Let $ys_1, ys_2, \ldots, ys_Z$ be linguistic variables. The ys is derived:

$$ys = \begin{Bmatrix} ys_0 = extremely\ poor, ys_1 = very\ poor, ys_2 = poor, ys_3 = medium, \\ ys_4 = good, ys_5 = very\ good, ys_6 = extremely\ good. \end{Bmatrix}$$

Definition 2[35]. Let $ys = \{ys_i \mid i = 0,1,2,\cdots,Y\}$ and $\beta \in [0, H]$ is the linguistic symbolic value. The Δ is employed to derive the 2-tuple linguistic equivalent to β in Eq. (1)-Eq. (2):

$$\Delta : [0, Y] \to ys \times [-0.5, 0.5), \tag{1}$$

$$\Delta(\beta) = \begin{cases} ys_i, i = round(\beta) \\ \alpha = \beta - i, \quad \alpha \in [-0.5, 0.5) \end{cases}, \tag{2}$$

where round(.) is the round operation.

Definition 3[35]. Let $ys = \{ys_i \mid i = 0,1,2,\cdots,Y\}$ be a LTSs and (ys_i, α) be a 2-tuple. The function Δ^{-1} returns the numerical information $\beta \in [0, Y] \subset R$ in Eq. (3)-Eq.(4).

$$\Delta^{-1}:\ ys\times\left[-0.5,0.5\right)\to\left[0,Y\right], \tag{3}$$

$$\Delta^{-1}\left(ys_i,\alpha\right)=i+\alpha=\beta, \tag{4}$$

2.2. SVNSs

Definition 4[32]. The SVNSs is derived in Eq. (5):

$$y\beta=\left\{\left(o,\phi_\beta\left(y\right),\varphi_\beta\left(y\right),\gamma_\beta\left(y\right)\right)\middle|\, y\in\Theta\right\} \tag{5}$$

where $\phi_\beta\left(y\right),\varphi_\beta\left(y\right),\gamma_\beta\left(y\right)\in\left[0,1\right]$ represent TM, IM and FM, $0\le\phi_\beta\left(y\right)+\varphi_\beta\left(y\right)+\gamma_\beta\left(y\right)\le 3$.

2.3. 2TLNSs

Definition 5 [34]**.** Let $ys=\left\{ys_i\middle|i=0,1,2,\cdots,Y\right\}$. If $y\delta=\left\langle\left(ys_t,y\xi\right),\left(ys_i,y\psi\right),\left(ys_f,y\zeta\right)\right\rangle$ is defined for $ys_t,ys_i,ys_f\in ys$, $y\xi,y\psi,y\zeta\in\left[0,0.5\right)$, where $\left(ys_t,y\xi\right),\left(ys_i,y\psi\right),\left(ys_f,y\zeta\right)$ depict the TM, IM and FM by 2TLSs, then the 2TLNSs are designed in Eq. (6):

$$y\delta_j=\left\langle\left(ys_{t_j},y\xi_j\right),\left(ys_{i_j},y\psi_j\right),\left(ys_{f_j},y\zeta_j\right)\right\rangle \tag{6}$$

where $0\le\Delta^{-1}\left(ys_{t_j},y\xi_j\right)\le Y,0\le\Delta^{-1}\left(ys_{i_j},y\psi_j\right)\le Y,0\le\Delta^{-1}\left(ys_{f_j},y\zeta_j\right)\le Y$, and $0\le\Delta^{-1}\left(ys_{t_j},y\xi_j\right)+\Delta^{-1}\left(ys_{i_j},y\psi_j\right)+\Delta^{-1}\left(ys_{f_j},y\zeta_j\right)\le 3Y$.

Definition 6 [34]**.** Let $y\delta=\left\langle\left(ys_t,y\xi\right),\left(ys_i,y\psi\right),\left(ys_f,y\zeta\right)\right\rangle$. The score function $SF\left(y\delta\right)$ and accuracy function $AF\left(y\delta\right)$ are designed in Eq. (7)- Eq. (8):

$$SF\left(y\delta\right)=\frac{\left(2Z+\Delta^{-1}\left(ys_t,y\xi\right)-\Delta^{-1}\left(ys_i,y\psi\right)-\Delta^{-1}\left(ys_f,y\zeta\right)\right)}{3Y},SF\left(y\delta\right)\in\left[0,1\right] \tag{7}$$

$$AF(y\delta)=\frac{\Delta^{-1}\left(ys_t,y\xi\right)-\Delta^{-1}\left(ys_f,y\zeta\right)}{Y},AF(y\delta)\in\left[-1,1\right]. \tag{8}$$

Definition 7 [34]**.** Let $y\delta_1=\left\langle\left(ys_{t_1},y\xi_1\right),\left(ys_{i_1},y\psi_1\right),\left(ys_{f_1},y\zeta_1\right)\right\rangle$ and $y\delta_2=\left\langle\left(ys_{t_2},y\xi_2\right),\left(ys_{i_2},y\psi_2\right),\left(ys_{f_2},y\zeta_2\right)\right\rangle$, then

(1) *if* $SF\left(y\delta_1\right)\prec SF\left(y\delta_2\right)$, we have: $y\delta_1\prec y\delta_2$;

(2) *if* $SF\left(y\delta_1\right)=SF\left(y\delta_2\right)$ and $AF\left(y\delta_1\right)\prec AF\left(y\delta_2\right)$, we have: $y\delta_1\prec y\delta_2$;

(3) *if* $SF\left(y\delta_1\right)=SF\left(y\delta_2\right)$ and $AF\left(y\delta_1\right)=AF\left(y\delta_2\right)$, we have: $y\delta_1=y\delta_2$.

Definition 8 [34]. Let $y\delta_1=\left\langle\left(ys_{t_1},y\xi_1\right),\left(ys_{i_1},y\psi_1\right),\left(ys_{f_1},y\zeta_1\right)\right\rangle$, $y\delta_2=\left\langle\left(ys_{t_2},y\xi_2\right),\left(ys_{i_2},y\psi_2\right),\left(ys_{f_2},y\zeta_2\right)\right\rangle$ and $y\delta=\left\langle\left(ys_t,y\xi\right),\left(ys_i,y\psi\right),\left(ys_f,y\zeta\right)\right\rangle$ be three 2TLNNs, then

$$(1)\quad y\delta_1\oplus y\delta_2=\left\{\begin{array}{l}\Delta\left(Y\left(\dfrac{\Delta^{-1}\left(ys_{t_1},y\xi_1\right)}{Y}+\dfrac{\Delta^{-1}\left(ys_{t_2},y\xi_2\right)}{Y}-\dfrac{\Delta^{-1}\left(ys_{t_1},y\xi_1\right)}{Y}\cdot\dfrac{\Delta^{-1}\left(ys_{t_2},y\xi_2\right)}{Y}\right)\right),\\ \Delta\left(Y\left(\dfrac{\Delta^{-1}\left(ys_{i_1},y\psi_1\right)}{Y}\cdot\dfrac{\Delta^{-1}\left(ys_{i_2},y\psi_2\right)}{Y}\right)\right),\\ \Delta\left(Y\left(\dfrac{\Delta^{-1}\left(ys_{f_1},y\zeta_1\right)}{Y}\cdot\dfrac{\Delta^{-1}\left(ys_{f_2},y\zeta_2\right)}{Y}\right)\right)\end{array}\right\};$$

$$(2)\quad y\delta_1\oplus y\delta_2=\left\{\begin{array}{l}\Delta\left(Y\left(\dfrac{\Delta^{-1}\left(ys_{t_1},y\xi_1\right)}{Y}\cdot\dfrac{\Delta^{-1}\left(ys_{t_2},y\xi_2\right)}{Y}\right)\right),\\ \Delta\left(Y\left(\dfrac{\Delta^{-1}\left(ys_{i_1},y\psi_1\right)}{Y}+\dfrac{\Delta^{-1}\left(ys_{i_2},y\psi_2\right)}{Y}-\dfrac{\Delta^{-1}\left(ys_{i_1},y\psi_1\right)}{Y}\cdot\dfrac{\Delta^{-1}\left(ys_{i_2},y\psi_2\right)}{Y}\right)\right),\\ \Delta\left(Y\left(\dfrac{\Delta^{-1}\left(ys_{f_1},y\zeta_1\right)}{Y}+\dfrac{\Delta^{-1}\left(ys_{f_2},y\zeta_2\right)}{Y}-\dfrac{\Delta^{-1}\left(ys_{f_1},y\zeta_1\right)}{Y}\cdot\dfrac{\Delta^{-1}\left(ys_{f_2},y\zeta_2\right)}{Y}\right)\right)\end{array}\right\};$$

$$(3)\quad \alpha y\delta=\left\{\begin{array}{l}\Delta\left(Y\left(1-\left(1-\dfrac{\Delta^{-1}\left(ys_t,y\xi\right)}{Y}\right)^{\alpha}\right)\right),\\ \Delta\left(Y\left(\dfrac{\Delta^{-1}\left(ys_i,y\psi\right)}{Y}\right)^{\alpha}\right),\Delta\left(Y\left(\dfrac{\Delta^{-1}\left(ys_f,y\zeta\right)}{Y}\right)^{\alpha}\right)\end{array}\right\},\alpha>0;$$

$$(4)\quad y\delta^{\alpha}=\left\{\begin{array}{l}\Delta\left(Y\left(\left(\frac{\Delta^{-1}\left(ys_t,y\xi\right)}{Y}\right)^{\alpha}\right),\Delta\left(Y\left(1-\left(1-\left(\frac{\Delta^{-1}\left(ys_i,y\psi\right)}{Y}\right)^{\alpha}\right)\right)\right),\\ \Delta\left(Y\left(1-\left(1-\left(\frac{\Delta^{-1}\left(ys_f,y\zeta\right)}{Y}\right)^{\alpha}\right)\right)\right)\end{array}\right\},\alpha>0.$$

2.4. HM techniques

The DHM technique [40] is defined.

Definition 9 [40]**.** The DHM technique is designed in Eq. (9).

$$\mathrm{DHM}^{(\rho)}\left(ya_1,ya_2,...,ya_m\right)=\left(\prod_{1\le i_1<\ldots<i_\rho\le n}\left(\frac{\sum_{j=1}^{\rho}ya_{i_j}}{\rho}\right)\right)^{\frac{1}{C_n^\rho}} \tag{9}$$

w where ρ is a parameter values and $\rho=1,2,\ldots,n$, $i_1,i_2,\ldots,i_\rho$ are ρ changed from $\{1,2,\ldots,n\}$, C_n^ρ is the binomial coefficient and $C_n^\rho=\frac{n!}{x!(n-x)!}$.

2.5. Prioritized Average (PA) technique

The prioritized average (PA) technique was constructed through Yager [41].

Definition 10[41]. Let $YG=\{YG_1,YG_2,\cdots,YG_n\}$ be attribute sets with prioritization order $YG_1\succ YG_2\succ YG_3\cdots\succ YG_n$, indicate attribute YG_j has a higher priority than YG_k, if $j<k$. The value $YG_j(zx)$ is the performance for YG_j, and meets $YG_j(yx)\in[0,1]$ in Eq. (10). If

$$PA\left(YG_i\left(yx\right)\right)=\sum_{j=1}^{n}yw_jYG_j\left(yx\right) \tag{10}$$

where $yw_j=\frac{YT_j}{\sum_{j=1}^{n}YT_j}$, $YT_j=\prod_{k=1}^{j-1}YG_k\left(yx\right)\left(j=2,\cdots,n\right)$, $YT_1=1$. Then PA is called the prioritized average (PA) technique.

3. Some Prioritized weighted DHM technique with 2TLNNs

3.1. 2TLNDHM technique

Wu et al. [40] constructed the 2TLNWDHM technique.

Definition 11 [40]. Let $y\delta_j = \left\langle \left(ys_{t_j}, y\xi_j \right), \left(ys_{i_j}, y\psi_j \right), \left(yhs_{f_j}, y\zeta_j \right) \right\rangle$ be 2TLNNs with weight values $yw_j = (yw_1, yw_2, \ldots, yw_n)^T$, $yw_j \in [0,1]$ and $\sum_{j=1}^{n} yw_j = 1$. The 2TLNWDHM technique is studied in Eq. (11):

$$\begin{aligned}
\text{2TLNWDHM}_{yw}^{(yx)}\left(y\delta_1, y\delta_2, \cdots, y\delta_n \right) &= \left(\underset{1\le i_1<\ldots<i_{yx}\le n}{\otimes} \left(\frac{\overset{x}{\underset{j=1}{\oplus}} yw_{i_j} y\varphi_{i_j}}{yx} \right) \right)^{\frac{1}{C_n^{yx}}} \\
&= \left\{ \begin{array}{l}
\Delta\left(Y\left(\prod_{1\le i_1<\ldots<i_{yx}\le n} \left(1-\left(\prod_{j=1}^{yx}\left(1-\frac{\Delta^{-1}\left(ys_{t_j}, y\xi_j \right)}{Y} \right)^{yw_{i_j}} \right)^{\frac{1}{yx}} \right) \right)^{\frac{1}{C_n^{yx}}} \right), \\
\Delta\left(Y\left(1-\left(\prod_{1\le i_1<\ldots<i_{yx}\le n} \left(1-\left(\prod_{j=1}^{yx}\left(\frac{\Delta^{-1}\left(ys_{i_j}, y\psi_j \right)}{Y} \right)^{yw_{i_j}} \right)^{\frac{1}{yx}} \right) \right)^{\frac{1}{C_n^{yx}}} \right) \right), \\
\Delta\left(Y\left(1-\left(\prod_{1\le i_1<\ldots<i_{yx}\le n} \left(1-\left(\prod_{j=1}^{yx}\left(\frac{\Delta^{-1}\left(ys_{f_j}, y\zeta_j \right)}{Y} \right)^{yw_{i_j}} \right)^{\frac{1}{yx}} \right) \right)^{\frac{1}{C_n^{yx}}} \right) \right)
\end{array} \right\},
\end{aligned} \tag{11}$$

The 2TLNWDHM supplies three properties[40].

Property 1. (Idempotency) If $y\delta_j = \left\langle \left(ys_{t_j}, y\xi_j \right), \left(ys_{i_j}, y\psi_j \right), \left(yhs_{f_j}, y\zeta_j \right) \right\rangle$ are same, then

$$\text{2TLNWDHM}_{yw}^{(yx)}\left(y\delta_1, y\delta_2, \cdots, y\delta_n \right) = y\delta \tag{12}$$

Property 2. (Monotonicity) Let $y\delta_{yx_j}$, $y\delta_{yy_j}$ be two group of 2TLNNs, if $y\delta_{yx_j} \le y\delta_{yy_j}$, for all j, then

$$\begin{aligned} &\text{2TLNWDHM}_{yw}^{(yx)}\left(y\delta_{yx_1}, y\delta_{yx_2}, \cdots, y\delta_{yx_n}\right) \\ &\le \text{2TLNWDHM}_{yw}^{(yy)}\left(y\delta_{yy_1}, y\delta_{yy_2}, \cdots, y\delta_{yy_n}\right) \end{aligned} \tag{13}$$

Property 3. (Boundedness) Let $y\delta_j = \left\langle \left(ys_{t_j}, y\xi_j\right), \left(ys_{i_j}, y\psi_j\right), \left(yhs_{f_j}, y\zeta_j\right)\right\rangle$ be a group of 2TLNNs. If $y\delta^+ = \left(\max_j\left(ys_{t_j}, y\xi_j\right), \min_j\left(ys_{i_j}, y\psi_j\right), \min_j\left(ys_{f_j}, y\zeta_j\right)\right)$ and $y\delta^- = \left(\min_j\left(ys_{t_j}, y\xi_j\right), \max_j\left(ys_{i_j}, y\psi_j\right), \max_j\left(ys_{f_j}, y\zeta_j\right)\right)$, then

$$y\delta^- \le \text{2TLNWDHM}_{yw}^{(yx)}\left(y\delta_1, y\delta_2, \cdots, y\delta_n\right) \le y\delta^+ \tag{14}$$

3.2 The 2TLNPWDHM technique

To consider the weight information, the 2TLN prioritized weighted DHM (2TLNPWDHM) information technique is built on 2TLNWDHM [40] and PA [41].

Definition 12. Let $y\delta_j = \left\langle \left(ys_{t_j}, y\xi_j\right), \left(ys_{i_j}, y\psi_j\right), \left(yhs_{f_j}, y\zeta_j\right)\right\rangle$ with weight values $yw_j = (yw_1, yw_2, \ldots, yw_n)^T$, $yw_j \in [0,1]$ and $\sum_{j=1}^{n} yw_j = 1$. Then

$$2TLNPWDHM_{yw}^{(yy)}(y\delta_1, y\delta_2, \ldots, y\delta_n) = \left(\bigotimes_{1\le i_1<\ldots<i_{yx}\le n} \left(\frac{\overset{yx}{\underset{j=1}{\oplus}} \frac{yw_j YT_{i_j}}{\sum_{i=1}^{n} yw_j YT_{i_j}} y\delta_{i_j}}{yx} \right) \right)^{\frac{1}{C_n^{yx}}} \tag{15}$$

where $YT_j = \prod_{k=1}^{j-1} yw_j SF\left(y\delta_j\right)\left(j = 2, \cdots, n\right)$, $YT_1 = yw_j$ and $SF\left(y\delta_j\right)$ is the score functions of $y\delta_j = \left\langle \left(ys_{t_j}, y\xi_j\right), \left(ys_{i_j}, y\psi_j\right), \left(ys_{f_j}, y\zeta_j\right)\right\rangle$.

Theorem 1. Assume $y\delta_j = \left\langle \left(ys_{t_j}, y\xi_j\right), \left(ys_{i_j}, y\psi_j\right), \left(yhs_{f_j}, y\zeta_j\right)\right\rangle$ with weight values $yw_j = (yw_1, yw_2, \ldots, yw_n)^T$, $yw_j \in [0,1]$ and $\sum_{j=1}^{n} yw_j = 1$. The fused value by 2TLNPWDHM is in Eq. (16):

$$
\begin{aligned}
2TLNPWDHM_{yw}^{(yy)}(y\delta_1, y\delta_2, \ldots, y\delta_n) &= \left(\bigotimes_{1\le i_1<\ldots<i_{yx}\le n} \left(\frac{\overset{yx}{\underset{j=1}{\oplus}} \frac{yw_j YT_{i_j}}{\sum_{i=1}^{n}\left(yw_j YT_{i_j}\right)} y\delta_{i_j}}{yx} \right) \right)^{\frac{1}{C_n^{yx}}} \\
&= \left\{ \begin{array}{l}
\Delta\left(Y\left(\prod_{1\le i_1<\ldots<i_{yx}\le n} \left(1-\left(\prod_{j=1}^{yx}\left(1-\frac{\Delta^{-1}\left(ys_{t_j}, y\xi_j\right)}{Y}\right)^{\left(yw_j YT_{i_j}\right)\Big/\sum_{i=1}^{n}\left(yw_j YT_{i_j}\right)}\right)^{\frac{1}{yx}}\right)^{\frac{1}{C_n^{yx}}}\right)\right), \\
\Delta\left(Y\left(1-\left(\prod_{1\le i_1<\ldots<i_{yx}\le n} \left(1-\left(\prod_{j=1}^{x}\left(\frac{\Delta^{-1}\left(ys_{i_j}, y\psi_j\right)}{Y}\right)^{\left(yw_j YT_{i_j}\right)\Big/\sum_{i=1}^{n}\left(yw_j YT_{i_j}\right)}\right)^{\frac{1}{yx}}\right)^{\frac{1}{C_n^{yx}}}\right)\right)\right), \\
\Delta\left(Y\left(1-\left(\prod_{1\le i_1<\ldots<i_{yx}\le n} \left(1-\left(\prod_{j=1}^{x}\left(\frac{\Delta^{-1}\left(ys_{f_j}, y\zeta_j\right)}{Y}\right)^{\left(yw_j YT_{i_j}\right)\Big/\sum_{i=1}^{n}\left(yw_j YT_{i_j}\right)}\right)^{\frac{1}{yx}}\right)^{\frac{1}{C_n^{yx}}}\right)\right)\right)
\end{array} \right\},
\end{aligned}
\tag{16}
$$

where $YT_j = \prod_{k=1}^{j-1} yw_j SF\left(y\delta_j\right)\left(j = 2, \cdots, n\right)$, $YT_1 = yw_j$ and $SF\left(y\delta_j\right)$ is the score functions of $y\delta_j = \left\langle \left(ys_{t_j}, y\xi_j\right), \left(ys_{i_j}, y\psi_j\right), \left(ys_{f_j}, y\zeta_j\right)\right\rangle$.

Proof. From Definition 8, we could have,

$$\frac{yw_jYT_{i_j}}{\sum_{i=1}^{n}\left(yw_jYT_{i_j}\right)}y\delta_{i_j}=\left\{\begin{array}{l}\Delta\left(Y\left(1-\left(1-\frac{\Delta^{-1}\left(ys_{t_j},y\xi_j\right)}{Z}\right)^{\left(yw_jYT_{i_j}\right)\Big/\sum_{i=1}^{n}\left(yw_jYT_{i_j}\right)}\right)\right),\\ \Delta\left(Y\left(\frac{\Delta^{-1}\left(ys_{i_j},y\psi_j\right)}{Y}\right)^{\left(yw_jYT_{i_j}\right)\Big/\sum_{i=1}^{n}\left(yw_jYT_{i_j}\right)}\right),\\ \Delta\left(Y\left(\frac{\Delta^{-1}\left(ys_{f_j},y\zeta_j\right)}{Y}\right)^{\left(yw_jYT_{i_j}\right)\Big/\sum_{i=1}^{n}\left(yw_jYT_{i_j}\right)}\right)\end{array}\right\} \tag{17}$$

Then,

$$\overset{yx}{\underset{j=1}{\oplus}}\frac{yw_jYT_{i_j}}{\sum_{i=1}^{n}\left(yw_jYT_{i_j}\right)}y\delta_{i_j}=\left\{\begin{array}{l}\Delta\left(Y\left(1-\prod_{j=1}^{yx}\left(\left(1-\frac{\Delta^{-1}\left(ys_{t_j},y\xi_j\right)}{Z}\right)^{\left(yw_jYT_{i_j}\right)\Big/\sum_{i=1}^{n}\left(yw_jYT_{i_j}\right)}\right)\right)\right),\\ \Delta\left(Y\left(\prod_{j=1}^{yx}\left(\frac{\Delta^{-1}\left(ys_{i_j},y\psi_j\right)}{Y}\right)^{\left(yw_jYT_{i_j}\right)\Big/\sum_{i=1}^{n}\left(yw_jYT_{i_j}\right)}\right)\right),\\ \Delta\left(Y\left(\prod_{j=1}^{yx}\left(\frac{\Delta^{-1}\left(ys_{f_j},y\zeta_j\right)}{Y}\right)^{\left(yw_jYT_{i_j}\right)\Big/\sum_{i=1}^{n}\left(yw_jYT_{i_j}\right)}\right)\right)\end{array}\right\} \tag{18}$$

Thus,

$$\frac{\overset{yx}{\underset{j=1}{\oplus}}\frac{yw_jYT_{i_j}}{\sum_{i=1}^{n}\left(yw_jYT_{i_j}\right)}y\delta_{i_j}}{yx}=\left\{\begin{array}{l}\Delta\left(Y\left(1-\left(\prod_{j=1}^{yx}\left(1-\frac{\Delta^{-1}\left(ys_{t_j},y\xi_j\right)}{Y}\right)^{\left(yw_jYT_{i_j}\right)\Big/\sum_{i=1}^{n}\left(yw_jYT_{i_j}\right)}\right)^{\frac{1}{yx}}\right)\right),\\ \Delta\left(Y\left(\prod_{j=1}^{yx}\left(\frac{\Delta^{-1}\left(ys_{i_j},y\psi_j\right)}{Y}\right)^{\left(yw_jYT_{i_j}\right)\Big/\sum_{i=1}^{n}\left(yw_jYT_{i_j}\right)}\right)^{\frac{1}{yx}}\right),\\ \Delta\left(Y\left(\prod_{j=1}^{yx}\left(\frac{\Delta^{-1}\left(ys_{f_j},y\zeta_j\right)}{Y}\right)^{\left(yw_jYT_{i_j}\right)\Big/\sum_{i=1}^{n}\left(yw_jYT_{i_j}\right)}\right)^{\frac{1}{yx}}\right)\end{array}\right\} \tag{19}$$

Therefore,

$$\bigotimes_{1\le i_1<\ldots<i_{yx}\le n}\left(\frac{\overset{yx}{\underset{j=1}{\oplus}}\frac{yw_jYT_{i_j}}{\sum_{i=1}^{n}\left(yw_jYT_{i_j}\right)}y\delta_{i_j}}{yx}\right)$$

$$=\left\{\begin{array}{l}\Delta\left(Y\left(\prod_{1\le i_1<\ldots<i_{yx}\le n}\left(1-\left(\prod_{j=1}^{yx}\left(1-\frac{\Delta^{-1}\left(ys_{t_j},y\xi_j\right)}{Y}\right)^{\left(yw_jYT_{i_j}\right)\Big/\sum_{i=1}^{n}\left(yw_jYT_{i_j}\right)}\right)^{\frac{1}{yx}}\right)\right)\right),\\ \Delta\left(Y\left(1-\prod_{1\le i_1<\ldots<i_{yx}\le n}\left(1-\left(\prod_{j=1}^{yx}\left(\frac{\Delta^{-1}\left(ys_{i_j},y\psi_j\right)}{Y}\right)^{\left(yw_jYT_{i_j}\right)\Big/\sum_{i=1}^{n}\left(yw_jYT_{i_j}\right)}\right)^{\frac{1}{yx}}\right)\right)\right),\\ \Delta\left(Y\left(1-\prod_{1\le i_1<\ldots<i_{yx}\le n}\left(1-\left(\prod_{j=1}^{yx}\left(\frac{\Delta^{-1}\left(ys_{f_j},y\zeta_j\right)}{Y}\right)^{\left(yw_jYT_{i_j}\right)\Big/\sum_{i=1}^{n}\left(yw_jYT_{i_j}\right)}\right)^{\frac{1}{yx}}\right)\right)\right)\end{array}\right\} \quad (20)$$

Thereafter,

$$
\begin{aligned}
&2TLNPWDHM_{yw}^{(yx)}(y\delta_1, y\delta_2, \ldots, y\delta_n) \\
&= \left(\bigotimes_{1\le i_1<\ldots<i_{yx}\le n} \left(\frac{\overset{yx}{\underset{j=1}{\oplus}} \frac{yw_j YT_{i_j}}{\sum_{i=1}^{n}\left(yw_j YT_{i_j}\right)} y\delta_{i_j}}{yx} \right) \right)^{\frac{1}{C_n^{yx}}} \\
&= \left\{ \begin{array}{l}
\Delta\left(Y\left(\prod_{1\le i_1<\ldots<i_{yx}\le n} \left(1-\left(\prod_{j=1}^{yx}\left(1-\frac{\Delta^{-1}\left(ys_{t_j}, y\xi_j\right)}{y}\right)^{\left(yw_j YT_{i_j}\right)\Big/\sum_{i=1}^{n}\left(yw_j YT_{i_j}\right)}\right)^{\frac{1}{yx}}\right)\right)^{\frac{1}{C_n^{yx}}}\right), \\
\Delta\left(Y\left(1-\left(\prod_{1\le i_1<\ldots<i_{yx}\le n} \left(1-\left(\prod_{j=1}^{x}\left(\frac{\Delta^{-1}\left(ys_{i_j}, y\psi_j\right)}{Y}\right)^{\left(yw_j YT_{i_j}\right)\Big/\sum_{i=1}^{n}\left(yw_j YT_{i_j}\right)}\right)^{\frac{1}{yx}}\right)\right)^{\frac{1}{C_n^{yx}}}\right)\right), \\
\Delta\left(Y\left(1-\left(\prod_{1\le i_1<\ldots<i_{yx}\le n} \left(1-\left(\prod_{j=1}^{x}\left(\frac{\Delta^{-1}\left(ys_{f_j}, y\zeta_j\right)}{Y}\right)^{\left(yw_j YT_{i_j}\right)\Big/\sum_{i=1}^{n}\left(yw_j YT_{i_j}\right)}\right)^{\frac{1}{yx}}\right)\right)^{\frac{1}{C_n^{yx}}}\right)\right)
\end{array} \right\},
\end{aligned}
\tag{21}
$$

Thus, (16) is fully proved.

The 2TLNPWDHM supplies three properties.

Property 4. (Idempotency) If $y\delta_j = \left\langle \left(ys_{t_j}, y\xi_j\right), \left(ys_{i_j}, y\psi_j\right), \left(yhs_{f_j}, y\zeta_j\right)\right\rangle$ are same, then

$$2TLNPWDHM_{yw}^{(yx)}\left(y\delta_1, y\delta_2, \cdots, y\delta_n\right) = y\delta \tag{22}$$

Property 5. (Monotonicity) Let $y\delta_{yx_j}$, $y\delta_{yy_j}$ be two set of 2TLNNs, if $y\delta_{yx_j} \le y\delta_{yy_j}$, for all j, then

$$2TLNPDHM_{yw}^{(yx)}\left(y\delta_{yx_1}, y\delta_{yx_2}, \cdots, y\delta_{yx_n}\right) \le 2TLNPDHM_{yw}^{(yy)}\left(y\delta_{yy_1}, y\delta_{yy_2}, \cdots, y\delta_{yy_n}\right) \tag{23}$$

Property 6. (Boundedness) Let $y\delta_j = \left\langle \left(ys_{t_j}, y\xi_j \right), \left(ys_{i_j}, y\psi_j \right), \left(yhs_{f_j}, y\zeta_j \right) \right\rangle$ be a set of 2TLNNs. If $y\delta^+ = \left(\max_j \left(ys_{t_j}, y\xi_j \right), \min_j \left(ys_{i_j}, y\psi_j \right), \min_j \left(ys_{f_j}, y\zeta_j \right) \right)$ and $y\delta^- = \left(\min_j \left(ys_{t_j}, y\xi_j \right), \max_j \left(ys_{i_j}, y\psi_j \right), \max_j \left(ys_{f_j}, y\zeta_j \right) \right)$, then

$$y\delta^- \le 2TLNPDHM_{yw}^{(yx)} \left(y\delta_1, y\delta_2, \cdots, y\delta_n \right) \le y\delta^+ \tag{24}$$

4. Method for MADM based on the 2TLNPWDHM

Then, the MADM model is built based on 2TLNPWDHM. There are m alternatives $\{YA_1, YA_2, \ldots, YA_m\}$, n attributes $\{YT_1, YT_2, \ldots, YT_n\}$. The decision steps of MADM are proposed based on 2TLNPWDHM technique (See Figure 1).

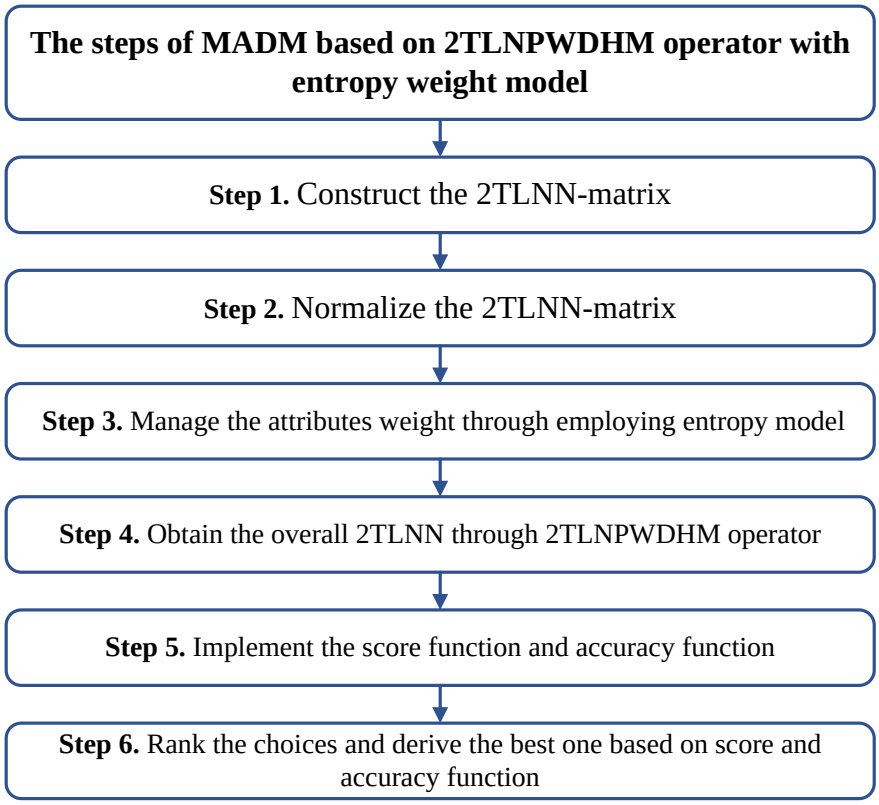

The steps of MADM based on 2TLNPWDHM operator with entropy weight model
Step 1. Construct the 2TLNN-matrix
Step 2. Normalize the 2TLNN-matrix
Step 3. Manage the attributes weight through employing entropy model
Step 4. Obtain the overall 2TLNN through 2TLNPWDHM operator
Step 5. Implement the score function and accuracy function
Step 6. Rank the choices and derive the best one based on score and accuracy function

Figure 1. The steps of MADM based on 2TLNPWDHM operator with entropy model

Step 1. Establish the 2TLNN-matrix $YM = \left[y\delta_{ij} \right]_{m \times n}$ in Eq. (25):

$$YM=\left[y\delta_{ij}\right]_{m\times n}=\begin{array}{c} \\ YA_1 \\ YA_2 \\ \vdots \\ YA_m \end{array}\begin{array}{c} \begin{array}{cccc} YT_1 & YT_2 & \cdots & YT_n \end{array} \\ \begin{bmatrix} y\delta_{11} & y\delta_{12} & \cdots & y\delta_{1n} \\ y\delta_{21} & y\delta_{22} & \cdots & y\delta_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ y\delta_{m1} & y\delta_{m2} & \cdots & y\delta_{mn} \end{bmatrix} \end{array} \tag{25}$$

where $y\delta_{ij}=\left\langle\left(ys_{t_{ij}},y\xi_{ij}\right),\left(ys_{i_{ij}},y\psi_{ij}\right),\left(ys_{f_{ij}},y\zeta_{ij}\right)\right\rangle$ is the 2TLNN.

Step 2. Normalize the $YM=\left[y\delta_{ij}\right]_{m\times n}$ to $NYM=\left[y\delta_{ij}\right]_{m\times n}$ in Eq. (26)-Eq. (27).

For benefit attributes:

$$\begin{aligned} y\delta_{ij}&=\left\langle\left(ys_{t_{ij}},y\xi_{ij}\right),\left(ys_{i_{ij}},y\psi_{ij}\right),\left(ys_{f_{ij}},y\zeta_{ij}\right)\right\rangle \\ &=\left\langle\left(ys_{t_{ij}},y\xi_{ij}\right),\left(ys_{i_{ij}},y\psi_{ij}\right),\left(ys_{f_{ij}},y\zeta_{ij}\right)\right\rangle \end{aligned} \tag{26}$$

For cost attributes:

$$\begin{aligned} y\delta_{ij}&=\left\langle\left(ys_{t_{ij}},y\xi_{ij}\right),\left(ys_{i_{ij}},y\psi_{ij}\right),\left(ys_{f_{ij}},y\zeta_{ij}\right)\right\rangle \\ &=\begin{Bmatrix}\Delta\left(Y-\Delta^{-1}\left(ys_{t_{ij}},y\xi_{ij}\right)\right),\Delta\left(Y-\Delta^{-1}\left(ys_{i_{ij}},y\psi_{ij}\right)\right),\\ \Delta\left(Y-\Delta^{-1}\left(ys_{f_{ij}},y\zeta_{ij}\right)\right)\end{Bmatrix} \end{aligned} \tag{27}$$

Step 3. Implement the attributes weight in line with entropy[42].

Entropy [43] is a conventional tool to implement weight values. Firstly, the normalized decision matrix NDM_{ij} is implemented in Eq. (28)- Eq. (32):

$$NDM_{ij}=\frac{\begin{pmatrix}SF\left\{\left(us_{t_{ij}},u\xi_{ij}\right),\left(us_{i_{ij}},u\psi_{ij}\right),\left(us_{f_{ij}},u\zeta_{ij}\right)\right\}\\ +AF\left\{\left(us_{t_{ij}},u\xi_{ij}\right),\left(us_{i_{ij}},u\psi_{ij}\right),\left(us_{f_{ij}},u\zeta_{ij}\right)\right\}\end{pmatrix}}{\sum_{i=1}^{m}\begin{pmatrix}SF\left\{\left(us_{t_{ij}},u\xi_{ij}\right),\left(us_{i_{ij}},u\psi_{ij}\right),\left(us_{f_{ij}},u\zeta_{ij}\right)\right\}\\ +AF\left\{\left(us_{t_{ij}},u\xi_{ij}\right),\left(us_{i_{ij}},u\psi_{ij}\right),\left(us_{f_{ij}},u\zeta_{ij}\right)\right\}\end{pmatrix}}, \tag{28}$$

Then, the Shannon entropy values $SEV=\left(SEV_1,SEV_2,\cdots,SEV_n\right)$ is inaugurated:

$$SEV_j=-\frac{1}{\ln m}\sum_{i=1}^{m}NDM_{ij}\ln NDM_{ij} \tag{29}$$

and $NDM_{ij}\ln NDM_{ij}=0$ if $NDM_{ij}=0$.

Then, the weights $yw=\left(yw_1, yw_2, \cdots, yw_n\right)$ is inaugurated:

$$yw_j=\frac{1-SEV_j}{\sum_{j=1}^{n}\left(1-SEV_j\right)},\ \ j=1,2,\cdots,n. \tag{30}$$

Step 3. Calculate the weight values of $YT_{ij}\left(i=1,2,\cdots,5,\ j=2,\cdots,4\right)$:

$$YT_{ij}=\prod_{\lambda=1}^{j-1}SF\left(y\delta_{i\lambda}\right)\left(i=1,2,\cdots,5,\ j=2,\cdots,4\right) \tag{31}$$

$$YT_{i1}=1,\ i=1,2,\cdots,5 \tag{32}$$

Step 4. Utilize $NYM=\left[y\delta_{ij}\right]_{m\times n}$ and 2TLNPWDHM to construct the overall 2TLNN $y\delta_i=\left\langle\left(ys_{t_i}, y\xi_i\right),\left(ys_{i_i}, y\psi_i\right),\left(ys_{f_i}, y\zeta_i\right)\right\rangle$ of YA_i in Eq. (33).

$$y\delta_i=\left\langle\left(ys_{t_i}, y\xi_i\right),\left(ys_{i_i}, y\psi_i\right),\left(ys_{f_i}, y\zeta_i\right)\right\rangle$$

$$=\left(\bigotimes_{1\le i_1<\ldots<i_{yx}\le n}\left(\frac{\overset{yx}{\underset{j=1}{\oplus}}\frac{yw_jYT_{i_j}}{\sum_{i=1}^{n}\left(yw_jYT_{i_j}\right)}y\delta_{i_j}}{yx}\right)\right)^{\frac{1}{C_n^{yx}}}$$

$$= \left\{ \begin{array}{l} \Delta\left(Y\left(\prod_{1\le i_1<\ldots<i_{yx}\le n}\left(1-\left(\prod_{j=1}^{yx}\left(1-\frac{\Delta^{-1}\left(ys_{t_{ij}}, y\xi_{ij}\right)}{Y}\right)^{\left(yw_jYT_{i_j}\right)\Big/\sum_{i=1}^{n}\left(yw_jYT_{i_j}\right)}\right)^{\frac{1}{yx}}\right)\right)^{\frac{1}{C_n^{yx}}}\right), \\ \Delta\left(Y\left(1-\left(\prod_{1\le i_1<\ldots<i_{yx}\le n}\left(1-\left(\prod_{j=1}^{yx}\left(\frac{\Delta^{-1}\left(ys_{i_{ij}}, y\psi_{ij}\right)}{Y}\right)^{\left(yw_jYT_{i_j}\right)\Big/\sum_{i=1}^{n}\left(yw_jYT_{i_j}\right)}\right)^{\frac{1}{yx}}\right)\right)^{\frac{1}{C_n^{yx}}}\right)\right), \\ \Delta\left(Y\left(1-\left(\prod_{1\le i_1<\ldots<i_{yx}\le n}\left(1-\left(\prod_{j=1}^{yx}\left(\frac{\Delta^{-1}\left(ys_{f_{ij}}, y\zeta_{ij}\right)}{Y}\right)^{\left(yw_jYT_{i_j}\right)\Big/\sum_{i=1}^{n}\left(yw_jYT_{i_j}\right)}\right)^{\frac{1}{yx}}\right)\right)^{\frac{1}{C_n^{yx}}}\right)\right) \end{array} \right\}, \tag{33}$$

Step 5. Obtain the $SF\left(y\delta_i\right), AF\left(y\delta_i\right)$ in Eq. (34)- Eq. (35).

$$SF\left(y\delta_i\right) = \frac{\left(2Y + \Delta^{-1}\left(ys_{t_i}, y\xi_i\right) - \Delta^{-1}\left(ys_{i_i}, y\psi_i\right) - \Delta^{-1}\left(ys_{f_i}, y\zeta_i\right)\right)}{3Y} \tag{34}$$

$$AF(y\delta_i) = \Delta^{-1}\left(ys_{t_i}, y\xi_i\right) - \Delta^{-1}\left(ys_{f_i}, y\zeta_i\right) \tag{35}$$

Step 6. Rank the choices $YA_i\left(i=1,2,\cdots,m\right)$ and derive the optimal one through employing the $SF\left(y\delta_i\right), AF\left(y\delta_i\right)$.

4. Numerical example and comparative analysis

4.1. Numerical example

In the era of informatization, computer information has become the core technology for global development. The demand for professional computer information talents in various

industries is becoming increasingly strong, especially for vocational education with less output of undergraduate level talents, which cannot meet the needs of the people and industrial development for high-level technical and skilled talents. In the report of the 20th National Congress of the Communist Party of China, it was pointed out that "we will comprehensively promote the collaborative innovation of vocational education, higher education, and continuing education, promote the integration of vocational education, industry and education, and science and education, optimize the positioning of vocational education types." This once again clarifies the important position of vocational education and the important role of industry education integration as a vocational education model, further pointing out the direction for the development of vocational education. The integration of industry and education, as well as school enterprise cooperation, are the basic modes of vocational education and the key to running vocational education well. This promotes the optimization of the computer information talent training mode under the background of talent transformation. To achieve the integration of industry, academia, and research, it is necessary to reform the teaching system, improve the direct connection with industry and experience transmission, help society to transport professional talents, and improve the practical application system of computer information teaching in universities. Although there are still some problems in the development under the background of the changing demand for talent cultivation, the integration of industry, academia, and research has become an important teaching model for computer information majors in vocational colleges. From the perspective of curriculum analysis, "integration of industry, academia, and research" allows enterprises

to participate in the design of professional courses, effectively connect with industry development, and reasonably set courses. Through the continuous development of courses, it can innovate the learning mode of theoretical knowledge for students, continuously promote and apply knowledge in classroom teaching, break traditional learning methods, stimulate the interest of course learning, enhance students' practical abilities, deepen their impression of computer information theory knowledge through practice, improve their professional skills and professional qualities, and cooperate with industry to enable students to understand timely evaluation of the industry, help them improve their technical deficiencies, and provide guidance for future employment. From the perspective of teacher team construction, the "industry university research integration" teaching model can promote the construction of the teacher team and improve the efficiency of talent resource utilization. Teachers can enter the industry for training, and students can enhance their ability to apply theoretical knowledge during the training. Employees in the industry can also create their own value through various ways of school teaching, achieving a win-win situation for multiple parties. The integrated teaching model of industry, academia, and research can not only help students integrate technical knowledge, but also help industry employees build their confidence in their own skills. The talent exchange between both parties can also provide scientific research guidance for universities, improve the overall level of teaching and research. The exchange of talents will undoubtedly promote the smooth development of scientific research work, and provide students with implementation space and development platforms for technological applications. Strong scientific research results can also provide

guarantees for students' learning, enhance their comprehensive strength, and enable industries to have a certain understanding of student quality, which can help students and industries achieve two-way choices. The "integration of industry, academia, and research" teaching model can promote structural changes in the talent supply side, promote deep cooperation between schools and enterprises, start from professional settings and courses, provide impetus for the transformation of computer information teaching mode, optimize course teaching mode, break through traditional boring theoretical teaching, and achieve more efficient practical teaching through new teaching innovation. The new teaching model can effectively utilize relevant funds and equipment, improve the efficiency of resource utilization, provide convenience for students' computer information application, promote the improvement of curriculum system and innovation of teaching methods, help outstanding computer information talents promote teaching optimization, achieve quality education reform, and enhance students' employment competitiveness. In terms of industrial talent strategy, it can also be effectively implemented. "Integration of industry, academia, and research" can enhance the connection between universities and industries, allowing excellent computer information talents cultivated by universities to directly connect with the industry, reducing the human and material investment and resource waste generated by the industry in talent selection. Students can have sufficient opportunities for technical practice, understand the value they create and the loopholes that need to be improved in the practice process, Helping students develop into applied talents, the industry can also take this opportunity to enhance social image and expand the positive impact of industry reputation. Students can

also quickly grasp the characteristics of their careers during the learning process, enhance their planning and goal setting for future careers, scientifically formulate learning goals, plan their careers, stimulate their own knowledge system construction and enthusiasm, gradually improve their skills, and give students a clearer motivation to learn, gradually improving their own value. In addition, the "integration of industry, academia, and research" teaching model can also help industries achieve transformation and development, facilitate economic reform, meet the demand for talent and technology, and provide lasting talent motivation for technological improvement and innovation within the industry. In this paper, we provide a numerical example for talent cultivation quality evaluation in practical aspects of computer information major through 2TLNPWDHM operator. There are five possible local engineering colleges and universities $YA_i\left(i=1,2,3,4,5\right)$ to be chosen in line with four attributes: ①YT_1 is the theoretical knowledge cultivation for computer information major; ②YT_2 is teaching costs computer information major; ③YT_3 is the team cooperation and coordination ability computer information major; ④YT_4 is the comprehensive literacy of college students computer information major. Evidently, YT_2 is the cost attributes.

Then, the 2TLNPWDHM technique is constructed for talent cultivation quality evaluation in practical aspects of computer information major.

Step 1. Construct the 2TLNN-matrix $YM=\left[y\delta_{ij}\right]_{5\times4}$. The five local engineering colleges and universities $YA_i\left(i=1,2,3,4,5\right)$ are assessed through 2TLNNs with four attributes, which are implemented in Table 1.

Table 1. 2TLNNs information matrix

	YT_1	YT_2
YA_1	<(ys_2, -0.17), (ys_4,0.21) (ys_4,0.27)>	<(ys_2,0.32), (ys_2,0.16) (ys_4,0.24)>
YA_2	<(ys_2,0.43), (ys_3, -0.29) (ys_4,0.36)>	<(ys_3, -0.18), (ys_2,0.23) (ys_3,0.42)>
YA_3	<(ys_4,0.25), (ys_2,0.31) (ys_2,0.28)>	<(ys_5,0.16), (ys_2,0.08) (ys_2, -0.24)>
YA_4	<(ys_2,0.49), (ys_3,0.34) (ys_4,-0.19)>	<(ys_2,0.13), (ys_2,0.21) (ys_4,0.27)>
YA_5	<(ys_4,0.23), (ys_2,0.37) (ys_2,0.46)>	<(ys_5,0.12), (ys_2,0.32) (ys_2, -0.16)>

	YT_3	YT_4
YA_1	<(ys_3,0.26), (ys_2,0.29) (ys_4, -0.32)>	<(ys_4, -0.35), (ys_2,0.39) (ys_2,0.47)>
YA_2	<(ys_2,0.14), (ys_4,0.28) (ys_3,0.25)>	<(ys_2,0.42), (ys_4,0.43) (ys_4,0.49)>
YA_3	<(ys_3,0.26), (ys_2, -0.13) (ys_2,0.24)>	<(ys_2,0.16), (ys_2,0.27) (ys_4,0.36)>
YA_4	<(ys_2, -0.23), (ys_2,0.14) (ys_2,0.25)>	<(ys_4,0.14), (ys_2, -0.19) (ys_4,0.43)>
YA_5	<(ys_4,0.27), (ys_3,0.19) (ys_3, -0.17)>	<(ys_2, -0.28), (ys_3,0.37) (ys_2,0.26)>

Step 2. Normalize the $YM = \left[y\delta_{ij} \right]_{5\times4}$ to $NYM = \left[y\delta_{ij} \right]_{5\times4}$ (see Table 2).

Table 2. The normalized 2TLNNs matrix

	YT_1	YT_2
YA_1	<(ys_2, -0.17), (ys_4,0.21) (ys_4,0.27)>	<(ys_4,-0.32), (ys_4,-0.16) (ys_2,-0.24)>
YA_2	<(ys_2,0.43), (ys_3, -0.29) (ys_4,0.36)>	<(ys_3, 0.18), (ys_4,-0.23) (ys_3,-0.42)>
YA_3	<(ys_4,0.25), (ys_2,0.31) (ys_2,0.28)>	<(ys_1,-0.16), (ys_4,-0.08) (ys_4, 0.24)>
YA_4	<(ys_2,0.49), (ys_3,0.34) (ys_4,-0.19)>	<(ys_4,-0.13), (ys_4,-0.21) (ys_2,-0.27)>
YA_5	<(ys_4,0.23), (ys_2,0.37) (ys_2,0.46)>	<(ys_1,-0.12), (ys_4,-0.32) (ys_4, 0.16)>

	YT_3	YT_4
YA_1	<(ys_3,0.26), (ys_2,0.29) (ys_4, -0.32)>	<(ys_4, -0.35), (ys_2,0.39) (ys_2,0.47)>

YA_2	<(ys_2,0.14), (ys_4,0.28) (ys_3,0.25)>	<(ys_2,0.42), (ys_4,0.43) (ys_4,0.49)>
YA_3	<(ys_3,0.26), (ys_2, -0.13) (ys_2,0.24)>	<(ys_2,0.16), (ys_2,0.27) (ys_4,0.36)>
YA_4	<(ys_2, -0.23), (ys_2,0.14) (ys_2,0.25)>	<(ys_4,0.14), (ys_2, -0.19) (ys_4,0.43)>
YA_5	<(ys_4,0.27), (ys_3,0.19) (ys_3, -0.17)>	<(ys_2, -0.28), (ys_3,0.37) (ys_2,0.26)>

Step 3. Implement the attributes weight.

$$yw_1 = 0.1760, yw_2 = 0.33781, yw_3 = 0.2751, yw_4 = 0.1708$$

Step 3. Utilize Eq. (24)- Eq. (25) to calculate the YT_{ij} $\left(i = 1,2,3,4,5;\ j = 1,2,3,4\right)$ (see Table 3-4).

Table 3. The YT_{ij} $\left(i = 1,2,3,4,5;\ j = 1,2,3,4\right)$

	YG_1	YG_2	YG_3	YG_4
YA_1	1.0000	0.4716	0.3537	0.2476
YA_2	1.0000	0.6118	0.4130	0.2478
YA_3	1.0000	0.4079	0.2397	0.1917
YA_4	1.0000	0.7138	0.5710	0.3640
YA_5	1.0000	0.7775	0.4665	0.3499

Table 4. The yw_jYT_{ij} $\left(i = 1,2,3,4,5;\ j = 1,2,3,4\right)$

	YG_1	YG_2	YG_3	YG_4
YA_1	0.1760	0.1783	0.0973	0.0423
YA_2	0.1760	0.2313	0.1136	0.0423
YA_3	0.1760	0.1542	0.0659	0.0327
YA_4	0.1760	0.2699	0.1571	0.0622

YA_5	0.1760	0.2940	0.1283	0.0598

Step 4. From table 2 and Table 4, the $y\delta_i\ (i=1,2,3,4,5)$ is derived through 2TLNPWDHM technique. Then, the final results are constructed in Table 5 ($yx=3$).

Table 5. The final results by 2TLNPWDHM technique

	2TLNPWDHM technique
YA_1	<(ys_2,0.4659), (ys_1, -0.2768), (ys_4,-0.3126) >
YA_2	<(ys_4,-0.2548), (ys_4,0.1217) , (ys_6,-0.3214) >
YA_3	<(ys_5, 0.2765), (ys_4, 0.4583) , (ys_6, -0.3315) >
YA_4	<(ys_3, -0.3257), (ys_4,-0.4347) , (ys_4,0.1658) >
YA_5	<(ys_2,-0.2436), (ys_4,0.2659) , (ys_6,-0.1432) >

Step 5. Through Table 5 and the score functions are constructed in Table 4.

Table 4. The score functions

	2TLNPWDHM
YA_1	0.6806
YA_2	0.5921
YA_3	0.4514
YA_4	0.5556
YA_5	0.6392

Step 6. From Table 4, the order is depicted in Table 5.

Table 5. The order for local applied higher education institutions

	Order
2TLNPWDHM technique	$YA_1 > YA_5 > YA_2 > YA_4 > YA_3$

4.2. Parameter influence

In order to fully depict the different parameters effects for 2TLNPWDHM technique, the results are fully implemented in Tables 6-7 and Figure 2-3.

Table 6. Scores for different parameters of 2TLNPWDHM technique

yx	$SF(YA_1)$	$SF(YA_2)$	$SF(YA_3)$	$SF(YA_4)$	$SF(YA_5)$
1	0.5881	0.5117	0.3901	0.4802	0.5524
2	0.6327	0.5504	0.4196	0.5165	0.5942
3	0.6806	0.5921	0.4514	0.5556	0.6392
4	0.7633	0.6641	0.5063	0.6232	0.7170

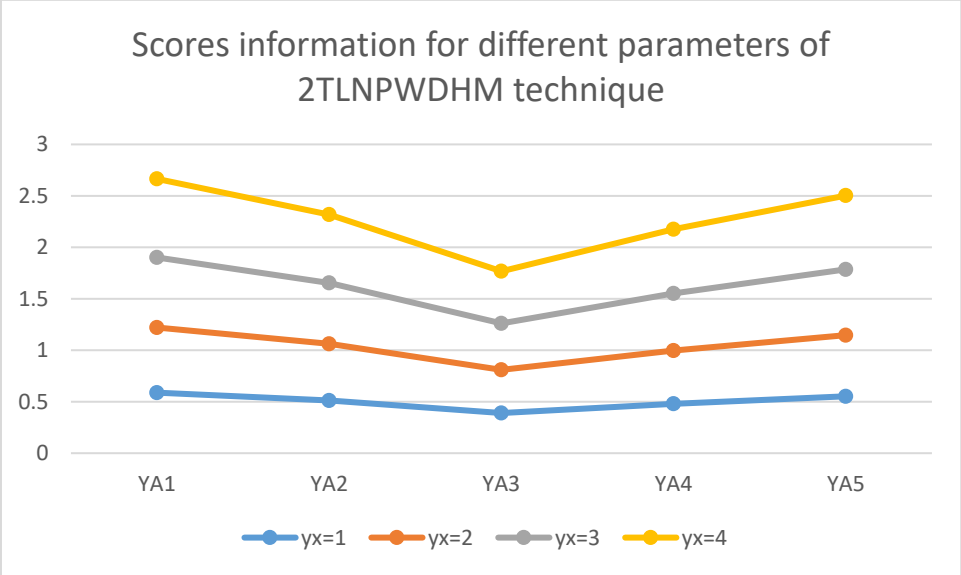


Figure 2. Scores for different parameters of 2TLNPWDHM technique

Table 7. Order for different parameters of 2TLNPDHM technique

yx	Order
1	$YA_1 > YA_5 > YA_2 > YA_4 > YA_3$
2	$YA_1 > YA_5 > YA_2 > YA_4 > YA_3$
3	$YA_1 > YA_5 > YA_2 > YA_4 > YA_3$
4	$YA_1 > YA_5 > YA_2 > YA_4 > YA_3$

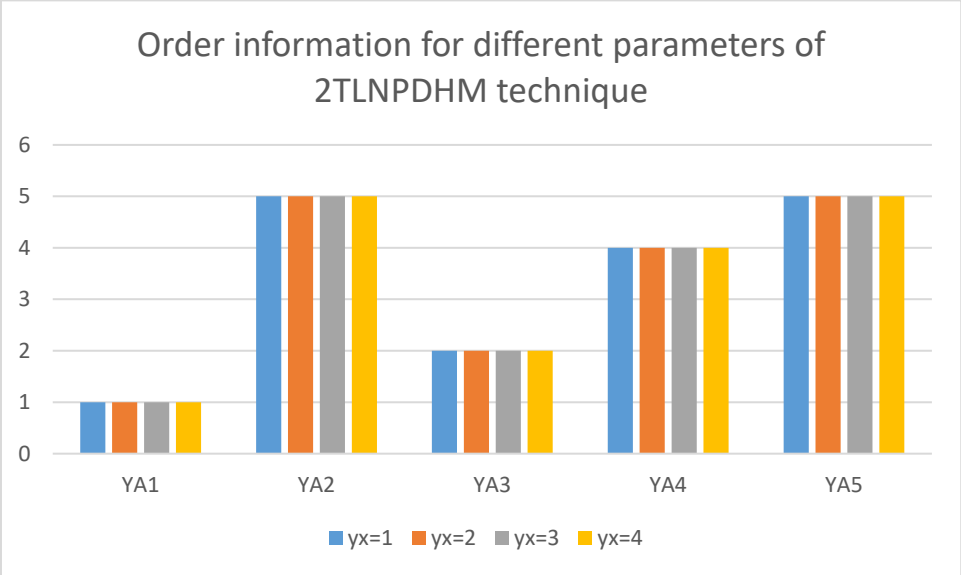


Figure 3. Scores information for different parameters of 2TLNPWDHM technique

4.3. Comparative analysis

Then, the 2TLNPWDHM is compared with 2TLNNWA technique [34], 2TLNNWG technique [34], 2TLNWHM technique [40] , 2TLNWDHM technique[40], 2TLNN-CODAS technique [39], 2TLNN-EDAS technique [44] and 2TLNN-TODIM technique [45]. The comparative studies are fully portrayed in Table 8.

Table 8. Order for different techniques

	Order
2TLNNWA technique [34]	$YA_1 > YA_5 > YA_2 > YA_4 > YA_3$
2TLNNWG technique [34]	$YA_1 > YA_5 > YA_4 > YA_2 > YA_3$
2TLNWHM technique [40]	$YA_1 > YA_5 > YA_2 > YA_4 > YA_3$
2TLNWDHM technique[40]	$YA_1 > YA_5 > YA_2 > YA_4 > YA_3$
2TLNN-CODAS technique [39]	$YA_1 > YA_5 > YA_2 > YA_4 > YA_3$
2TLNN-EDAS technique [44]	$YA_1 > YA_5 > YA_2 > YA_4 > YA_3$
2TLNN-TODIM method [45]	$YA_1 > YA_2 > YA_5 > YA_4 > YA_3$
The proposed 2TLNPWDHM technique	$YA_1 > YA_5 > YA_2 > YA_4 > YA_3$

On the basis of the WS coefficients [46, 47], the WS coefficient between 2TLNNWA technique [34], 2TLNNWG technique [34], 2TLNWHM technique [40] , 2TLNWDHM technique[40], 2TLNN-CODAS technique [39], 2TLNN-EDAS technique [44], 2TLNN-TODIM technique [45] and the proposed 2TLNPWDHM technique is 1.0000, 0.7917, 1.0000, 1.0000, 1.0000, 1.0000, 0.7266, respectively. The WS coefficient shows the order of the proposed 2TLNPWDHM technique are same to the order of the 2TLNNWA technique [34], 2TLNWHM technique [40], 2TLNWDHM technique[40], 2TLNN-CODAS technique [39] and 2TLNN-EDAS technique [44]; the WS coefficient shows the order of the proposed 2TLNPWDHM technique are slightly different to the ranking results of the 2TLNNWG technique [34] and 2TLNN-TODIM technique [45]. This validates the effectiveness of the 2TLNPWDHM technique. The main advantages of 2TLNPWDHM technique are

summarized: (1) The 2TLNPWDHM technique could consider the prioritization among the attributes. (2) The 2TLNPWDHM could take into account the interrelationship between any arguments and could eliminate the influence of unfairly information on the final outcome.

5. Conclusion

The computer information industry is a strategic, fundamental, and pioneering industry of the national economy. The cultivation of high-level and compound talents plays an important role in promoting the vigorous development of the computer information industry, driving economic growth, adjusting industrial structure, and transforming development methods. New engineering has put forward new requirements for the construction of computer information majors in comprehensive universities, promoting the integration of computer information majors with emerging industries, accelerating the transformation and upgrading of traditional majors, especially in exploring and innovating high-level engineering talent training programs and models. The professional characteristics of the Computer Information Science and Technology major are not obvious, the curriculum is too focused on basics, and is not closely integrated with the national strategic emerging computer information industry and market demand. The proportion of theoretical courses is too high, and the proportion of practical courses is low. Computer information majors are highly practical majors, and this curriculum that emphasizes theory over practice is increasingly detrimental to the cultivation of students' innovative awareness and ability. The number of elective courses is too few, and the speed of product updates and upgrades in the field of computer information is fast. New knowledge and technologies are constantly emerging, and a single course setting can no longer meet the diverse needs of the industry. Practical activities related to innovation and entrepreneurship mainly involve student participation in competitions, with limited corresponding curriculum and insufficient integration of industry,

academia, and research. The talent cultivation quality evaluation in practical aspects of computer information major is looked as the MADM issue. In this paper, the WDHM and PA operator are combined with 2TLNNs to construct the 2TLNPWDHM operator. Finally, a real-life example for talent cultivation quality evaluation in practical aspects of computer information major is employed to depict the built methods. The main study contributions of current study are constructed: (1) the 2TLNPWDHM operator is built; (2) the 2TLNPWDHM operator is designed to cope with the MADM with 2TLNNs; (3) an empirical real-life example for talent cultivation quality evaluation in practical aspects of computer information major is constructed to verify the 2TLNPWDHM operator; (4) some comparative decision studies are constructed to verify the rationality of the 2TLNPWDHM operator.

This study may have some limitations that could be implemented for talent cultivation quality evaluation in practical aspects of computer information major in future studies:(1) The MADM technique proposed doesn't investigate the consensus issues of DMs and applying consensus theory to talent cultivation quality evaluation in practical aspects of computer information major with 2TLNS is a worthwhile research topic; (2) In subsequent decision studies for talent cultivation quality evaluation in practical aspects of computer information major, the MADM techniques of 2TLNS need to be studied in any other uncertain environment.

Acknowledgment

The work was supported by the Anhui Provincial Education Commission, Anhui Provincial Department of Education under Grant No. JNFX2023093 with Project Title《 Domestic Visiting and Training Funding Project for Young Backbone Teachers》.

References

[1] J. Jiang, W. Tan, X. Zhu, J. Liu, T. Liu, Innovative models and practices for engineering and technology talent cultivation in transnational higher education, Transformations in Business & Economics, 20 (2021) 37-52.

[2] X. Liu, P. Atlantis, Research on characteristic energy specialty curriculum system suiting low-carbon talent

cultivation, in: 6th International Conference on Management, Education, Information and Control (MEICI), Shenyang, PEOPLES R CHINA, 2016, pp. 1176-1180.

[3] F. Zhao, Z. Shi, H. Wang, Research on application-oriented undergraduate talents cultivation program of engineering management specialty, in: International Conference on Construction and Real Estate Management (ICCREM), Guangzhou, PEOPLES R CHINA, 2017, pp. 405-410.

[4] Y. Zhou, F. Zheng, Study of engineering talent cultivation in emerging engineering education, Transformations in Business & Economics, 20 (2021) 21-36.

[5] L. Qiang, Z. Na, Electronic information engineering "3+4" research on the cultivation of the talents training of the educational system of the secondary and higher vocational colleges, in: International Conference on Education, Management, Computer and Society (EMCS), Shenyang, PEOPLES R CHINA, 2016, pp. 1530-1533.

[6] Y.-Y. Wei, W.-F. Chen, T. Xie, J.-J. Peng, Cross-disciplinary curriculum integration spaces for emergency management engineering talent cultivation in higher education, Computer Applications in Engineering Education, 30 (2022) 1175-1189.

[7] H. Cui, Innovation of talent cultivation education mode of modern apprenticeship participated by industry associations, Educational Sciences-Theory & Practice, 18 (2018) 3628-3634.

[8] H. Qiu, Research on talent introduction and cultivation strategy of film and television industry in new era, in: 4th International Conference on Education Management and Information Technology (ICEMIT), Changchun, PEOPLES R CHINA, 2018, pp. 1253-1256.

[9] X. Chen, Network technician talent cultivation mode discover "competency-based, highlighting the ability of students to vocational training", in: International Conference on Information Technology and Scientific Management, Tianjin Polytechn Univ, Tianjin, PEOPLES R CHINA, 2010, pp. 210-212.

[10] S. Gong, H. Ma, Research on the mechanism of the cultivation of innovative high-skilled talents in electronic commerce, in: 5th International Conference on Information Engineering for Mechanics and Materials (ICIMM), Hohhot, PEOPLES R CHINA, 2015, pp. 1226-1229.

[11] X.-d. Zhou, B. Shi, H.-y. Yu, Construction of stomatological text books and cultivation of innovative talents, Hua xi kou qiang yi xue za zhi = Huaxi kouqiang yixue zazhi = West China journal of stomatology, 26 (2008) 1-3.

[12] H. Ji, T. Li, H. Tian, Research on the evaluation system of college students' practice ability - based on the "three oriented" talent cultivation mode, in: 4th International Conference on Education, Management and Computing Technology (ICEMCT), Hangzhou, PEOPLES R CHINA, 2017, pp. 54-58.

[13] S. Tian, Y. Wang, Q. Hou, Y. Du, Ieee, Research and practice on the cultivation of innovative talents in automation specialty under the background of engineering education, in: 9th International Conference on Information Technology in Medicine and Education (ITME), Hangzhou, PEOPLES R CHINA, 2018, pp. 926-930.

[14] L. Yang, Reform of basic computer education and cultivation of innovation talent in higher education, in: International Conference on Advances in Computer Science, Environment, Ecoinformatics, and Education,

Wuhan, PEOPLES R CHINA, 2011, pp. 499-503.

[15] M.X. Yao, Analysis on the construction of the "student-centered" teaching quality evaluation system of college teachers, Basic & Clinical Pharmacology & Toxicology, 126 (2020) 406-406.

[16] Z. Zerihun, J. Beishuizen, W. Van Os, Student learning experience as indicator of teaching quality, Educational Assessment Evaluation and Accountability, 24 (2012) 99-111.

[17] L. Zhang, J. Jiao, A study on effective math teaching strategy design in hybrid learning environment, in: Hybrid Learning - 4th International Conference, ICHL 2011, Proceedings, Springer Verlag, 2011, pp. 212-223.

[18] S. Faizi, W. Salabun, S. Ullah, T. Rashid, J. Wieckowski, A new method to support decision-making in an uncertain environment based on normalized interval-valued triangular fuzzy numbers and comet technique, Symmetry-Basel, 12 (2020) 16.

[19] B. Kizielewicz, J. Wieckowski, B. Paradowski, W. Salabun, Dealing with nonmonotonic criteria in decision-making problems using fuzzy normalization, in: 4th International Conference on Intelligent and Fuzzy Systems (INFUS), Springer International Publishing Ag, Bornova, TURKEY, 2022, pp. 27-35.

[20] H.A.E.-W. Khalifa, P. Kumar, F. Smarandache, On optimizing neutrosophic complex programming using lexicographic order, Infinite Study, 2020.

[21] H.A.E.-W. Khalifa, P. Kumar, S. Mirjalili, A kkm approach for inverse capacitated transportation problem in neutrosophic environment, Sādhanā, 46 (2021) 166.

[22] S. Mandour, An exhaustive review of neutrosophic logic in addressing image processing issues, Neutrosophic Systems with Applications, 12 (2023) 36-55.

[23] F. Smarandache, Superhyperfunction, superhyperstructure, neutrosophic superhyperfunction and neutrosophic superhyperstructure: Current understanding and future directions, Neutrosophic Systems with Applications, 12 (2023) 68-76.

[24] M. Jdid, F. Smarandache, An efficient optimal solution method for neutrosophic transport models: Analysis, improvements, and examples, Neutrosophic Systems with Applications, 12 (2023) 56-67.

[25] S.S. Appadoo, M. Makhan, A. Kumar, Commentary on "utilizing linguistic picture fuzzy aggregation operators for multiple-attribute decision-making problems", International Journal of Fuzzy Systems, 23 (2021) 1914-1917.

[26] M. Yahya, S. Abdullah, R. Chinram, Y.D. Al-Otaibi, M. Naeem, Frank aggregation operators and their application to probabilistic hesitant fuzzy multiple attribute decision-making, International Journal of Fuzzy Systems, 23 (2021) 194-215.

[27] K. Kumar, S.M. Chen, Multiple attribute group decision making based on advanced linguistic intuitionistic fuzzy weighted averaging aggregation operator of linguistic intuitionistic fuzzy numbers, Information Sciences, 587 (2022) 813-824.

[28] M. Palanikumar, K. Arulmozhi, C. Jana, Multiple attribute decision-making approach for pythagorean neutrosophic normal interval-valued fuzzy aggregation operators, Computational & Applied Mathematics, 41 (2022) 27.

[29] S.S. Rawat, Komal, Multiple attribute decision making based on q-rung orthopair fuzzy hamacher muirhead

mean operators, Soft Computing, 26 (2022) 2465-2487.

[30] L.A. Zadeh, Fuzzy sets, Information and Control, 8 (1965) 338-353.

[31] F. Smarandache, A unifying field in logics: Neutrosophic logic, Multiple-Valued Logic, 8 (1999).

[32] F. Smarandache, A unifying field in logics: Neutrosophic logic. Neutrosophy, neutrosophic set, neutrosophic probability: Neutrsophic logic, Infinite Study, 2005.

[33] H. Wang, F. Smarandache, Y.Q. Zhang, R. Sunderraman, Interval neutrosophic sets and logic: Theory and applications in computing, Hexis: Phoenix, AZ, USA, (2005).

[34] J. Wang, G.W. Wei, Y. Wei, Models for green supplier selection with some 2-tuple linguistic neutrosophic number bonferroni mean operators, Symmetry-Basel, 10 (2018) 36.

[35] F. Herrera, L. Martinez, A 2-tuple fuzzy linguistic representation model for computing with words, Ieee Transactions on Fuzzy Systems, 8 (2000) 746-752.

[36] F. Herrera, L. Martinez, An approach for combining linguistic and numerical information based on the 2-tuple fuzzy linguistic representation model in decision-making, International Journal of Uncertainty Fuzziness and Knowledge-Based Systems, 8 (2000) 539-562.

[37] F. Herrera, L. Martinez, The 2-tuple linguistic computational model. Advantages of its linguistic description, accuracy and consistency, International Journal of Uncertainty Fuzziness and Knowledge-Based Systems, 9 (2001) 33-48.

[38] H. Wang, F. Smarandache, Y.Q. Zhang, R. Sunderraman, Single valued neutrosophic sets, Multispace Multistruct, (2010) 410-413.

[39] P. Wang, J. Wang, G.W. Wei, J. Wu, C. Wei, Y. Wei, Codas method for multiple attribute group decision making under 2-tuple linguistic neutrosophic environment, Informatica, 31 (2020) 161-184.

[40] S.J. Wu, J. Wang, G.W. Wei, Y. Wei, Research on construction engineering project risk assessment with some 2-tuple linguistic neutrosophic hamy mean operators, Sustainability, 10 (2018) 26.

[41] R.R. Yager, Prioritized aggregation operators, International Journal of Approximate Reasoning, 48 (2008) 263-274.

[42] Z. Yan, Todim-vikor method for quality evaluation of tourism rural landscape planning and design with 2-tuple linguistic neutrosophic numbers, Journal of Intelligent & Fuzzy Systems, Preprint (2023) https://doi.org/10.3233/JIFS-231400.

[43] C.E. Shannon, A mathematical theory of communication, Bell System Technical Journal, 27 (1948) 379-423.

[44] P. Wang, J. Wang, G.W. Wei, Edas method for multiple criteria group decision making under 2-tuple linguistic neutrosophic environment, Journal of Intelligent & Fuzzy Systems, 37 (2019) 1597-1608.

[45] J. Wang, G.W. Wei, M. Lu, Todim method for multiple attribute group decision making under 2-tuple linguistic neutrosophic environment, Symmetry-Basel, 10 (2018) 486.

[46] W. Salabun, K. Urbaniak, A new coefficient of rankings similarity in decision-making problems, in: 20th Annual International Conference on Computational Science (ICCS), Springer International Publishing Ag, Amsterdam, NETHERLANDS, 2020, pp. 632-645.

[47] W. Sałabun, J. Wątróbski, A. Shekhovtsov, Are mcda methods benchmarkable? A comparative study of

topsis, vikor, copras, and promethee ii methods, Symmetry, 12 (2020) 1549.

Received: June 27, 2024. Accepted: Oct 11, 2024

NSS
Neutrosophic Sets and Systems, Vol. 75, 2025
University of New Mexico

Enhanced TFNN-GRA Technique for Multiple-Attribute Decision-Making with Triangular Fuzzy Neutrosophic Information and Applications to Performance Evaluation of Internally Cured High-Strength Concrete

Huayang Sun, Zhongyu Liu*

Zhengzhou University, Zhengzhou, 450001, Henan, China

*Corresponding author, E-mail: m18943699056@163.com

Abstract: Internal curing technology involves the integration of internal curing materials into concrete, typically under pre-water storage conditions. This method works by gradually releasing water to increase the internal humidity of the concrete, which enhances the hydration of cementitious materials, reduces self-shrinkage in high-strength concrete (HSC), and improves its overall performance. Commonly used internal curing materials include absorbent resin and lightweight aggregates. However, using these materials in excessive amounts can compromise the strength or durability of the concrete, making it essential to determine the optimal dosage for effective application. When applied to HSC, internal curing helps mitigate shrinkage and enhances frost resistance. As noted, accurately determining the appropriate dosage of internal curing materials is critical to maximizing these benefits. The performance evaluation of internally cured HSC is a multiple-attribute decision-making (MADM) problem, and to address the uncertainties involved in this process, triangular fuzzy neutrosophic sets (TFNSs) are particularly suitable. TFNSs offer a more precise way to represent uncertain information. Grey relational analysis (GRA), a widely used technique in grey system theory, assesses the similarity between sequence curves based on their geometric shapes. In this paper, we introduce a triangular fuzzy neutrosophic number GRA (TFNN-GRA) technique under TFNSs, where no prior weight information is available. Weight values under TFNSs are determined using information entropy. By combining GRA with TFNSs, we propose the TFNN-GRA method and outline the steps for solving MADM problems. Finally, a numerical example is presented to demonstrate the performance evaluation of internally cured HSC, along with comparative studies that highlight the advantages of the TFNN-GRA technique.

Keywords: Multiple-attribute decision-making (MAGM); TFNSs; GRA technique; performance evaluation of internally cured HSC

1. Introduction

HSC typically utilizes a low water-cement ratio and a high content of cementitious materials, making it denser than ordinary concrete and providing superior durability [1, 2]. As a result, it is often

recommended for outdoor structures exposed to seawater erosion, super-tall or large-span buildings, as well as engineering projects prone to impact damage. HSC is especially prevalent in infrastructure projects, where its ability to support large spans and heavy loads in harsh environments is highly valued [3, 4]. In China, high-strength self-compacting concrete is also widely used in steel-concrete composite columns for super high-rise buildings, as well as in other critical projects[5-7]. However, HSC is prone to early-stage cracking, which has become a significant focus of research within engineering and academic communities. Although admixtures like silica fume and ultrafine mineral powder are used to improve workability, they can also bring negative effects [8-10]. During both the early and later stages of curing, HSC requires more water to hydrate the large amounts of admixtures, which exacerbates self-shrinkage. Moreover, the elastic modulus of HSC is significantly higher than that of lower-strength concrete. This means that, even at the same shrinkage rate, HSC experiences much greater tensile stress without a corresponding increase in tensile strength, making it more susceptible to early cracking. Once cracks occur, the durability of HSC is compromised. High-performance concrete is well-suited for harsh environments and large-span structures [11, 12]. However, due to its low water-cement ratio, there is insufficient internal moisture to meet hydration requirements. Under normal curing conditions, its dense structure makes it difficult for external water to penetrate. As hydration progresses, this leads to self-shrinkage, which can cause cracks and diminish performance. While methods such as steam curing exist, they are expensive and often ineffective [13, 14]. After years of experimental research, it has been found that adding a certain proportion of internal curing materials can significantly reduce self-shrinkage, improve strength, and enhance the durability of high-performance concrete, achieving the desired results. There are two primary methods of curing concrete: internal curing and external curing[15, 16]. Initially, only external curing methods were used. However, with the development of concrete technology, HSC with a compressive strength of 100 MPa was first introduced in the 1940s. Despite its advantages, the low water-cement ratio of HSC causes shrinkage and cracking. The fundamental requirement for internal curing using lightweight aggregates is that the material must be porous and capable of retaining water. Common lightweight aggregates used internationally include ceramic particles, expanded shale, diatomaceous earth, and zeolites. Prior to being added to concrete, these aggregates

are pre-saturated with water. As the hydration of cementitious materials progresses, the moisture within the concrete decreases. Due to differences in humidity and capillary pressure, the water stored in the lightweight aggregates is slowly released, creating an optimal environment for the continued hydration of the cementitious materials [17]. However, it is important to note that the capillary water supply only occurs when the distance between the cement paste and the lightweight aggregate, known as the capillary pore distance, is between 100-200 μm. The effect is limited to this range. High water-absorbent resin (also known as superabsorbent polymer, or SAP) is another widely used internal curing material. This polymer has a complex, interwoven internal structure, containing a large number of hydrophilic groups such as carboxyl and hydroxyl groups. Its three-dimensional network of chemical chains allows it to absorb water at a rate thousands of times its own weight[18]. Because of its exceptional water retention properties at room temperature, high water-absorbent resin is more effective in maintaining moisture than lightweight aggregates. The typical particle size of SAP ranges from 100 to 250 μm. There are various types of SAP, including natural starch and cellulose, synthetic polyacrylic acid resin, polyacrylamide resin, acrylamide-acrylic acid copolymer, and hydroxyethyl cellulose ether. These materials are commonly produced using either dissolution or suspension polymerization methods [19]. Like lightweight aggregates, the purpose of using high water-absorbent resin in concrete is to provide an internal humidity environment that supports the ongoing hydration of cementitious materials. However, unlike lightweight aggregates, the release of water from SAP is not only driven by differences in humidity and capillary pressure but is also influenced by factors such as rising temperature and pH levels as the hydration process continues. Due to its superior water retention properties, SAP is often more suitable for concrete preparation than lightweight aggregates.

MADM plays a critical role in contemporary human society. In today's democratic and open social environments, its applications extend from purchasing essential household items and selecting corporate investment projects to determining locations for logistics distribution centers [20, 21]. These decision-making scenarios cover a wide range of sectors, involving varying scales of decision-makers, from a few family members to thousands of individuals. Well-informed decisions contribute to the growth and success of families, businesses, and even nations, whereas poor decisions can hinder

progress and lead to adverse consequences [22-24]. As society, the economy, and technology continue to evolve, the decision-making landscape grows more intricate, with increasingly diverse evaluation criteria for decision problems. Often, these criteria can be conflicting or restrictive with respect to one another. A critical challenge that scholars face is how to accurately represent the preference information of decision-makers. Due to the pressures of limited time, the complexity of decision-making tasks, and gaps in decision-makers' expertise, the preference information provided for alternative options tends to be vague. Traditional mathematical models, based on precise data, fail to capture this uncertainty [25, 26]. In response, various fuzzy set theories have been introduced by scholars globally, including fuzzy sets [27], single-valued neutrosophic sets (SVNSs) [28-30], and interval-valued neutrosophic sets (IVNSs) [31, 32]. These theories have been continuously refined, allowing for more accurate representations of decision-makers' fuzzy preferences. As such, efficiently aggregating uncertain information has become a key research focus in the realm of fuzzy MADM. In this study, the performance evaluation of internally cured HSC is considered as an example of MADM. TFNSs [33] are better suited for representing uncertainty in this context. Meanwhile, GRA [34] is utilized to assess the relational degree between sequences based on the geometric similarity of their curves. However, current methods related to the modified GRA technique [35, 36] within TFNNs are lacking. Therefore, it is crucial to integrate the TFNN-GRA approach into this context. The primary goal of this study is to develop the TFNN-GRA technique to enhance the MADM process. A numerical example involving the performance evaluation of internally cured HSC is provided, along with several comparative analyses to demonstrate the technique's effectiveness. The study's research objectives are as follows: (1) proposing the TFNN-GRA approach for MADM with TFNSs under completely unknown weights; (2) applying the entropy method to determine the weights within TFNSs; (3) utilizing the TFNN-GRA for performance evaluation of internally cured HSC; and (4) conducting comparative studies to verify that the TFNN-GRA technique is effective for this application.

To achieve this, the research framework is structured: Section 2 introduces the TFNNs. Section 3 presents the TFNN-GRA technique for MADM. In Section 4, numerical example is demonstrated the

application of TFNN-GRA technique in the performance evaluation of internally cured HSC. Section 5 offers the conclusions.

2. Preliminaries

Biswas et al. [33] constructed the TFNSs.

Definition 1[33]. The TFNSs UU is put forward:

$$UU=\left\{\left(\theta,UA(\theta),UB(\theta),UC(\theta)\right)\middle|\theta\in\Theta\right\} \tag{1}$$

where $UA(\theta),UB(\theta),UC(\theta)\in[0,1]$ is truth-membership, indeterminacy-membership and falsity-membership by TFNs.

$$UA(\theta)=\left(UA^{L}(\theta),UA^{M}(\theta),UA^{U}(\theta)\right),0\le UA^{L}(\theta)\le UA^{M}(\theta)\le UA^{U}(\theta)\le 1 \tag{2}$$

$$UB(\theta)=\left(UB^{L}(\theta),UB^{M}(\theta),UB^{U}(\theta)\right),0\le UB^{L}(\theta)\le UB^{M}(\theta)\le UB^{U}(\theta)\le 1 \tag{3}$$

$$UC(\theta)=\left(UC^{L}(\theta),UC^{M}(\theta),UC^{U}(\theta)\right),0\le UC^{L}(\theta)\le UC^{M}(\theta)\le UC^{U}(\theta)\le 1 \tag{4}$$

For convenience, $UU=\left\{\left(UA^{L},UA^{M},UA^{U}\right),\left(UB^{L},UB^{M},UB^{U}\right),\left(UC^{L},UC^{M},UC^{U}\right)\right\}$ is TFNN, and meet $0\le UA^{U}+UB^{U}+UC^{U}\le 3$.

Definition 2[33]. Let $UU_1=\left\{\left(UA_1^{L},UA_1^{M},UA_1^{U}\right),\left(UB_1^{L},UB_1^{M},UB_1^{U}\right),\left(UC_1^{L},UC_1^{M},UC_1^{U}\right)\right\}$, $UU_2=\left\{\left(UA_2^{L},UA_2^{M},UA_2^{U}\right),\left(UB_2^{L},UB_2^{M},UB_2^{U}\right),\left(UC_2^{L},UC_2^{M},UC_2^{U}\right)\right\}$ and $UU=\left\{\left(UA^{L},UA^{M},UA^{U}\right),\left(UB^{L},UB^{M},UB^{U}\right),\left(UC^{L},UC^{M},UC^{U}\right)\right\}$, the operational laws are put forward:

$$(1)\ UU_1 \oplus UU_2 = \left\{ \begin{pmatrix} UA_1^L + UA_2^L - UA_1^L UA_2^L, \\ UA_1^M + UA_2^M - UA_1^M UA_2^M, UA_1^U + UA_2^U - UA_1^U UA_2^U \end{pmatrix}, \right.$$
$$\left. \left(UB_1^L UB_2^L, UB_1^M UB_2^M, UB_1^U UB_2^U\right), \left(UC_1^L UC_2^L, UC_1^M UC_2^M, UC_1^U UC_2^U\right) \right\};$$

$$(2)\ UU_1 \otimes UU_2 = \left\{ \left(UA_1^L UA_2^L, UA_1^M UA_2^M, UA_1^U UA_2^U\right), \right.$$
$$\begin{pmatrix} UB_1^L + UB_2^L - UB_1^L UB_2^L, \\ UB_1^M + UB_2^M - UB_1^M UB_2^M, UB_1^U + UB_2^U - UB_1^U UB_2^U \end{pmatrix},$$
$$\left. \begin{pmatrix} UC_1^L + UC_2^L - UC_1^L UC_2^L, \\ UC_1^M + UC_2^M - UC_1^M UC_2^M, UC_1^U + UC_2^U - UC_1^U UC_2^U \end{pmatrix} \right\};$$

$$(3)\ \xi UU = \left\{ \begin{pmatrix} 1-\left(1-UA^L\right)^\xi, 1-\left(1-UA^M\right)^\xi, 1-\left(1-UA^U\right)^\xi \end{pmatrix}, \right.$$
$$\left(\left(UB^L\right)^\xi, \left(UB^M\right)^\xi, \left(UB^U\right)^\xi \right),$$
$$\left. \left(\left(UC^L\right)^\xi, \left(UC^M\right)^\xi, \left(UC^U\right)^\xi \right) \right\}, \xi > 0;$$

$$(4)\ UU^\xi = \left\{ \left(\left(UA^L\right)^\xi, \left(UA^M\right)^\xi, \left(UA^U\right)^\xi \right), \right.$$
$$\left(1-\left(1-UB^L\right)^\xi, 1-\left(1-UB^M\right)^\xi, 1-\left(1-UB^U\right)^\xi \right),$$
$$\left. \left(1-\left(1-UC^L\right)^\xi, 1-\left(1-UC^M\right)^\xi, 1-\left(1-UC^U\right)^\xi \right) \right\}, \xi > 0.$$

Through Definition 2, serval properties are constructed.

$$(1)\ UU_1 \oplus UU_2 = UU_2 \oplus UU_1, UU_1 \otimes UU_2 = UU_2 \otimes UU_1, \left(\left(UU_1\right)^{\xi_1} \right)^{\xi_2} = \left(UU_1\right)^{\xi_1 \xi_2}; \quad (5)$$

$$(2)\ \xi\left(UU_1 \oplus UU_2\right) = \xi UU_1 \oplus \xi UU_2, \left(UU_1 \otimes UU_2\right)^\xi = \left(UU_1\right)^\xi \otimes \left(UU_2\right)^\xi; \quad (6)$$

$$(3) \xi_1 UU_1 \oplus \xi_2 UU_1 = \left(\xi_1 + \xi_2\right) UU_1, \left(UU_1\right)^{\xi_1} \otimes \left(UU_1\right)^{\xi_2} = \left(UU_1\right)^{\left(\xi_1 + \xi_2\right)}. \quad (7)$$

Definition 3[33]. Let $UU = \left\{ \left(UA^L, UA^M, UA^U\right), \left(UB^L, UB^M, UB^U\right), \left(UC^L, UC^M, UC^U\right) \right\}$, the score functions (SF) and accuracy functions (AF) are constructed:

$$SF\left(UU\right)=\frac{1}{12}\begin{bmatrix}8+\left(UA^{L}+2UA^{M}+UA^{U}\right)\\-\left(UB^{L}+2UB^{M}+UB^{U}\right)\\-\left(UC^{L}+2UC^{M}+UC^{U}\right)\end{bmatrix},SF\left(UU\right)\in\left[0,1\right] \tag{8}$$

$$AF\left(UU\right)=\frac{1}{8}\begin{bmatrix}4+\left(UA^{L}+2UA^{M}+UA^{U}\right)\\-\left(UB^{L}+2UB^{M}+UB^{U}\right)\end{bmatrix},AF\left(UU\right)\in\left[0,1\right] \tag{9}$$

For two TFNNs UU_1 and UU_2, then

(1) *if* $SF\left(UU_1\right)\prec SF\left(UU_2\right),\ UU_1\prec UU_2$;
(2) *if* $SF\left(UU_1\right)=SF\left(UU_2\right),AF\left(UU_1\right)\prec AF\left(UU_2\right),UU_1\prec UU_2$;
(3) *if* $SF\left(UU_1\right)=SF\left(UU_2\right),AF\left(UU_1\right)=AF\left(UU_2\right),UU_1=UU_2$.

Definition 4[37]. Let $UU_1=\left\{\left(UA_1^L,UA_1^M,UA_1^U\right),\left(UB_1^L,UB_1^M,UB_1^U\right),\left(UC_1^L,UC_1^M,UC_1^U\right)\right\}$, $UU_2=\left\{\left(UA_2^L,UA_2^M,UA_2^U\right),\left(UB_2^L,UB_2^M,UB_2^U\right),\left(UC_2^L,UC_2^M,UC_2^U\right)\right\}$, the mixed distance is put forward based on the Euclidean distance and Hamming distance:

$$\begin{aligned}&MD\left(UU_1,UU_2\right)=ED\left(UU_1,UU_2\right)+HD\left(UU_1,UU_2\right)\\&=\frac{1}{2}\left(\sqrt{\frac{1}{9}\left(\begin{array}{l}\left|UA_1^L-UA_2^L\right|^2+\left|UA_1^M-UA_2^M\right|^2+\left|UA_1^U-UA_2^U\right|^2\\+\left|UB_1^L-UB_2^L\right|^2+\left|UB_1^M-UB_2^M\right|^2+\left|UB_1^U-UB_2^U\right|^2\\+\left|UC_1^L-UC_2^L\right|^2+\left|UC_1^M-UC_2^M\right|^2+\left|UC_1^U-UC_2^U\right|^2\end{array}\right)}+\right.\\&\left.\frac{1}{9}\left(\begin{array}{l}\left|UA_1^L-UA_2^L\right|+\left|UA_1^M-UA_2^M\right|+\left|UA_1^U-UA_2^U\right|\\+\left|UB_1^L-UB_2^L\right|+\left|UB_1^M-UB_2^M\right|+\left|UB_1^U-UB_2^U\right|\\+\left|UC_1^L-UC_2^L\right|+\left|UC_1^M-UC_2^M\right|+\left|UC_1^U-UC_2^U\right|\end{array}\right)\right)\end{aligned} \tag{10}$$

3. GRA technique for MADM issue with TFNNs

The TFNN-GRA is put forward for MADM. There are m alternatives $\left\{UX_1,UX_2,\ldots,UX_m\right\}$, n devised attributes $\left\{UG_1,UG_2,\ldots,UG_n\right\}$ and TFNN-GRA technique for MADM are constructed (**Figure 1**).

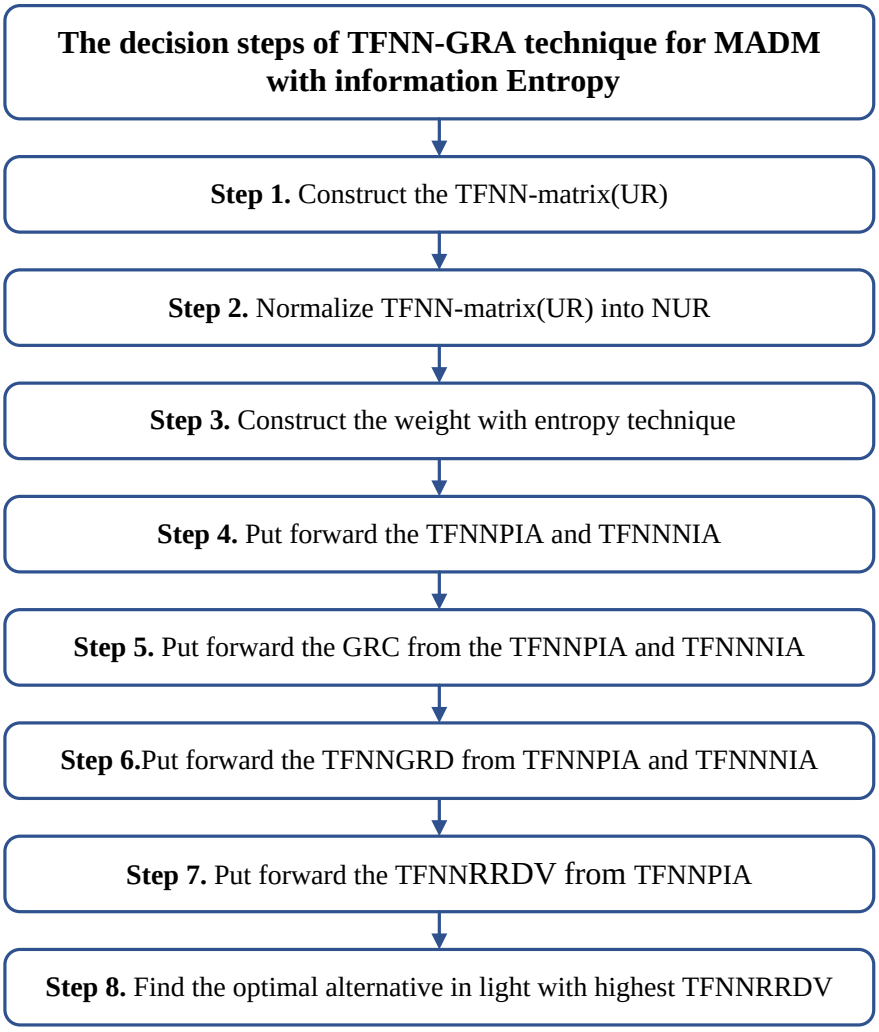

The decision steps of TFNN-GRA technique for MADM with information Entropy
Step 1. Construct the TFNN-matrix(UR)
Step 2. Normalize TFNN-matrix(UR) into NUR
Step 3. Construct the weight with entropy technique
Step 4. Put forward the TFNNPIA and TFNNNIA
Step 5. Put forward the GRC from the TFNNPIA and TFNNNIA
Step 6.Put forward the TFNNGRD from TFNNPIA and TFNNNIA
Step 7. Put forward the TFNNRRDV from TFNNPIA
Step 8. Find the optimal alternative in light with highest TFNNRRDV

Figure 1. Framework of TFNN-GRA for MADM

Step 1. Construct the TFNN-matrix $UR = \left[UR_{ij} \right]_{m\times n}$:

$$UR = \left[UR_{ij} \right]_{m\times n} = \begin{array}{c} \\ UX_1 \\ UX_2 \\ \vdots \\ UX_m \end{array} \begin{array}{c} \begin{array}{cccc} UG_1 & UG_2 & \ldots & UG_n \end{array} \\ \begin{bmatrix} UR_{11} & UR_{12} & \ldots & UR_{1n} \\ UR_{21} & UR_{22} & \ldots & UR_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ UR_{m1} & UR_{m2} & \ldots & UR_{mn} \end{bmatrix} \end{array} \tag{11}$$

where $UR_{ij} = \begin{Bmatrix} \left(\left(UA_{ij}^{L} \right), \left(UA_{ij}^{M} \right), \left(UA_{ij}^{U} \right) \right), \\ \left(\left(UB_{ij}^{L} \right), \left(UB_{ij}^{M} \right), \left(UB_{ij}^{U} \right) \right), \\ \left(\left(UC_{ij}^{L} \right), \left(UC_{ij}^{M} \right), \left(UC_{ij}^{U} \right) \right) \end{Bmatrix}$.

Step 2. Normalize the $UR = \left[UR_{ij} \right]_{m\times n}$ to $NUR = \left[NUR_{ij} \right]_{m\times n}$.

Aimed at benefit attributes:

$$NUR_{ij}=\begin{Bmatrix}\left(\left(NUA_{ij}^{L}\right),\left(NUA_{ij}^{M}\right),\left(NUA_{ij}^{U}\right)\right),\\ \left(\left(NUB_{ij}^{L}\right),\left(NUB_{ij}^{M}\right),\left(NUB_{ij}^{U}\right)\right),\\ \left(\left(NUC_{ij}^{L}\right),\left(NUC_{ij}^{M}\right),\left(NUC_{ij}^{U}\right)\right)\end{Bmatrix}=\begin{Bmatrix}\left(\left(UA_{ij}^{L}\right),\left(UA_{ij}^{M}\right),\left(UA_{ij}^{U}\right)\right),\\ \left(\left(UB_{ij}^{L}\right),\left(UB_{ij}^{M}\right),\left(UB_{ij}^{U}\right)\right),\\ \left(\left(UC_{ij}^{L}\right),\left(UC_{ij}^{M}\right),\left(UC_{ij}^{U}\right)\right)\end{Bmatrix} \tag{12}$$

Aimed at cost attributes:

$$NUR_{ij}=\begin{Bmatrix}\left(\left(NUA_{ij}^{L}\right),\left(NUA_{ij}^{M}\right),\left(NUA_{ij}^{U}\right)\right),\\ \left(\left(NUB_{ij}^{L}\right),\left(NUB_{ij}^{M}\right),\left(NUB_{ij}^{U}\right)\right),\\ \left(\left(NUC_{ij}^{L}\right),\left(NUC_{ij}^{M}\right),\left(NUC_{ij}^{U}\right)\right)\end{Bmatrix}=\begin{Bmatrix}\left(\left(UC_{ij}^{L}\right),\left(UC_{ij}^{M}\right),\left(UC_{ij}^{U}\right)\right),\\ \left(\left(UB_{ij}^{L}\right),\left(UB_{ij}^{M}\right),\left(UB_{ij}^{U}\right)\right),\\ \left(\left(UA_{ij}^{L}\right),\left(UA_{ij}^{M}\right),\left(UA_{ij}^{U}\right)\right)\end{Bmatrix} \tag{13}$$

Step 3. Obtain the weight with entropy technique.

Entropy [38] is a better technique to obtain weight. The normalized TFNN-matrix $NTFNN_{ij}$ is constructed:

$$NTFNN_{ij}=\frac{SF\begin{Bmatrix}\left(\left(NUA_{ij}^{L}\right),\left(NUA_{ij}^{M}\right),\left(NUA_{ij}^{U}\right)\right),\\ \left(\left(NUB_{ij}^{L}\right),\left(NUB_{ij}^{M}\right),\left(NUB_{ij}^{U}\right)\right),\\ \left(\left(NUC_{ij}^{L}\right),\left(NUC_{ij}^{M}\right),\left(NUC_{ij}^{U}\right)\right)\end{Bmatrix}+AF\begin{Bmatrix}\left(\left(NUA_{ij}^{L}\right),\left(NUA_{ij}^{M}\right),\left(NUA_{ij}^{U}\right)\right),\\ \left(\left(NUB_{ij}^{L}\right),\left(NUB_{ij}^{M}\right),\left(NUB_{ij}^{U}\right)\right),\\ \left(\left(NUC_{ij}^{L}\right),\left(NUC_{ij}^{M}\right),\left(NUC_{ij}^{U}\right)\right)\end{Bmatrix}+1}{\sum_{i=1}^{m}\left(SF\begin{Bmatrix}\left(\left(NUA_{ij}^{L}\right),\left(NUA_{ij}^{M}\right),\left(NUA_{ij}^{U}\right)\right),\\ \left(\left(NUB_{ij}^{L}\right),\left(NUB_{ij}^{M}\right),\left(NUB_{ij}^{U}\right)\right),\\ \left(\left(NUC_{ij}^{L}\right),\left(NUC_{ij}^{M}\right),\left(NUC_{ij}^{U}\right)\right)\end{Bmatrix}+AF\begin{Bmatrix}\left(\left(NUA_{ij}^{L}\right),\left(NUA_{ij}^{M}\right),\left(NUA_{ij}^{U}\right)\right),\\ \left(\left(NUB_{ij}^{L}\right),\left(NUB_{ij}^{M}\right),\left(NUB_{ij}^{U}\right)\right),\\ \left(\left(NUC_{ij}^{L}\right),\left(NUC_{ij}^{M}\right),\left(NUC_{ij}^{U}\right)\right)\end{Bmatrix}+1\right)}, \tag{14}$$

Then, TFNN Shannon decision entropy (TFNNSDE) is put forward through Eq. (15):

$$TFNNSDE_{j}=-\frac{1}{\ln m}\sum_{i=1}^{m}NTFNN_{ij}\ln NTFNN_{ij} \tag{15}$$

and $NTFNN_{ij}\ln NTFNN_{ij}=0$ if $NTFNN_{ij}=0$.

Then, the weights $UW=\left(UW_{1},UW_{2},\cdots,UW_{n}\right)$ is derived:

$$UW_j = \frac{1-TPNNSE_j}{\sum_{j=1}^{n}\left(1-TPNNSE_j\right)}, \quad j=1,2,\cdots,n. \tag{16}$$

Step 4. Put forward the TFNN positive ideal alternative (TFNNPIA) and TFNN negative ideal alternative (TFNNNIA)[39]:

$$TPNNPIA = \begin{Bmatrix} \left(\left(NUA_j^L\right)^+, \left(NUA_j^M\right)^+, \left(NUA_j^U\right)^+\right), \\ \left(\left(NUB_j^L\right)^+, \left(NUB_j^M\right)^+, \left(NUB_j^U\right)^+\right), \\ \left(\left(NUC_j^L\right)^+, \left(NUC_j^M\right)^+, \left(NUC_j^U\right)^+\right) \end{Bmatrix} \tag{17}$$

$$TFNNNIA = \begin{Bmatrix} \left(\left(NUA_j^L\right)^-, \left(NUA_j^M\right)^-, \left(NUA_j^U\right)^-\right), \\ \left(\left(NUB_j^L\right)^-, \left(NUB_j^M\right)^-, \left(NUB_j^U\right)^+\right), \\ \left(\left(NUC_j^L\right)^-, \left(NUC_j^M\right)^-, \left(NUC_j^U\right)^-\right) \end{Bmatrix} \tag{18}$$

$$\begin{aligned} &SF\begin{Bmatrix} \left(\left(NUA_j^L\right)^+, \left(NUA_j^M\right)^+, \left(NUA_j^U\right)^+\right), \\ \left(\left(NUB_j^L\right)^+, \left(NUB_j^M\right)^+, \left(NUB_j^U\right)^+\right), \\ \left(\left(NUC_j^L\right)^+, \left(NUC_j^M\right)^+, \left(NUC_j^U\right)^+\right) \end{Bmatrix} \\ &= \max_i SF\begin{Bmatrix} \left(\left(NUA_{ij}^L\right), \left(NUA_{ij}^M\right), \left(NUA_{ij}^U\right)\right), \\ \left(\left(NUB_{ij}^L\right), \left(NUB_{ij}^M\right), \left(NUB_{ij}^U\right)\right), \\ \left(\left(NUC_{ij}^L\right), \left(NUC_{ij}^M\right), \left(NUC_{ij}^U\right)\right) \end{Bmatrix} \end{aligned} \tag{19}$$

$$\begin{aligned} &SF\begin{Bmatrix} \left(\left(NUA_j^L\right)^-, \left(NUA_j^M\right)^-, \left(NUA_j^U\right)^-\right), \\ \left(\left(NUB_j^L\right)^-, \left(NUB_j^M\right)^-, \left(NUB_j^U\right)^+\right), \\ \left(\left(NUC_j^L\right)^-, \left(NUC_j^M\right)^-, \left(NUC_j^U\right)^-\right) \end{Bmatrix} \\ &= \min_i SF\begin{Bmatrix} \left(\left(NUA_{ij}^L\right), \left(NUA_{ij}^M\right), \left(NUA_{ij}^U\right)\right), \\ \left(\left(NUB_{ij}^L\right), \left(NUB_{ij}^M\right), \left(NUB_{ij}^U\right)\right), \\ \left(\left(NUC_{ij}^L\right), \left(NUC_{ij}^M\right), \left(NUC_{ij}^U\right)\right) \end{Bmatrix} \end{aligned} \tag{20}$$

Step 5. Put forward the grey rational coefficients (GRC) from TFNNPIA and TFNNNIA:

$$TFNNPIA\left(\xi_{ij}\right)=\frac{\left(\begin{array}{l}\min\limits_{1\le i\le m} MD\left(\left\{\begin{array}{l}\left(\left(NUA_j^L\right)^+,\left(NUA_j^M\right)^+,\left(NUA_j^U\right)^+\right),\\ \left(\left(NUB_j^L\right)^+,\left(NUB_j^M\right)^+,\left(NUB_j^U\right)^+\right),\\ \left(\left(NUC_j^L\right)^+,\left(NUC_j^M\right)^+,\left(NUC_j^U\right)^+\right)\end{array}\right\},\left\{\begin{array}{l}\left(\left(NUA_{ij}^L\right),\left(NUA_{ij}^M\right),\left(NUA_{ij}^U\right)\right),\\ \left(\left(NUB_{ij}^L\right),\left(NUB_{ij}^M\right),\left(NUB_{ij}^U\right)\right),\\ \left(\left(NUC_{ij}^L\right),\left(NUC_{ij}^M\right),\left(NUC_{ij}^U\right)\right)\end{array}\right\}\right)+\\ \rho\max\limits_{1\le i\le m} MD\left(\left\{\begin{array}{l}\left(\left(NUA_j^L\right)^+,\left(NUA_j^M\right)^+,\left(NUA_j^U\right)^+\right),\\ \left(\left(NUB_j^L\right)^+,\left(NUB_j^M\right)^+,\left(NUB_j^U\right)^+\right),\\ \left(\left(NUC_j^L\right)^+,\left(NUC_j^M\right)^+,\left(NUC_j^U\right)^+\right)\end{array}\right\},\left\{\begin{array}{l}\left(\left(NUA_{ij}^L\right),\left(NUA_{ij}^M\right),\left(NUA_{ij}^U\right)\right),\\ \left(\left(NUB_{ij}^L\right),\left(NUB_{ij}^M\right),\left(NUB_{ij}^U\right)\right),\\ \left(\left(NUC_{ij}^L\right),\left(NUC_{ij}^M\right),\left(NUC_{ij}^U\right)\right)\end{array}\right\}\right)\end{array}\right)}{\left(\begin{array}{l}MD\left(\left\{\begin{array}{l}\left(\left(NUA_j^L\right)^+,\left(NUA_j^M\right)^+,\left(NUA_j^U\right)^+\right),\\ \left(\left(NUB_j^L\right)^+,\left(NUB_j^M\right)^+,\left(NUB_j^U\right)^+\right),\\ \left(\left(NUC_j^L\right)^+,\left(NUC_j^M\right)^+,\left(NUC_j^U\right)^+\right)\end{array}\right\},\left\{\begin{array}{l}\left(\left(NUA_{ij}^L\right),\left(NUA_{ij}^M\right),\left(NUA_{ij}^U\right)\right),\\ \left(\left(NUB_{ij}^L\right),\left(NUB_{ij}^M\right),\left(NUB_{ij}^U\right)\right),\\ \left(\left(NUC_{ij}^L\right),\left(NUC_{ij}^M\right),\left(NUC_{ij}^U\right)\right)\end{array}\right\}\right)\\ +\rho\max\limits_{1\le i\le m} MD\left(\left\{\begin{array}{l}\left(\left(NUA_j^L\right)^+,\left(NUA_j^M\right)^+,\left(NUA_j^U\right)^+\right),\\ \left(\left(NUB_j^L\right)^+,\left(NUB_j^M\right)^+,\left(NUB_j^U\right)^+\right),\\ \left(\left(NUC_j^L\right)^+,\left(NUC_j^M\right)^+,\left(NUC_j^U\right)^+\right)\end{array}\right\},\left\{\begin{array}{l}\left(\left(NUA_{ij}^L\right),\left(NUA_{ij}^M\right),\left(NUA_{ij}^U\right)\right),\\ \left(\left(NUB_{ij}^L\right),\left(NUB_{ij}^M\right),\left(NUB_{ij}^U\right)\right),\\ \left(\left(NUC_{ij}^L\right),\left(NUC_{ij}^M\right),\left(NUC_{ij}^U\right)\right)\end{array}\right\}\right)\end{array}\right)} \tag{21}$$

$$TFNNNIA\left(\xi_{ij}\right)=\frac{\left(\begin{array}{l}\min\limits_{1\le i\le m} MD\left(\left\{\begin{array}{l}\left(\left(NUA_j^L\right)^-,\left(NUA_j^M\right)^-,\left(NUA_j^U\right)^-\right),\\ \left(\left(NUB_j^L\right)^-,\left(NUB_j^M\right)^-,\left(NUB_j^U\right)^+\right),\\ \left(\left(NUC_j^L\right)^-,\left(NUC_j^M\right)^-,\left(NUC_j^U\right)^-\right)\end{array}\right\},\left\{\begin{array}{l}\left(\left(NUA_{ij}^L\right),\left(NUA_{ij}^M\right),\left(NUA_{ij}^U\right)\right),\\ \left(\left(NUB_{ij}^L\right),\left(NUB_{ij}^M\right),\left(NUB_{ij}^U\right)\right),\\ \left(\left(NUC_{ij}^L\right),\left(NUC_{ij}^M\right),\left(NUC_{ij}^U\right)\right)\end{array}\right\}\right)+\\ \rho\max\limits_{1\le i\le m} MD\left(\left\{\begin{array}{l}\left(\left(NPA_j^L\right)^-,\left(NPA_j^M\right)^-,\left(NPA_j^U\right)^-\right),\\ \left(\left(NPB_j^L\right)^-,\left(NPB_j^M\right)^-,\left(NPB_j^U\right)^+\right),\\ \left(\left(NPC_j^L\right)^-,\left(NPC_j^M\right)^-,\left(NPC_j^U\right)^-\right)\end{array}\right\},\left\{\begin{array}{l}\left(\left(NPA_{ij}^L\right),\left(NPA_{ij}^M\right),\left(NPA_{ij}^U\right)\right),\\ \left(\left(NPB_{ij}^L\right),\left(NPB_{ij}^M\right),\left(NPB_{ij}^U\right)\right),\\ \left(\left(NPC_{ij}^L\right),\left(NPC_{ij}^M\right),\left(NPC_{ij}^U\right)\right)\end{array}\right\}\right)\end{array}\right)}{\left(\begin{array}{l}MD\left(\left\{\begin{array}{l}\left(\left(NUA_j^L\right)^-,\left(NUA_j^M\right)^-,\left(NUA_j^U\right)^-\right),\\ \left(\left(NUB_j^L\right)^-,\left(NUB_j^M\right)^-,\left(NUB_j^U\right)^+\right),\\ \left(\left(NUC_j^L\right)^-,\left(NUC_j^M\right)^-,\left(NUC_j^U\right)^-\right)\end{array}\right\},\left\{\begin{array}{l}\left(\left(NUA_{ij}^L\right),\left(NUA_{ij}^M\right),\left(NUA_{ij}^U\right)\right),\\ \left(\left(NUB_{ij}^L\right),\left(NUB_{ij}^M\right),\left(NUB_{ij}^U\right)\right),\\ \left(\left(NUC_{ij}^L\right),\left(NUC_{ij}^M\right),\left(NUC_{ij}^U\right)\right)\end{array}\right\}\right)\\ +\rho\max\limits_{1\le i\le m} MD\left(\left\{\begin{array}{l}\left(\left(NPA_j^L\right)^-,\left(NPA_j^M\right)^-,\left(NPA_j^U\right)^-\right),\\ \left(\left(NPB_j^L\right)^-,\left(NPB_j^M\right)^-,\left(NPB_j^U\right)^+\right),\\ \left(\left(NPC_j^L\right)^-,\left(NPC_j^M\right)^-,\left(NPC_j^U\right)^-\right)\end{array}\right\},\left\{\begin{array}{l}\left(\left(NPA_{ij}^L\right),\left(NPA_{ij}^M\right),\left(NPA_{ij}^U\right)\right),\\ \left(\left(NPB_{ij}^L\right),\left(NPB_{ij}^M\right),\left(NPB_{ij}^U\right)\right),\\ \left(\left(NPC_{ij}^L\right),\left(NPC_{ij}^M\right),\left(NPC_{ij}^U\right)\right)\end{array}\right\}\right)\end{array}\right)} \tag{22}$$

The distinguishing coefficient ρ =0.5.

Step 6. Put forward the TFNN grey relation degree (TFNNGRD) from TFNNPIA and TFNNNIA:

$$TFNNGRD\left(\xi_i^+\right)=\sum_{j=1}^{n} UW_j\times TFNNPIA\left(\xi_{ij}\right) \tag{23}$$

$$TFNNGRD\left(\xi_i^-\right)=\sum_{j=1}^{n} UW_j\times TFNNNIA\left(\xi_{ij}\right) \tag{24}$$

Step 7. Put forward the TFNN relative relational degree values (TFNNRRDV) from TFNNPIA:

$$\begin{aligned} TFNNRRDV\left(\xi_i\right)&=\frac{TFNNGRD\left(\xi_i^+\right)}{TFNNGRD\left(\xi_i^-\right)+TFNNGRD\left(\xi_i^+\right)}\\ &=\frac{\sum_{j=1}^{n} UW_j\times TFNNPIA\left(\xi_{ij}\right)}{\sum_{j=1}^{n} UW_j\times TFNNPIA\left(\xi_{ij}\right)+\sum_{j=1}^{n} UW_j\times TFNNNIA\left(\xi_{ij}\right)} \end{aligned} \tag{25}$$

Step 8. Find the optimal alternative in light with highest $TFNNRRDV\left(\xi_i\right)$.

4. Numerical example and comparative analysis

4.1. Numerical example

HSC is known for its early cracking and unstable volume shape, which significantly impact its durability. Among the factors causing cracks, only about 20% are due to external loads. The majority of structural cracks in HSC occur after the initial setting and are primarily due to changes in self-generated volume and external environmental influences. HSC has a higher density than ordinary concrete, but its low water-cement ratio results in lower porosity. This characteristic suggests that incorporating internal curing materials in HSC can effectively reduce crack formation and mitigate cracking. Internal curing materials, such as high-water absorbent resin, play a crucial role in this process. These materials have a strong water absorption capacity. When mixed with HSC, the high-water absorbent resin absorbs free water during the early stages of concrete hydration, preventing its loss. As the hydration process progresses and the hydration storage capacity decreases, the resin releases the absorbed water, allowing hydration to continue. This mechanism is particularly beneficial for HSC, which has a lower water-cement ratio compared to ordinary concrete. The primary cause of self-shrinkage in HSC is the change in internal humidity. The introduction of internal curing materials like Super Absorbent Polymers (SAP) can effectively address this issue by alleviating the drying problem inside the concrete during hydration. SAP absorbs excess water and releases it as needed, ensuring continuous hydration and reducing self-shrinkage. This internal curing method helps to control crack formation more effectively in HSC than in ordinary concrete, enhancing its overall durability and structural integrity. In summary, the use of internal curing materials in HSC is a promising approach to mitigate early cracking and unstable volume changes.

By managing internal humidity and supporting continuous hydration, these materials help to reduce self-shrinkage and improve the durability of HSC. The performance evaluation of internally cured HSC is MADM. Numerical example for performance evaluation of internally cured HSC based on TFNN-GRA. Five internally cured HSCs $UX_i\left(i=1,2,3,4,5\right)$ are evaluated with four attributes: ①UG_1 is compressive strength; ②UG_2 is the mass loss rate; ③UG_3 is relative dynamic modulus of elasticity; ④UG_4 is self-shrinkage shrinkage rate. The five internally cured HSCs $UX_i\left(i=1,2,3,4,5\right)$ are evaluated with four attributes through employing the TFNNs. The TFNN-GRA technique is used for performance evaluation of internally cured HSC.

Step 1. Put forward the TFNN-matrix $UR=\left[UR_{ij}\right]_{m\times n}$ (See Table 1).

Table 1. The TFNN data

	PG_1	PG_2
PX_1	$\begin{Bmatrix}(0.51,0.65,0.76),\\(0.59,0.81,0.87),\\(0.46,0.63,0.72)\end{Bmatrix}$	$\begin{Bmatrix}(0.57,0.63,0.69),\\(0.57,0.61,0.72),\\(0.36,0.42,0.54)\end{Bmatrix}$
PX_2	$\begin{Bmatrix}(0.63,0.68,0.73),\\(0.47,0.53,0.64),\\(0.36,0.42,0.56)\end{Bmatrix}$	$\begin{Bmatrix}(0.53,0.65,0.69),\\(0.34,0.42,0.57),\\(0.26,0.32,0.45)\end{Bmatrix}$
PX_3	$\begin{Bmatrix}(0.63,0.76,0.84),\\(0.48,0.53,0.64),\\(0.59,0.67,0.72)\end{Bmatrix}$	$\begin{Bmatrix}(0.48,0.52,0.85),\\(0.25,0.56,0.67),\\(0.31,0.43,0.74)\end{Bmatrix}$
PX_4	$\begin{Bmatrix}(0.54,0.68,0.74),\\(0.34,0.57,0.63),\\(0.46,0.63,0.72)\end{Bmatrix}$	$\begin{Bmatrix}(0.43,0.62,0.71),\\(0.54,0.68,0.79),\\(0.45,0.54,0.63)\end{Bmatrix}$
PX_5	$\begin{Bmatrix}(0.53,0.69,0.81),\\(0.64,0.72,0.76),\\(0.45,0.58,0.74)\end{Bmatrix}$	$\begin{Bmatrix}(0.34,0.53,0.76),\\(0.39,0.48,0.56),\\(0.32,0.45,0.64)\end{Bmatrix}$

	PG_3	PG_4
PX_1	$\begin{Bmatrix}(0.53,0.61,0.76),\\(0.42,0.54,0.65),\\(0.74,0.81,0.86)\end{Bmatrix}$	$\begin{Bmatrix}(0.28,0.43,0.67),\\(0.34,0.58,0.89),\\(0.63,0.75,0.83)\end{Bmatrix}$
PX_2	$\begin{Bmatrix}(0.43,0.57,0.84),\\(0.46,0.51,0.62),\\(0.37,0.43,0.57)\end{Bmatrix}$	$\begin{Bmatrix}(0.31,0.42,0.46),\\(0.54,0.65,0.74),\\(0.56,0.64,0.83)\end{Bmatrix}$
PX_3	$\begin{Bmatrix}(0.62,0.74,0.86),\\(0.29,0.56,0.69),\\(0.39,0.46,0.76)\end{Bmatrix}$	$\begin{Bmatrix}(0.73,0.76,0.82),\\(0.57,0.68,0.75),\\(0.43,0.56,0.62)\end{Bmatrix}$
PX_4	$\begin{Bmatrix}(0.63,0.84,0.89),\\(0.23,0.38,0.57),\\(0.28,0.36,0.46)\end{Bmatrix}$	$\begin{Bmatrix}(0.35,0.52,0.73),\\(0.67,0.73,0.78),\\(0.29,0.34,0.45)\end{Bmatrix}$
PX_5	$\begin{Bmatrix}(0.57,0.62,0.71),\\(0.46,0.53,0.67),\\(0.53,0.62,0.74)\end{Bmatrix}$	$\begin{Bmatrix}(0.43,0.52,0.63),\\(0.36,0.46,0.54),\\(0.48,0.52,0.63)\end{Bmatrix}$

Step 2. Normalize the $UR=\left[UR_{ij}\right]_{5\times4}$ to $NUR=\left[NUR_{ij}\right]_{5\times4}$. All the attributes are benefit, the $NUR=\left[NUR_{ij}\right]_{5\times4}$ is same to $UR=\left[UR_{ij}\right]_{5\times4}$ (See Table 2).

Table 2. The normalized TFNN data

	PG_1	PG_2
PX_1	$\begin{Bmatrix}(0.51,0.65,0.76),\\(0.59,0.81,0.87),\\(0.46,0.63,0.72)\end{Bmatrix}$	$\begin{Bmatrix}(0.57,0.63,0.69),\\(0.57,0.61,0.72),\\(0.36,0.42,0.54)\end{Bmatrix}$
PX_2	$\begin{Bmatrix}(0.63,0.68,0.73),\\(0.47,0.53,0.64),\\(0.36,0.42,0.56)\end{Bmatrix}$	$\begin{Bmatrix}(0.53,0.65,0.69),\\(0.34,0.42,0.57),\\(0.26,0.32,0.45)\end{Bmatrix}$

PX_3	$\begin{Bmatrix}(0.63,0.76,0.84),\\(0.48,0.53,0.64),\\(0.59,0.67,0.72)\end{Bmatrix}$	$\begin{Bmatrix}(0.48,0.52,0.85),\\(0.25,0.56,0.67),\\(0.31,0.43,0.74)\end{Bmatrix}$
PX_4	$\begin{Bmatrix}(0.54,0.68,0.74),\\(0.34,0.57,0.63),\\(0.46,0.63,0.72)\end{Bmatrix}$	$\begin{Bmatrix}(0.43,0.62,0.71),\\(0.54,0.68,0.79),\\(0.45,0.54,0.63)\end{Bmatrix}$
PX_5	$\begin{Bmatrix}(0.53,0.69,0.81),\\(0.64,0.72,0.76),\\(0.45,0.58,0.74)\end{Bmatrix}$	$\begin{Bmatrix}(0.34,0.53,0.76),\\(0.39,0.48,0.56),\\(0.32,0.45,0.64)\end{Bmatrix}$

	PG_3	PG_4
PX_1	$\begin{Bmatrix}(0.53,0.61,0.76),\\(0.42,0.54,0.65),\\(0.74,0.81,0.86)\end{Bmatrix}$	$\begin{Bmatrix}(0.28,0.43,0.67),\\(0.34,0.58,0.89),\\(0.63,0.75,0.83)\end{Bmatrix}$
PX_2	$\begin{Bmatrix}(0.43,0.57,0.84),\\(0.46,0.51,0.62),\\(0.37,0.43,0.57)\end{Bmatrix}$	$\begin{Bmatrix}(0.31,0.42,0.46),\\(0.54,0.65,0.74),\\(0.56,0.64,0.83)\end{Bmatrix}$
PX_3	$\begin{Bmatrix}(0.62,0.74,0.86),\\(0.29,0.56,0.69),\\(0.39,0.46,0.76)\end{Bmatrix}$	$\begin{Bmatrix}(0.73,0.76,0.82),\\(0.57,0.68,0.75),\\(0.43,0.56,0.62)\end{Bmatrix}$
PX_4	$\begin{Bmatrix}(0.63,0.84,0.89),\\(0.23,0.38,0.57),\\(0.28,0.36,0.46)\end{Bmatrix}$	$\begin{Bmatrix}(0.35,0.52,0.73),\\(0.67,0.73,0.78),\\(0.29,0.34,0.45)\end{Bmatrix}$
PX_5	$\begin{Bmatrix}(0.57,0.62,0.71),\\(0.46,0.53,0.67),\\(0.53,0.62,0.74)\end{Bmatrix}$	$\begin{Bmatrix}(0.43,0.52,0.63),\\(0.36,0.46,0.54),\\(0.48,0.52,0.63)\end{Bmatrix}$

Step 3. Construct the weight with entropy:

$$UW_1 = 0.1933, UW_2 = 0.3410, UW_3 = 0.2578, UW_4 = 0.2079.$$

Step 4. Construct the TFNNPIA and TFNNNIA (Table 3).

Table 3. The TFNNPIA and TFNNNIA

	UG_1	UG_2
TFNNPIA	$\begin{Bmatrix}(0.63,0.68,0.73),\\(0.47,0.53,0.64),\\(0.36,0.42,0.56)\end{Bmatrix}$	$\begin{Bmatrix}(0.57,0.63,0.69),\\(0.57,0.61,0.72),\\(0.36,0.42,0.54)\end{Bmatrix}$
TFNNNIA	$\begin{Bmatrix}(0.53,0.69,0.81),\\(0.64,0.72,0.76),\\(0.45,0.58,0.74)\end{Bmatrix}$	$\begin{Bmatrix}(0.34,0.53,0.76),\\(0.39,0.48,0.56),\\(0.32,0.45,0.64)\end{Bmatrix}$
	UG_3	UG_4
TFNNPIA	$\begin{Bmatrix}(0.62,0.74,0.86),\\(0.29,0.56,0.69),\\(0.39,0.46,0.76)\end{Bmatrix}$	$\begin{Bmatrix}(0.73,0.76,0.82),\\(0.57,0.68,0.75),\\(0.43,0.56,0.62)\end{Bmatrix}$
TFNNNIA	$\begin{Bmatrix}(0.43,0.57,0.84),\\(0.46,0.51,0.62),\\(0.37,0.43,0.57)\end{Bmatrix}$	$\begin{Bmatrix}(0.28,0.43,0.67),\\(0.34,0.58,0.89),\\(0.63,0.75,0.83)\end{Bmatrix}$

Step 5. Construct the $TFNNPIA\left(\xi_{ij}\right)$ and $TFNNNIA\left(\xi_{ij}\right)$ (Table 4-5).

Table 4. The $TFNNPIA\left(\xi_{ij}\right)$

Alternatives	UG_1	UG_2	UG_3	UG_4
UX_1	0.6941	1.0000	0.5346	0.6312
UX_2	1.0000	0.8662	0.6236	0.6167
UX_3	0.6605	0.6307	1.0000	1.0000
UX_4	0.8942	0.4872	0.6529	0.6599
UX_5	0.6933	0.5532	0.7780	0.6570

Table 5. The $TFNNNIA\left(\xi_{ij}\right)$

Alternatives	UG_1	UG_2	UG_3	UG_4
UX_1	0.5494	0.4499	0.5343	1.0000

UX_2	0.9995	0.5817	1.0000	0.8297
UX_3	0.5652	0.4712	0.4303	0.5056
UX_4	0.6599	0.6599	0.9055	0.5332
UX_5	1.0000	1.0000	0.6182	0.6045

Step 6. Construct the $TFNNGRD\left(\xi_i^+\right)$ and $TFNNGRD\left(\xi_i^-\right)$ (Table 6):

Table 6. $TFNNGRD\left(\xi_i^+\right)$ and $TFNNGRD\left(\xi_i^-\right)$

	$TFNNGRD\left(\xi_i^+\right)$	$TFNNGRD\left(\xi_i^-\right)$
UX_1	0.7442	0.6053
UX_2	0.7777	0.8219
UX_3	0.8085	0.4860
UX_4	0.6445	0.6969
UX_5	0.6598	0.8193

Step 7. Put forward the $TFNNRRDV\left(\xi_i\right)$ (Table 7).

Table 7. The $TFNNRRDV\left(\xi_i\right)$

	$TFNNRRDV\left(\xi_i\right)$	Order
UX_1	0.5515	2
UX_2	0.4862	3
UX_3	0.6246	1
UX_4	0.4805	4
UX_5	0.4461	5

Step 8. From $TFNNRRDV\left(\xi_i\right)$, the order is constructed: $UX_3 > UX_1 > UX_2 > UX_4 > UX_5$ and UX_3 is the best internally cured HSC.

4.2. Comparative analysis

The TFNN-GRA approach is compared with TFNNWA approach [33], TFNNWG approach [33], SVTNDPNBM approach [40], TFNN-CE technique[41], TFNN-VIKOR technique[42], TFNN-MABAC technique[37] and TFN-EDAS approach [43]. The order of these approaches is constructed in Table 8 and Figure 2.

Table 8. Order for different approaches

	order
TFNNWA technique [33]	$UX_3 > UX_1 > UX_2 > UX_4 > UX_5$
TFNNWG technique [33]	$UX_3 > UX_1 > UX_4 > UX_2 > UX_5$
SVTNDPNBM technique [40]	$UX_3 > UX_1 > UX_4 > UX_2 > UX_5$
TFNN-CE technique[41]	$UX_3 > UX_1 > UX_2 > UX_4 > UX_5$
TFNN-VIKOR technique[42]	$UX_3 > UX_1 > UX_2 > UX_4 > UX_5$
TFNN-MABAC technique[37]	$UX_3 > UX_1 > UX_2 > UX_4 > UX_5$
TFN-EDAS technique [43]	$UX_3 > UX_1 > UX_2 > UX_4 > UX_5$
TFNN-GRA technique	$UX_3 > UX_1 > UX_2 > UX_4 > UX_5$

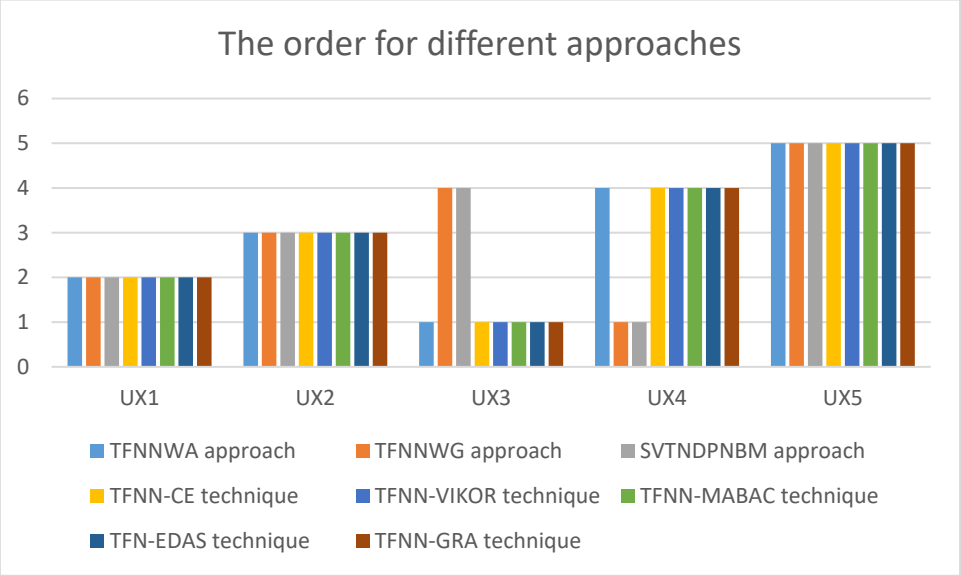


Figure 2. Order for different techniques

Based on the WS coefficients [44, 45], the WS coefficient between the existing techniques and the newly developed TFNN-GRA technique are constructed: 1.0000, 0.9236, 0.9236, 1.0000, 1.0000, 1.0000, and 1.0000, respectively. From this analysis, it is evident that both models identify the same optimal and worst internally cured HSC, confirming the effectiveness of the TFNN-GRA technique.

The key differences between the newly developed TFNN-GRA technique and the existing TFNN-GRA technique [39] are as follows:

(1) **Weight Determination**: The newly developed TFNN-GRA uses objective weights, while the existing TFNN-GRA[39] employs subjective weights.

(2) **Weight Derivation Method**: The new TFNN-GRA utilizes entropy, score, and accuracy functions to derive objective weights, whereas the existing TFNN-GRA[39] relies on subjective weights without applying a specific weighting method.

(3) **Distance Metric**: The new TFNN-GRA applies a mixed distance measure to calculate the GRC and GRD, while TFNN-GRA [39] uses the Hamming distance for these calculations.

One limitation of the newly developed TFNN-GRA technique is that it fails to incorporate consensus measures and prospect theory for the performance evaluation of internally cured HSC.

5. Conclusion

The curing conditions of concrete can significantly affect its early strength development. Inadequate early curing of high-performance concrete with a low water-cement ratio not only results in increased volume shrinkage and early cracking due to self-drying but also severely reduces early strength because of internal water shortages and a slower hydration reaction rate. In response, most domestic scholars and engineers have focused on improving the early hydration and strength of concrete by increasing the curing temperature and humidity of the external environment. However, there have been no reports in China on producing self-curing concrete that is less sensitive to external curing conditions by incorporating self-curing agents. Exploring the production of self-curing high-performance concrete is of great significance. It can improve construction conditions, reduce concrete curing costs, and expand the application fields and scope of high-performance concrete. Additionally, it offers a new approach to minimizing shrinkage in high-performance concrete. The performance evaluation of internally cured HSC is treated as a MADM problem. In this study, the TFNN-GRA technique is proposed under the framework of TFNSs. The entropy method is applied to derive the weight under TFNSs. Following this, the GRA technique is adapted to work with TFNSs, and the TFNN-GRA process for MADM is outlined step by step. Finally, numerical example is constructed to demonstrate the performance evaluation of internally cured HSC, and comparative studies are validated the TFNN-GRA technique.

References

[1] M. Lopez, L.F. Kahn, K.E. Kurtis, Effect of internally stored water on creep of high-performance concrete, Aci Materials Journal, 105 (2008) 265-273.

[2] W.J. Weiss, P. Lura, Special section on advances in internally cured concrete introduction, Journal of Materials in Civil Engineering, 24 (2012) 959-960.

[3] D.J. Shen, J.L. Jiang, M.Y. Zhang, P.P. Yao, G.Q. Jiang, Tensile creep and cracking potential of high performance concrete internally cured with super absorbent polymers at early age, Construction and Building Materials, 165 (2018) 451-461.

[4] F.J. Vázquez-Rodríguez, A. Arato, D.I. Martinez-Delgado, A. Guzmán, N. Elizondo-Villarreal, R. Puente-Ornelas, E. Rodríguez, Evaluation of chloride diffusion and corrosion resistance in reinforced concrete using internal curing and shrinkage reducing admixtures, International Journal of Electrochemical Science, 13 (2018) 6027-6047.

[5] J. Zhang, X. Zheng, Q. Wang, Mixture optimization on internally cured high strength engineered cementitious composite with pre-wetted sand-like zeolite, Journal of Advanced Concrete Technology, 16 (2018) 485-497.

[6] T. Manzur, S. Rahman, T. Torsha, M.A. Noor, K.M.A. Hossain, Burnt clay brick aggregate for internal curing of concrete under adverse curing conditions, Ksce Journal of Civil Engineering, 23 (2019) 5143-5153.

[7] D.J. Shen, Z.Z. Feng, J.C. Kang, C.Y. Wen, H.F. Shi, Effect of barchip fiber on stress relaxation and cracking potential of concrete internally cured with super absorbent polymers, Construction and Building Materials, 249 (2020) 13.

[8] M. Rajczakowska, K. Habermehl-Cwirzen, H. Hedlund, A. Cwirzen, The effect of exposure on the autogenous self-healing of ordinary portland cement mortars, Materials, 12 (2019) 23.

[9] D.J. Shen, C. Liu, Z.Z. Feng, S.S. Zhu, C. Liang, Influence of ground granulated blast furnace slag on the early-age anti-cracking property of internally cured concrete, Construction and Building Materials, 223 (2019) 233-243.

[10] D.J. Shen, C. Liu, C.C. Li, X.G. Zhao, G.Q. Jiang, Influence of barchip fiber length on early-age behavior and cracking resistance of concrete internally cured with super absorbent polymers, Construction and Building Materials, 214 (2019) 219-231.

[11] A. Danish, M.A. Mosaberpanah, M.U. Salim, Robust evaluation of superabsorbent polymers as an internal curing agent in cementitious composites, Journal of Materials Science, 56 (2021) 136-172.

[12] N. Hamzah, H.M. Saman, M.H. Baghban, A.R.M. Sam, I. Faridmehr, M.N.M. Sidek, O. Benjeddou, G.F. Huseien, A review on the use of self-curing agents and its mechanism in high-performance cementitious materials, Buildings, 12 (2022) 27.

[13] T.G. Atsbha, S. Zhutovsky, Optimising internal curing parameters for autonomous curing of normal-strength concrete, Magazine of Concrete Research, 75 (2023) 888-905.

[14] Z.Z. Feng, D.J. Shen, C.Y. Wen, X.J. Tang, G.Q. Jiang, Effect of water-to-cement ratios on performance of concrete with prewetted lightweight aggregates, Journal of Materials in Civil Engineering, 35 (2023) 14.

[15] M.M. Jaberizadeh, P.A. Danoglidis, S.P. Shah, M.S. Konsta-Gdoutos, Eco-efficient cementitious composites using waste cellulose fibers: Effects on autogenous shrinkage, strength and energy absorption capacity,

Construction and Building Materials, 408 (2023) 14.

[16] Q. Wang, W.J. Huang, J. Sun, J. Zhang, Shrinkage of high-strength engineered cementitious composite with zeolite particles as internal curing agent, Advances in Cement Research, 35 (2023) 287-296.

[17] Y. Wei, W.Q. Guo, L. Ma, Y.L. Liu, B.Y. Yang, Materials, structure, and construction of a low-shrinkage uhpc overlay on concrete bridge deck, Construction and Building Materials, 406 (2023) 19.

[18] J.T. Xu, X. Qin, Y.K. Lin, C.F. Cao, J.H. Liu, Q.J. Huang, Research on performance deterioration of internally cured pavement concrete under the coupling effect of salt freeze-thaw, Polymers, 15 (2023) 25.

[19] Y.T. Zhang, J.H. Lin, Z.W. Fan, X.Y. Zhu, Durability of internally cured concrete with silicon-aluminum based fine lightweight aggregate exposed to combined sulfate and wetting-drying attack, Materials and Structures, 56 (2023) 14.

[20] K. Zhang, Y.J. Xie, S.A. Noorkhah, M. Imeni, S.K. Das, Neutrosophic management evaluation of insurance companies by a hybrid todim-bsc method: A case study in private insurance companies, Management Decision, 61 (2023) 363-381.

[21] Z.Y. Wang, Q. Cai, J.P. Lu, G.W. Wei, Sustainable supplier selection by using dual probabilistic linguistic edas and itara method, Journal of Intelligent & Fuzzy Systems, 44 (2023) 9495-9512.

[22] T. Senapati, G.Y. Chen, R. Mesiar, R.R. Yager, Intuitionistic fuzzy geometric aggregation operators in the framework of aczel-alsina triangular norms and their application to multiple attribute decision making, Expert Systems with Applications, 212 (2023) 15.

[23] T. Senapati, An aczel-alsina aggregation-based outranking method for multiple attribute decision-making using single-valued neutrosophic numbers, Complex & Intelligent Systems, (2023) 15.

[24] P. Rani, S.M. Chen, A.R. Mishra, Multiple attribute decision making based on mairca, standard deviation-based method, and pythagorean fuzzy sets, Information Sciences, 644 (2023) 15.

[25] V. Ulucay, Q-neutrosophic soft graphs in operations management and communication network, Soft Computing, 25 (2021) 8441-8459.

[26] D. Stanujkic, D. Karabasevic, G. Popovic, F. Smarandache, E.K. Zavadskas, I. Meidute-Kavaliauskiene, Multiple-criteria decision-making based on the use of single-valued neutrosophic sets and similarity measures, Economic Computation and Economic Cybernetics Studies and Research, 55 (2021) 5-22.

[27] L.A. Zadeh, Fuzzy sets, Information and Control, 8 (1965) 338-353.

[28] F. Smarandache, Neutrosophic probability, set, and logic, proquest information & learning, Ann Arbor, Michigan, USA, 105 (1998) 118-123.

[29] F. Smarandache, Neutrosophy: Neutrosophic probability, set, and logic: Analytic synthesis & synthetic analysis, American Research Press, 1998.

[30] F. Smarandache, A unifying field in logics: Neutrosophic logic. Neutrosophy, neutrosophic set, neutrosophic probability: Neutrsophic logic, Infinite Study, 2005.

[31] H.Y. Zhang, J.Q. Wang, X.H. Chen, An outranking approach for multi-criteria decision-making problems with interval-valued neutrosophic sets, Neural Computing & Applications, 27 (2016) 615-627.

[32] P.K. Singh, Interval-valued neutrosophic graph representation of concept lattice and its ()-decomposition,

Arabian Journal for Science and Engineering, 43 (2018) 723-740.

[33] P. Biswas, S. Pramanik, B.C. Giri, Aggregation of triangular fuzzy neutrosophic set information and its application to multi-attribute decision making, Neutrosophic Sets and Systems, 12 (2016) 20-40.

[34] J.L. Deng, Introduction to grey system, The Journal of Grey System, 1 (1989) 1-24.

[35] G. Wei, New method of grey relational analysis to multiple attribute decision making with intervals, Systems Engineering and Electronics, 28 (2006) 1358-1359+1383.

[36] G. Wei, H. Yao, New method of grey related analysis to multiple attribute decision making problem with interval numbers, Statistics and Decision, 2006 (2006) 131.

[37] I. Irvanizam, N.N. Zi, R. Zuhra, A. Amrusi, H. Sofyan, An extended mabac method based on triangular fuzzy neutrosophic numbers for multiple-criteria group decision making problems, Axioms, 9 (2020) 18.

[38] C.E. Shannon, A mathematical theory of communication, Bell System Technical Journal, 27 (1948) 379-423.

[39] B. Xie, Modified gra methodology for madm under triangular fuzzy neutrosophic sets and applications to blended teaching effect evaluation of college english courses, Soft Computing, (2023) https://doi.org/10.1007/s00500-00023-08891-00506.

[40] J.P. Fan, X.F. Jia, M.Q. Wu, Green supplier selection based on dombi prioritized bonferroni mean operator with single-valued triangular neutrosophic sets, International Journal of Computational Intelligence Systems, 12 (2019) 1091-1101.

[41] L. Li, Cross-entropy method for efficiency evaluation of integrated development of agriculture and tourism to promote rural revitalization under the triangular fuzzy neutrosophic sets, Journal of Intelligent & Fuzzy Systems, 44 (2023) 6151-6161.

[42] J. Wang, G.W. Wei, M. Lu, An extended vikor method for multiple criteria group decision making with triangular fuzzy neutrosophic numbers, Symmetry-Basel, 10 (2018) 15.

[43] J.P. Fan, X.F. Jia, M.Q. Wu, A new multi-criteria group decision model based on single-valued triangular neutrosophic sets and edas method, Journal of Intelligent & Fuzzy Systems, 38 (2020) 2089-2102.

[44] W. Salabun, K. Urbaniak, A new coefficient of rankings similarity in decision-making problems, in: 20th Annual International Conference on Computational Science (ICCS), Springer International Publishing Ag, Amsterdam, NETHERLANDS, 2020, pp. 632-645.

[45] W. Sałabun, J. Wątróbski, A. Shekhovtsov, Are mcda methods benchmarkable? A comparative study of topsis, vikor, copras, and promethee ii methods, Symmetry, 12 (2020) 1549.

Received: June 28, 2024. Accepted: Oct 12, 2024

Neutrosophic Sets and Systems, Vol. 75, 2025
University of New Mexico

Introduction to algorithmic ethics under a neutrosophic framework: Re-mapping the debate

Antonios Paraskevas [1,*] and Florentin Smarandache [2]

[1]University of Macedonia, Department of Applied Informatics, 156, Egnatia Str., 54636, Thessaloniki, Greece; aparaskevas@uom.edu.gr;
[2]University of New Mexico, Mathematics Department, Gallup, NM 87301, USA; smarand@unm.edu;

* Correspondence: aparaskevas@uom.edu.gr;

Abstract: Algorithms have become an integral part of professional life and social interactions, in health, transport, commerce and industry. Moreover, they are bringing about changes in the natural, social and human sciences, enriching our knowledge and testing the limits of technology. In the age of artificial intelligence (AI), the ethical behavior of algorithms is facing increasing scrutiny. Traditional models that attempt to evaluate algorithm ethics using binary or probabilistic methods often fall short in addressing the complexity and uncertainties present in real-world situations. This study introduces a novel approach to assessing algorithmic ethics by utilizing neutrosophic indeterminacy. Neutrosophic logic, which considers three parameters - truth (T), falsity (F), and indeterminacy (I) - provides a robust framework for capturing the ambiguity and inconsistencies that arise in ethical decision-making processes. By utilizing this approach, we attempt to develop a method for measuring ethical ambiguity in algorithmic decision-making. By applying the proposed conceptual framework to an illustrative example we highlight the capacity of neutrosophic logic to capture and measure ethical uncertainties in a more comprehensive manner, thus offering a new tool for evaluating the ethical integrity of algorithms in complex environments.

Keywords: Neutrosophic ethical integrity score; neutrosophic logic; algorithmic ethics; fairness; transparency; accountability.

1. Introduction

Algorithms subtly influence our lives in various ways. Operations, judgements, and choices that were previously left to people are increasingly being delegated to algorithms, which may advise, if not determine, how data should be evaluated and what actions should be performed in response. Examples are abundant. Profiling and categorization algorithms influence how people and groups are formed and managed [1]. Recommendation systems guide users on when and how to exercise, what to buy, which route to take, and who to contact [2]. Data mining techniques are believed to have promise for making sense of growing streams of behavioral data provided by the 'Internet of Things' [3]. Personalization and filtering algorithms continue to be used by online service providers to manage information access [4-5]. Machine learning algorithms automatically detect misleading, biased, or erroneous knowledge at the time of generation.

In this paper we adapt the following definition about an algorithm: "a finite, abstract, effective, compound control structure, imperatively given, accomplishing a given purpose under given provisions" [6].

In the broader context, the rapid expansion of algorithms in crucial industries such as healthcare, banking, and law enforcement has raised significant ethical concerns. Algorithms have the potential to worsen biases, diminish transparency, and weaken accountability. There are many factors that make it difficult to determine the potential and actual ethical influence of an algorithm. Identifying the role of human subjectivity in algorithm design and configuration often requires an examination of long-term, multi-user development processes. Learning algorithms, which are frequently cited as the "future" of algorithms and analytics [7], add uncertainty into how and why judgements are made because of their ability to change operational parameters and decision-making rules "in the wild" [8]. The potential for algorithms to improve individual and social welfare comes with significant ethical risks [9]. In conclusion, algorithms are not ethically neutral.

In recent years, researchers have attempted to identify and categorize the ethical problems that algorithms give rise to and the solutions that have been proposed in recent relevant literature. The conceptual map proposed by [10] (Fig.1) remains a fruitful framework for reviewing the current debate on the ethics of algorithms.

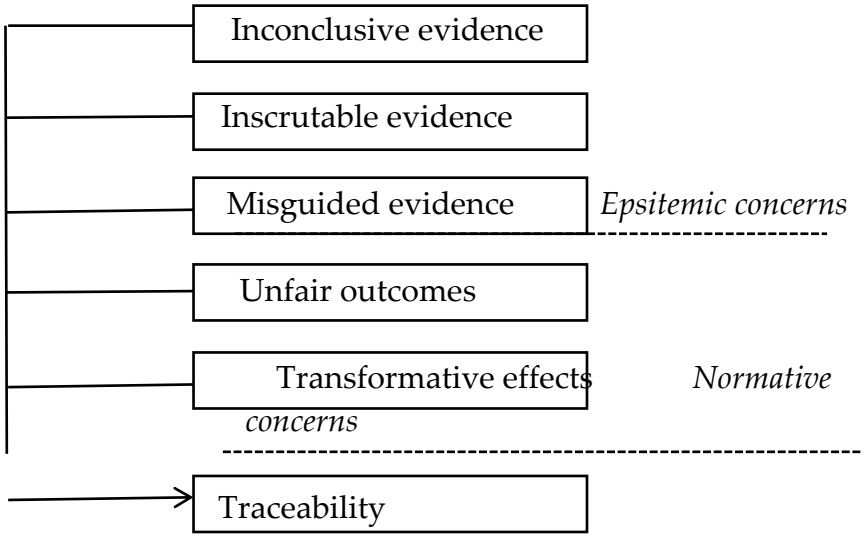


Figure 1. Six types of ethical concerns raised by algorithms [10]

Inconclusive evidence refers to the inevitably uncertain knowledge when algorithms draw conclusions from the data they process using inferential statistics and/or machine learning techniques.

Inscrutable evidence refers to the accessibility between the data and the conclusion in the case data are used as (or processed to produce) evidence for a conclusion.

Misguided evidence means that conclusions can only be as reliable (but also as neutral) as the data they are based on.

Unfair outcomes refers to the observer-dependent 'fairness' of the action and its effects.

Transformative effects denotes the unintended and significant changes that algorithms can impose on individuals or society, such as altering behaviors, decision-making processes, or social structures, often without transparency or user consent.

Traceability implies that harm caused by algorithmic activity is hard to debug (i.e. to detect the harm and find its cause), but also that it is rarely straightforward to identify who should be held responsible for the harm caused.

Driven by the definition of inconclusive evidence and its observed uncertainty, we are re-mapping the debate by introducing the concept of indeterminacy into all aspects of the ethical map (Fig.1). This approach strengthens our ability to address the complexity and ambiguity inherent in ethical decision-making for algorithms. By providing a structured approach to assessing how uncertainty affects ethical decisions, we increase accountability and transparency. This research

introduces a new neutrosophic methodological framework for evaluating ethical correctness (truth), ethical breaches (falsehood), and uncertainty (indeterminacy) in algorithms, offering a more comprehensive approach to ethical assessment. Thus, the conceptual idea of our study is depicted in Figure 2 and is posed as an organizing structure that allows a neutrosophic rigorous diagnosis of ethical challenges related to the use of algorithms.

Furthermore, aligning the ethical evaluation process with the intricacies of real-world scenarios makes the framework more useful and effective for guiding responsible and informed algorithm design and implementation. Introducing an "ethical score" metric that deals with indeterminacy can help tackle problematic situations such as ambiguous outcomes, data quality and representation, complex interactions, conflicting ethical principles, and human interpretation.

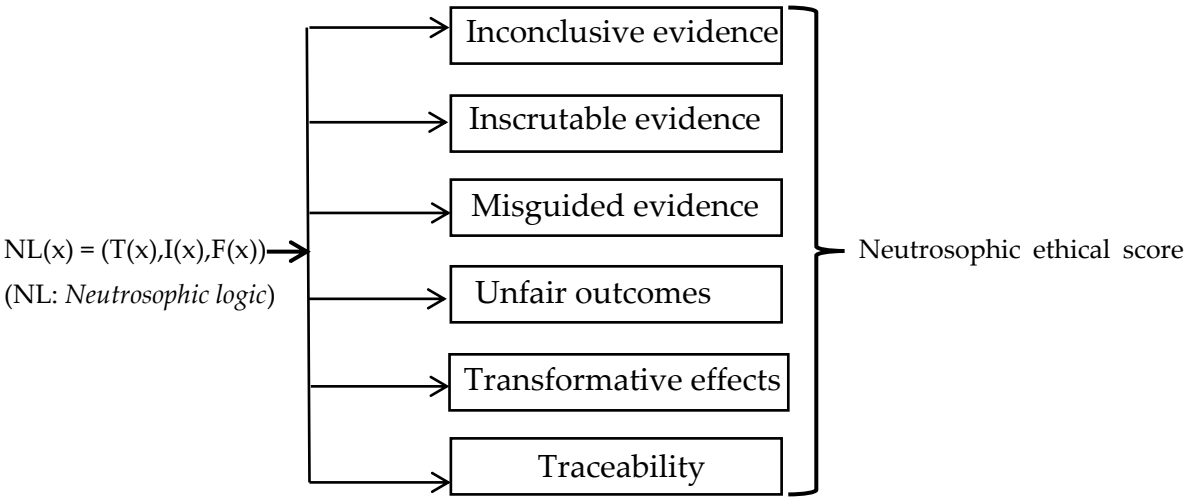


Figure 2. Neutrosophic algorithmic ethical map

Our review of the literature has revealed no prior studies that utilize this approach to systematically evaluate ethical ambiguity in algorithms, particularly in scenarios where principles like fairness, transparency, and accountability are in conflict. For this reason, this paper examines the significance of indeterminacy in ethical evaluations of algorithms and explores how a neutrosophic approach could impact the design of future algorithms. Actually, in our study, we aim to map out the ethical problems that arise from algorithmic decision-making by posing two main questions: 1) how can we systematically evaluate ethical uncertainty in algorithmic decision-making using neutrosophic logic? and 2) how does indeterminacy manifest in the conflicting ethical principles of justice, transparency, and accountability in algorithmic design?

While significant emphasis has been placed on concerns regarding prejudice, fairness, and openness [10-13], existing ethical frameworks sometimes struggle to navigate situations of uncertainty or conflicting ethical principles. In this way, our study also pinpoints the overlooked concept of indeterminacy and offers strategies for identifying ethical dilemmas, addressing a gap in the literature where traditional binary and fuzzy logic methods may fall short.

The structure of this article is as follows: In Section 2, we define and explain the neutrosophic mathematical framework needed to "construct" our proposed algorithmic “ethical index”, namely the Overall Neutrosophic Ethical Integrity Score (OvNEIS). This score will help us quantify the ethical performance of an algorithm in a neutrosophic environment. Next, in Section 3, we highlight the applicability of the suggested OvNEIS in an illustrative example from the field of healthcare. We examine an algorithmic decision-making process under three ethical criteria: fairness, transparency, and accountability and we briefly comment on the obtained results. Lastly, the "Concluding Remarks" section wraps up the key points of our study and proposes potential research work.

2. Materials and Methods

In neutrosophic logic, a concept A is T% true, I% indeterminate, and F% false, with (T, I, F) ⊂]]-0, 1+[[^{3}, where]]-0, 1+[[is an interval of hyperreals.

In this model, truth, falsehood, and indeterminacy may coexist, allowing for a more comprehensive representation of complex and ambiguous information. Sets containing neutrosophic components are employed in neutrosophic logic, with constituents having degrees of truth, falsehood, and indeterminacy. Its capacity to deal with ambiguity and uncertainty makes it useful in circumstances where standard logic systems may fail to offer correct representations.

In this framework, a formula φ is characterized by a triplet of truth-values, called the neutrosophical value defined as [14]:

NL(φ) = (T(φ), I(φ), F(φ)) where (T(φ), I(φ), F(φ)) ⊂]]-0, 1+[[^{3} (1)

For each algorithmic ethical criterion C_i $(i = 1,2 \dots, n)$, we can represent its performance by utilizing the concept of neutrosophic set in a similar way as given in (1):

$$C_i = \{(x, T_i(x), I_i(x), F_i(x)) | x \in U\} \quad (2)$$

where U : universe of discourse (e.g. algorithm outputs)

Next we define a formula, namely Neutrosophic Ethical Integrity Score (NEIS), which will be utilized for evaluating the ethical performance of algorithms based on the principles of neutrosophic logic. It is designed in such a way so as to balance positive and negative aspects (subtraction of falsehood) and capture the complexity of ethical evaluation (indeterminacy as a modifier)

Definition 1. The Neutrosophic Ethical Integrity Score (NEIS) for a given ethical criterion $\boldsymbol{i}$ is defined as:

$NEIS_i : \mathbb{R}^3 \rightarrow \mathbb{R}$

where $\mathbb{R}^3$ represents the three-dimensional space of the components of ethical evaluation, specifically truth, falsehood, and indeterminacy.

The $\boldsymbol{NEIS_i}$ function is given as follows:

$$NEIS_i = T_i\text{-}F_i\text{+}I_i \quad (3)$$

where :

$T_i \in [0,1]$ represents the degree to which the ethical criterion is satisfied by the algorithm. It is obvious that a value of T_i = 1 means complete satisfaction and T_i = 0 indicates complete dissatisfaction.

Following the same logic, F_i = 1 indicates complete violation to ethical criterion i and F_i = 0 means no violation.

Extra care should be given when evaluating the value of I_i. In our context, I_i= 1 indicates complete indeterminacy (uncertainty or ambiguity) in the ethical evaluation, while I_i= 0 means no indeterminacy.

Once the individual NEIS values are calculated for each criterion, the overall NEIS can be computed. This may involve taking a weighted average or simple average of the individual scores, depending on the specific context and importance of each criterion.

Definition 2. Let n be the number of criteria for assessing algorithmic ethics. The Overall NEIS can be defined as:

$$\text{Overall NEIS (OvNEIS)} = \sum_{i=1}^{n} w_i * NEIS_i \quad (4)$$

where w_i is the weight assigned to algorithmic ethical criterion i such as $\sum_{i=1}^{n} w_i = 1$.

3. Results

Let us now study an illustrative example that applies the proposed Neutrosophic Ethical Integrity Score (NEIS) to an algorithmic decision-making process in the context of healthcare. Let us assume that we wish to measure the performance of an algorithm designed to recommend treatment plans for patients in the following three ethical criteria: (i) *fairness*, (ii) *transparency*, and (iii) *accountability*.

To assess its ethical integrity, we evaluate its performance using the three aforesaid criteria in the next framework:

(i) Fairness: Does the algorithm treat all patients similarly, regardless of demographics (such as age, gender, or ethnicity)?

(ii) Transparency: Are healthcare practitioners able to comprehend the algorithm's decision-making process?

(iii)Accountability: Can the algorithm's outputs and decisions be tracked back to the responsible stakeholders?

Next, we will calculate the NEIS for each criterion.

Let us assume that based on historical data or surveys and expert evaluations we have the following assignment of values for each criterion (Equation 2):

(i) Fairness: C_f = (0.8, 0.1, 0.1)

(ii) Transparency: C_t = (0.7, 0.1, 0.2)

(iii) Accountability: C_a = (0.6, 0.1, 0.3)

The NEIS for each criterion is calculated using Equation (3):

$NEIS_{C_f} = T_{C_f} - F_{C_f} + I_{C_f} = 0.8$

In the same way, we get :

$NEIS_{C_t} = 0.6$ and

$NEIS_{C_a} = 0.4$

Let us know make the fair assumption that the weights of each ethical criterion is equal, i.e. $w_{C_f} = w_{C_t} = w_{C_a} = 0.33$.

Remark: depending on the context of the problem we are studying, different weights could be assigned to the criteria. This could be useful in cases where the fairness criteria is considered more important than the accountability and transparency criteria (e.g. hiring algorithms, loan approval algorithms, etc.). Therefore, a higher weight value for the fairness criterion will be assigned.

In our scenario, the overall NEIS is calculated by applying Equation (4):

OvNEIS = $w_{C_f} * NEIS_{C_f} + w_{C_t} * NEIS_{C_t} + w_{C_a} * NEIS_{C_a} = 0.6$

The Overall NEIS of 0.6 reflects an average ethical performance of the algorithm across all three criteria (Fairness, Transparency, and Accountability), given that all criteria are equally important.

A NEIS of 0.6 indicates moderate ethical integrity, with some space for improvement, notably in criteria such as accountability, which scored lower (0.4) compared to ther ethical criteria.

Furthermore, we can make the following observations based on our proposed method:

- The high fairness score indicates that the algorithm is unlikely to perpetuate considerable prejudice or discrimination, a critical worry in many algorithmic systems.
- The algorithm's decision-making process is only partly transparent, weakening confidence and making it difficult for consumers to comprehend or question its suggestions.
- The low accountability score raises questions about who is responsible for the algorithm's results, particularly in times of error or harm.

Next steps for the improvement of the algorithm:

- Continue to check for prejudice and fairness concerns, particularly in circumstances where indeterminacy is high.
- Improve the algorithm's capacity to explain its decision-making process to users, particularly when the decision is not clear.
- Involve users (e.g., healthcare experts) in assessing the algorithm's explanations to ensure that transparency has significance.
- Implement organizational policies that define accountability for algorithmic judgements, especially when they may cause harm.

4. Conclusions

Since 2016, the ethics of algorithms has been a major issue of debate among academics, technology providers, and regulators. The topic also gained pace as a result of the so-called "summer of AI" and the widespread deployment of Machine Learning algorithms. One aspect that was not explicitly captured by the research debate, and which is becoming a central point of discussion in the relevant literature, is the increasing focus on the use of algorithms, AI and digital technologies more broadly, to deliver socially good outcomes [15-17].

Starting from an admittedly simplistic notion of ethics as ''the study of what we ought to do,'' our aim has been to sketch a model that could "quantify" algorithmic ethics. In this perpserctive, we propose a formal mathematical neutrosophic framework that assists stakeholders in making key decisions based on a measurable indicator, OvNEIS. The latter, which is intended to assess uncertainty in ethical judgements in algorithmic decision-making, incorporates neutrosophic logic concepts such as truth, falsehood, and indeterminacy, allowing for a more comprehensive representation of complex and ambiguous data.

Specifically, OvEIS can address and add value, by considering the key role of indeterminacy, to the following issues concerning the algorithmic ethics:

- Inconclusive and Misguided Evidence: it helps to systematically quantify and address cases where evidence is either inconclusive or misleading.
- Inscrutable Evidence: it can enhance traceability and make algorithmic processes less inaccessible, answering concerns about traceability.
- Unfair Outcomes: it can analyse fairness as one of its primary aspects, guaranteeing that algorithms are more thoroughly reviewed for bias and fairness by taking into account not just truth and falsity, but also indeterminacy in their outcomes.
- Transformative Effects: it can consider the uncertainty in their long-term impacts on society, highlighting potential ethical concerns before they manifest.
- Traceability and Accountability: by scoring algorithms on this criterion, we ensure that ethical violations are not only discovered but also quantified, supporting responsible design and implementation.

Future study should focus on extending this technique to a broader range of case studies from a variety of sectors. Validation should be enhanced and a comparative study with other fuzzy/neutrosophic MCDM methods could shed light on issues such as efficiency, complexity and robustness of the proposed method.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Floridi, L. (2012). Big data and their epistemological challenge. *Philosophy & Technology*, 25(4), 435–437.
2. de Vries, K. (2010). Identity, profiling algorithms and a world of ambient intelligence. *Ethics and Information Technology*, 12(1), 71–85.
3. Portmess, L., & Tower, S. (2014). Data barns, ambient intelligence and cloud computing: The tacit epistemology and linguistic representation of Big Data. *Ethics and Information Technology*, 17(1), 1–9.
4. Newell, S., & Marabelli, M. (2015). Strategic opportunities (and challenges) of algorithmic decision-making: A call for action on the long-term societal effects of 'datification'. *The Journal of Strategic Information Systems*, 24(1), 3–14.
5. Taddeo, M., & Floridi, L. (2015). The debate on the moral responsibilities of online service providers. *Science and Engineering Ethics*, 1–29.
6. Hill, R. K. (2016). What an algorithm is. *Philosophy & Technology*, 29, 35-59.
7. Tutt, A. (2016, March 1). An FDA for Algorithms. *69 Admin. L. Rev. 83 (2017)*, Available at SSRN: https://ssrn.com/abstract=2747994 or http://dx.doi.org/10.2139/ssrn.2747994.
8. Burrell, J. (2016). How the machine 'thinks:' Understanding opacity in machine learning algorithms. *Big Data & Society*, 3(1): 1–12.

9. Floridi, L., & Taddeo, M. (2016). What is data ethics?. Philosophical Transactions of the Royal Society A: Mathematical, *Physical and Engineering Sciences*, 374(2083), 20160360.
10. Mittelstadt, B. D., Allo, P., Taddeo, M., Wachter, S., & Floridi, L. (2016). The ethics of algorithms: Mapping the debate. *Big Data & Society*, 3(2), 2053951716679679.
11. Binns, R. (2018, January). Fairness in machine learning: Lessons from political philosophy. In *Conference on fairness, accountability and transparency (pp. 149-159)*. PMLR.
12. Crawford, K. (2016). Can an algorithm be agonistic? Ten scenes from life in calculated publics. *Science, Technology, & Human Values*, 41(1), 77-92.
13. Martin, K. (2019). Designing ethical algorithms. *MIS Quarterly Executive June*.
14. Smarandache, F. (1999). *A unifying field in Logics: Neutrosophic Logic*. In Philosophy, American Research Press, pp. 1–141.
15. Hager, G. D., Drobnis, A., Fang, F., Ghani, R., Greenwald, A., Lyons, T., & Tambe, M. (2019). Artificial intelligence for social good. *arXiv preprint arXiv:1901.05406*.
16. Paraman, P., & Anamalah, S. (2023). Ethical artificial intelligence framework for a good AI society: principles, opportunities and perils. *AI & SOCIETY*, 38(2), 595-611.
17. Cowls, J., Tsamados, A., Taddeo, M., & Floridi, L. (2021). A definition, benchmark and database of AI for social good initiatives. *Nature Machine Intelligence*, 3(2), 111-115.

Received: June 27, 2024. Accepted: Oct 13, 2024

NSS

Neutrosophic Sets and Systems, Vol. 75, 2025

University of New Mexico

$\mathcal{N}g^{\#}-$ Irresolvable and Strongly $\mathcal{N}g^{\#}-$ Irresolvable Spaces in Neutrosophic Topological Spaces

Martina Jency J[1], *Babisha Julit R L[2] and Pious Missier S[3]

[1]Assistant Professor, PG and Research Department of Mathematics,
V. O.Chidambaram College,Thoothukudi-628008, Tamil Nadu, India.; martina.mat@voccollege.ac.in

[2] Assistant Professor, Department of Mathematics,
St.Alphonsa College of Arts and Science,Karinkal,Kanyakumari-629157, Tamil Nadu, India.;
babishajulit@stalphonsa.edu.in

[3] Head & Associate Professor, Department of Mathematics, Don Bosco College of Arts and Science, KeelaEral, Thoothukudi-628 908, Tamil Nadu, India.; spmissier@gmail.com

*Correspondence: babishajulit@stalphonsa.edu.in

Abstract. The notion of breaking down a topological space into smaller, disjoint subsets is the basis for the topological concepts of 'resolvable space' and 'irresolvable space'. By synergizing topology and neutrosophy, we have introduced a new paradigm in neutrosophic topological spaces, represented by $\mathcal{N}g^{\#}-$ resolvable spaces, $\mathcal{N}g^{\#}-$ irresolvable spaces and strongly $\mathcal{N}g^{\#}-$ irresolvable spaces. This pioneering fusion enables the exploration of novel mathematical structures and their characterizations.

Keywords: $\mathcal{N}g^{\#}-$ closed set; $\mathcal{N}g^{\#}-$ dense set; $\mathcal{N}g^{\#}-$ Irresolvable spaces ; Strongly $\mathcal{N}g^{\#}-$ Irresolvable spaces

1. Introduction

In our day to day life we come across uncertainties while dealing with real life problems in Science, Engineering, Business and Finance Management. Traditional mathematical methods failed to solve uncertainties. In order to overcome this difficulty, Zadeh [**?**] introduced the concept of fuzzy sets in 1965. In 1986, a generalization of fuzzy set was introduced by K.Atanassov [**?**] as intuitionistic fuzzy sets. The approaches such as fuzzy sets, intuitionistic fuzzy sets, vague sets and rough sets can be treated as Mathematical tools to avert obstacles dealing with ambiguous data. But all these approaches have their failures in solving the problems involving indeterminate and inconsistent data due to inadequacy of parametrization.

Neutrality the degree of indeterminacy as an independent concept was introduced by Florentine Smarandache [**?**].He defined the generalization of fuzzy sets and intuitionistic fuzzy sets as a Neutrosophic set on the three components, namely Truth (membership), Indeterminacy, Falsehood (non-membership). Neutrosophic sets found its place into contemporary research since it helps to study indeterminacy in real life problems.

The concept of Neutrosophy has been developed into Neutrosophic topological spaces by Salama and Alblowi [**?**] in 2012. Additionally, they studied Neutrosophic continuity which has been shown to be quite significant in the realm of Neutrosophic topology.

Salama and Smarandache [**?**] initiated Neutrosophic closed sets and continuous functions.R. Dhavaseelan et.al. [**?**] introduced Neutrosophic generalized closed sets. Pious Missier et.al. [**?**], introduced the concept of $\mathcal{N}g^{\#}-$ closed and open sets in Neutrosophic Topological Spaces.

A topological space X is called resolvable if it can be expressed as the union of two or more disjoint, non-empty open subsets. In other words, there exist open sets A and B (with $A \cap B = \phi$) such that $X = A \cup B$. Otherwise, it is irresolvable. Resolvable spaces can be broken into non-overlapping open sets, whereas irresolvable spaces cannot be partitioned in a meaningful way within the given topology. Resolvable spaces and irresolvable spaces help topologists to classify and characterize topological spaces based on their "splitting" properties.(i.e.,) To distinguish between spaces that can be decomposed into disjoint open sets (resolvable) and those that cannot (irresolvable).

In this article we introduce $\mathcal{N}g^{\#}-$ resolvable spaces, and $\mathcal{N}g^{\#}-$ irresolvable spaces and strongly $\mathcal{N}g^{\#}-$ irresolvable spaces and analyse their characterizations.

2. **Preliminaries**

Definition 2.1. [**?**] A Neutrosophic set $\mathcal{NSA}_{\mathcal{N}}$ is an object having the form $\mathcal{A}_{\mathcal{N}} = \{\langle \lambda, \mu_{\mathcal{A}_{\mathcal{N}}}(\lambda\lambda), \sigma_{\mathcal{A}_{\mathcal{N}}}(\lambda), \gamma_{\mathcal{A}_{\mathcal{N}}}(\lambda)\rangle : \lambda \in \mathcal{X}_{\mathcal{N}}\}$. Here

(1) $\mu_{\mathcal{A}_{\mathcal{N}}}(\lambda)-$ degree of membership
(2) $\sigma_{\mathcal{A}_{\mathcal{N}}}(\lambda)-$ degree of indeterminacy
(3) $\gamma_{\mathcal{A}_{\mathcal{N}}}(\lambda)-$ degree of non-membership

A Neutrosophic set $\mathcal{A}_{\mathcal{N}} = \{\langle \lambda, \mu_{\mathcal{A}_{\mathcal{N}}}(\lambda), \sigma_{\mathcal{A}_{\mathcal{N}}}(\lambda), \gamma_{\mathcal{A}_{\mathcal{N}}}(\lambda)\rangle : \lambda \in \mathcal{X}_{\mathcal{N}}\}$ can be identified as an ordered triple $\langle \mu_{\mathcal{A}_{\mathcal{N}}}(\lambda), \sigma_{\mathcal{A}_{\mathcal{N}}}(\lambda), \gamma_{\mathcal{A}_{\mathcal{N}}}(\lambda)\rangle$ in $]-0, 1+[$ on $\mathcal{X}_{\mathcal{N}}$.

Definition 2.2. [**?**] For any two Neutrosophic sets $\mathcal{A}_{\mathcal{N}} = \{\langle \lambda, \mu_{\mathcal{A}_{\mathcal{N}}}(\lambda), \sigma_{\mathcal{A}_{\mathcal{N}}}(\lambda), \gamma_{\mathcal{A}_{\mathcal{N}}}(\lambda)\rangle : \lambda \in \mathcal{X}_{\mathcal{N}}\}$ and $\mathcal{B}_{\mathcal{N}} = \{\langle \lambda, \mu_{\mathcal{B}_{\mathcal{N}}}(\lambda), \sigma_{\mathcal{B}_{\mathcal{N}}}(\lambda), \gamma_{\mathcal{B}_{\mathcal{N}}}(\lambda)\rangle : \lambda \in \mathcal{X}_{\mathcal{N}}\}$ we have

(1) $\mathcal{A}_{\mathcal{N}} \subseteq \mathcal{B}_{\mathcal{N}} \iff \mu_{\mathcal{A}_{\mathcal{N}}}(\lambda) \leq \mu_{\mathcal{B}_{\mathcal{N}}}(\lambda), \sigma_{\mathcal{A}_{\mathcal{N}}}(\lambda) \leq \sigma_{\mathcal{B}_{\mathcal{N}}}(\lambda)$ and $\gamma_{\mathcal{A}_{\mathcal{N}}}(\lambda) \geq \gamma_{\mathcal{B}_{\mathcal{N}}}(\lambda)$
(2) $\mathcal{A}_{\mathcal{N}} \cap \mathcal{B}_{\mathcal{N}} = \langle \lambda, \mu_{\mathcal{A}_{\mathcal{N}}}(\lambda) \wedge \mu_{\mathcal{B}_{\mathcal{N}}}(\lambda), \sigma_{\mathcal{A}_{\mathcal{N}}}(\lambda) \wedge \sigma_{\mathcal{B}_{\mathcal{N}}}(\lambda)$ and $\gamma_{\mathcal{A}_{\mathcal{N}}}(\lambda) \vee \gamma_{\mathcal{B}_{\mathcal{N}}}(\lambda)\rangle$
(3) $\mathcal{A}_{\mathcal{N}} \cup \mathcal{B}_{\mathcal{N}} = \langle \lambda, \mu_{\mathcal{A}_{\mathcal{N}}}(\lambda) \vee \mu_{\mathcal{B}_{\mathcal{N}}}(\lambda), \sigma_{\mathcal{A}_{\mathcal{N}}}(\lambda) \vee \sigma_{\mathcal{B}_{\mathcal{N}}}(\lambda)$ and $\gamma_{\mathcal{A}_{\mathcal{N}}}(\lambda) \wedge \gamma_{\mathcal{B}_{\mathcal{N}}}(\lambda)\rangle$

Definition 2.3. [?] Let $\mathcal{A}_{\mathcal{N}} = \langle \mu_{\mathcal{A}_{\mathcal{N}}}(\lambda), \sigma_{\mathcal{A}_{\mathcal{N}}}(\lambda), \gamma_{\mathcal{A}_{\mathcal{N}}}(\lambda) \rangle$ be a $\mathcal{N}S$ on $\mathcal{X}_{\mathcal{N}}$, then the complement ${\mathcal{A}_{\mathcal{N}}}^c$ defined as

- ${\mathcal{A}_{\mathcal{N}}}^c = \{\langle \lambda, \gamma_{\mathcal{A}_{\mathcal{N}}}(\lambda), 1 - \sigma_{\mathcal{A}_{\mathcal{N}}}(\lambda), \mu_{\mathcal{A}_{\mathcal{N}}}(\lambda) \rangle : \lambda \in \mathcal{X}_{\mathcal{N}}\}$

Note that for any two Neutrosophic sets $\mathcal{A}_{\mathcal{N}}$ and $\mathcal{B}_{\mathcal{N}}$,

- $(\mathcal{A}_{\mathcal{N}} \cup \mathcal{B}_{\mathcal{N}})^c = {\mathcal{A}_{\mathcal{N}}}^c \cap {\mathcal{B}_{\mathcal{N}}}^c$
- $(\mathcal{A}_{\mathcal{N}} \cap \mathcal{B}_{\mathcal{N}})^c = {\mathcal{A}_{\mathcal{N}}}^c \cup {\mathcal{B}_{\mathcal{N}}}^c$.

Definition 2.4. [?] A Neutrosophic topology ($\mathcal{NT}$) on a non-empty set $\mathcal{X}_{\mathcal{N}}$ is a family τ_N of Neutrosophic subsets in $\mathcal{X}_{\mathcal{N}}$ satisfies the following axioms:

(1) $\mathbf{0}_{\mathcal{N}}, \mathbf{1}_{\mathcal{N}} \in \tau_N$
(2) $\mathcal{R}_{N_1} \cap \mathcal{R}_{N_2} \in \tau_N$ for any $\mathcal{R}_{N_1}, R_{N_2} \in \tau_N$
(3) $\bigcup \mathcal{R}_{N_i} \in \tau_N \quad \forall \quad \mathcal{R}_{N_i} : i \in I \subseteq \tau_N$

Here the empty set $\mathbf{0}_{\mathcal{N}}$ and the whole set $\mathbf{1}_{\mathcal{N}}$ may be defined as follows:

(1) $\mathbf{0}_{\mathcal{N}} = \{\langle \lambda, 0, 0, 1 \rangle : \lambda \in \mathcal{X}_{\mathcal{N}}\}$
(2) $\mathbf{1}_{\mathcal{N}} = \{\langle \lambda, 1, 1, 0 \rangle : \lambda \in \mathcal{X}_{\mathcal{N}}\}$

Definition 2.5. [?] Let $\mathcal{A}_{\mathcal{N}}$ be a $\mathcal{NS}$ in $\mathcal{NTS}$ $\mathcal{X}_{\mathcal{N}}$. Then

(1) $\mathcal{N}int(\mathcal{A}_{\mathcal{N}}) = \bigcup\{\mathcal{G}_{\mathcal{N}} : \mathcal{G}_{\mathcal{N}}$ is a $\mathcal{NOS}$ in $\mathcal{X}_{\mathcal{N}}$ and $\mathcal{G}_{\mathcal{N}} \subseteq \mathcal{A}_{\mathcal{N}}\}$ is called a Neutrosophic interior of $\mathcal{A}_{\mathcal{N}}$.
(2) $\mathcal{N}cl(\mathcal{A}_{\mathcal{N}}) = \bigcap\{\mathcal{K}_{\mathcal{N}} : \mathcal{K}_{\mathcal{N}}$ is a $\mathcal{NCS}$ in $\mathcal{X}_{\mathcal{N}}$ and $\mathcal{A}_{\mathcal{N}} \subseteq \mathcal{K}_{\mathcal{N}}\}$ is called Neutrosophic closure of $\mathcal{A}_{\mathcal{N}}$.

Definition 2.6. [?] A Neutrosophic set $\mathcal{A}_{\mathcal{N}}$ of a $\mathcal{NTS}$ $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is called a neutrosophic $\mathcal{N}\alpha gCS$ if $\mathcal{N}\alpha cl(\mathcal{A}_{\mathcal{N}}) \subseteq \mathcal{U}_{\mathcal{N}}$, whenever $\mathcal{A}_{\mathcal{N}} \subseteq \mathcal{U}_{\mathcal{N}}$ and $\mathcal{U}_{\mathcal{N}}$ is a $\mathcal{NOS}$ in $\mathcal{X}_{\mathcal{N}}$. The complement of $\mathcal{N}\alpha gCS$ is $\mathcal{N}\alpha gOS$.

Definition 2.7. [?]

A Neutrosophic set $\mathcal{A}_{\mathcal{N}}$ of a $\mathcal{NTS}$ $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is called a Neutrosophic $g^{\#}-$ closed ($\mathcal{N}g^{\#}CS$) if $\mathcal{N}cl(\mathcal{A}_{\mathcal{N}}) \subseteq \mathcal{Q}_{\mathcal{N}}$ whenever $\mathcal{A}_{\mathcal{N}} \subseteq \mathcal{Q}_{\mathcal{N}}$ and $\mathcal{Q}_{\mathcal{N}}$ is $\mathcal{N}\alpha gOS$ in $\mathcal{X}_{\mathcal{N}}$. The complement of $\mathcal{N}g^{\#}CS$ is $\mathcal{N}g^{\#}OS$.

Definition 2.8. [?] Let $\mathcal{A}_{\mathcal{N}}$ be a $\mathcal{NS}$ in $\mathcal{NTS}$ $\mathcal{X}_{\mathcal{N}}$. Then

(1) $\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}}) = \bigcup\{\mathcal{G}_{\mathcal{N}} : \mathcal{G}_{\mathcal{N}}$ is a $\mathcal{N}g^{\#}OS$ in $\mathcal{X}_{\mathcal{N}}$ and $\mathcal{G}_{\mathcal{N}} \subseteq \mathcal{A}_{\mathcal{N}}\}$ is called a Neutrosophic $g^{\#}-$ interior of $\mathcal{A}_{\mathcal{N}}$.
(2) $\mathcal{N}g^{\#}cl(\mathcal{A}_{\mathcal{N}}) = \bigcap\{\mathcal{K}_{\mathcal{N}} : \mathcal{K}_{\mathcal{N}}$ is a $\mathcal{N}g^{\#}CS$ in $\mathcal{X}_{\mathcal{N}}$ and $\mathcal{A}_{\mathcal{N}} \subseteq \mathcal{K}_{\mathcal{N}}\}$ is called Neutrosophic $g^{\#}-$ closure of $\mathcal{A}_{\mathcal{N}}$.

Definition 2.9. [?] A function $f_{\mathcal{N}} : (\mathcal{X}_{\mathcal{N}}, \tau_N) \longrightarrow (\mathcal{Y}_{\mathcal{N}}, \zeta_N)$ is said to be $\mathcal{N}g^{\#}-$ continuous function if $f_{\mathcal{N}}^{-1}(\mathcal{V}_{\mathcal{N}})$ is a $\mathcal{N}g^{\#}-$ closed set of $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ for every neutrosophic closed set $\mathcal{V}_{\mathcal{N}}$ of $(\mathcal{Y}_{\mathcal{N}}, \zeta_N)$.

Definition 2.10. [?] A function $f_{\mathcal{N}} : (\mathcal{X}_{\mathcal{N}}, \tau_N) \longrightarrow (\mathcal{Y}_{\mathcal{N}}, \zeta_N)$ is said to be Neutrosophic $g^{\#}-$ irresolute function if $f_{\mathcal{N}}^{-1}(\mathcal{V}_{\mathcal{N}})$ is a $\mathcal{N}g^{\#}CS$ of $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ for every $\mathcal{N}g^{\#}CS$ $\mathcal{V}_{\mathcal{N}}$ of $(\mathcal{Y}_{\mathcal{N}}, \zeta_N)$.

Definition 2.11. [?] A Neutrosophic Topological space $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is called a $T_{\mathcal{N}}g^{\#}-$ space if every $\mathcal{N}g^{\#}CS$ in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is $\mathcal{NCS}$ in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$.

Definition 2.12. [?] A Neutrosophic set $\mathcal{A}_{\mathcal{N}}$ in a Neutrosophic topological space $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is called Neutrosophic dense set if there exists no $\mathcal{NCS}$ $\mathcal{B}_{\mathcal{N}}$ in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ such that $\mathcal{A}_{\mathcal{N}} \subset \mathcal{B}_{\mathcal{N}} \subset \mathbf{1}_{\mathcal{N}}$.

Definition 2.13. [?] A Neutrosophic set $\mathcal{A}_{\mathcal{N}}$ in a Neutrosophic topological space $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is called Neutrosophic nowhere dense set if there exists no non- zero $\mathcal{NOS}$ $\mathcal{B}_{\mathcal{N}}$ in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ such that $\mathcal{B}_{\mathcal{N}} \subset \mathcal{N}cl(\mathcal{A}_{\mathcal{N}})$. That is $\mathcal{N}int(\mathcal{N}cl(\mathcal{A}_{\mathcal{N}})) = \mathbf{0}_{\mathcal{N}}$.

Definition 2.14. [?] A Neutrosophic set $\mathcal{A}_{\mathcal{N}}$ in a Neutrosophic topological space $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is called $\mathcal{N}g^{\#}-$ dense set if there exists no $\mathcal{N}g^{\#}CS$ $\mathcal{B}_{\mathcal{N}}$ in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ such that $\mathcal{A}_{\mathcal{N}} \subset \mathcal{B}_{\mathcal{N}} \subset \mathbf{1}_{\mathcal{N}}$. That is $\mathcal{N}g^{\#}cl(\mathcal{A}_{\mathcal{N}}) = \mathbf{1}_{\mathcal{N}}$.

3. $\mathcal{N}g^{\#}-$ Irresolvable Spaces

Definition 3.1. A Neutrosophic topological space $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is called $\mathcal{N}g^{\#}-$ resolvable space if there exists a $\mathcal{N}g^{\#}-$ dense set $\mathcal{A}_{\mathcal{N}}$ in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ such that $\mathcal{N}g^{\#}cl(\mathcal{A}_{\mathcal{N}}{}^c) = \mathbf{1}_{\mathcal{N}}$. Otherwise $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is called $\mathcal{N}g^{\#}-$ Irresolvable Space.

Example

3.2. Let $\mathcal{X}_{\mathcal{N}} = \{p, q\}$. Consider the Neutrosophic sets, $\mathcal{M}_{\mathcal{N}_1} = \langle(0.4, 0.5, 0.4), (0.3, 0.6, 0.4)\rangle$, $\mathcal{M}_{\mathcal{N}_2} = \langle(0.4, 0.5, 0.4), (0.4, 0.4, 0.3)\rangle$, $\mathcal{M}_{\mathcal{N}_3} = \langle(0.4, 0.4, 0.4), (0.5, 0.5, 0.4)\rangle$, $\mathcal{M}_{\mathcal{N}_4} = \langle(0.4, 0.6, 0.4), (0.4, 0.5, 0.5)\rangle$. Now $(\mathcal{X}_{\mathcal{N}}, \tau_N) = \{\mathbf{0}_{\mathcal{N}}, \mathcal{M}_{\mathcal{N}_1}, \mathcal{M}_{\mathcal{N}_2}, \mathcal{M}_{\mathcal{N}_3}, \mathcal{M}_{\mathcal{N}_4}, \mathbf{1}_{\mathcal{N}}\}$ is Neutrosophic topological space. Then, $\tau_N = \{\mathbf{0}_{\mathcal{N}}, \mathcal{M}_{\mathcal{N}_1}, \mathbf{1}_{\mathcal{N}}\}$ is $\mathcal{NT}$ on $\mathcal{X}_{\mathcal{N}}$. Here, $\mathcal{N}g^{\#}CS(\mathcal{X}_{\mathcal{N}}) = \{\mathbf{0}_{\mathcal{N}}, \mathcal{M}_{\mathcal{N}_2}, \mathbf{1}_{\mathcal{N}}\}, \mathcal{N}g^{\#}DS(\mathcal{X}_{\mathcal{N}}) = \{\mathcal{M}_{\mathcal{N}_3}, \mathcal{M}_{\mathcal{N}_4}, \mathbf{1}_{\mathcal{N}}\}$. Now $\mathcal{M}_{\mathcal{N}_3}$ is a $\mathcal{N}g^{\#}-$ dense set in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ and $\mathcal{N}g^{\#}cl(\mathcal{M}_{\mathcal{N}_3}^c) = \mathbf{1}_{\mathcal{N}}$. Hence, $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is $\mathcal{N}g^{\#}-$ resolvable space.

Example 3.3. Let $\mathcal{X}_{\mathcal{N}} = \{p, q\}$. Consider the Neutrosophic sets, $\mathcal{M}_{\mathcal{N}_1} = \langle(0.4, 0.5, 0.6), (0.4, 0.3, 0.5)\rangle$, $\mathcal{M}_{\mathcal{N}_2} = \langle(0.6, 0.5, 0.4), (0.5, 0.7, 0.4)\rangle$, $\mathcal{M}_{\mathcal{N}_3} = \langle(0.6, 0.5, 0.3), (0.6, 0.7, 0.3)\rangle$, $\mathcal{M}_{\mathcal{N}_4} = \langle(0.3, 0.5, 0.6), (0.3, 0.3, 0.6)\rangle$, $\mathcal{M}_{\mathcal{N}_5} = \langle(0.7, 0.5, 0.4), (0.6, 0.8, 0.4)\rangle$, $\mathcal{M}_{\mathcal{N}_6} = \langle(0.4, 0.5, 0.7), (0.4, 0.2, 0.6)\rangle$. Now, $(\mathcal{X}_{\mathcal{N}}, \tau_N) = \{\mathbf{0}_{\mathcal{N}}, \mathcal{M}_{\mathcal{N}_1}, \mathcal{M}_{\mathcal{N}_2}, \mathcal{M}_{\mathcal{N}_3}, \mathcal{M}_{\mathcal{N}_4}, \mathcal{M}_{\mathcal{N}_5}, \mathcal{M}_{\mathcal{N}_6}, \mathbf{1}_{\mathcal{N}}\}$ is Neutrosophic topological space. Then, $\tau_N = \{\mathbf{0}_{\mathcal{N}}, \mathcal{M}_{\mathcal{N}_1}, \mathbf{1}_{\mathcal{N}}\}$ is $\mathcal{NT}$ on $\mathcal{X}_{\mathcal{N}}$. Here $\mathcal{N}g^{\#}CS(\mathcal{X}_{\mathcal{N}}) =$

$\{\mathbf{0}_{\mathcal{N}}, \mathcal{M}_{\mathcal{N}_2}, \mathcal{M}_{\mathcal{N}_3}, \mathcal{M}_{\mathcal{N}_5}, \mathbf{1}_{\mathcal{N}}\}$, $\mathcal{N}g^{\#}DS(\mathcal{X}_{\mathcal{N}}) = \{\mathcal{M}_{\mathcal{N}_3}, \mathcal{M}_{\mathcal{N}_5}, \mathbf{1}_{\mathcal{N}}\}$. Now $\mathcal{M}_{\mathcal{N}_3}$ and $\mathcal{M}_{\mathcal{N}_5}$ both are $\mathcal{N}g^{\#}-$ dense sets in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ but $\mathcal{N}g^{\#}cl(\mathcal{M}_{\mathcal{N}_3}{}^c) = \mathcal{M}_{\mathcal{N}_2} \neq \mathbf{1}_{\mathcal{N}}$,$\mathcal{N}g^{\#}cl(\mathcal{M}_{\mathcal{N}_5}{}^c) = \mathcal{M}_{\mathcal{N}_2} \neq \mathbf{1}_{\mathcal{N}}$. Therefore, $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is not a $\mathcal{N}g^{\#}-$ resolvable space and hence $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is $\mathcal{N}g^{\#}-$ irresolvable space.

Theorem 3.4. *A $\mathcal{NTS}$ $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is a $\mathcal{N}g^{\#}-$ resolvable space if and only if $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ has a pair of $\mathcal{N}g^{\#}-$ dense sets $\mathcal{M}_1$ and $\mathcal{M}_2$ such that $\mathcal{M}_1 \subseteq \mathcal{M}_2{}^c$.*

Proof. Let $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ be a $\mathcal{N}g^{\#}-$ resolvable space. Suppose that for all $\mathcal{N}g^{\#}-$ dense sets $\mathcal{M}_i$ and $\mathcal{M}_j$, we have $\mathcal{M}_i \nsubseteq \mathcal{M}_j{}^c$. Then $\mathcal{M}_i \supset \mathcal{M}_j{}^c$. Then $\mathcal{N}g^{\#}cl(\mathcal{M}_i) \supset \mathcal{N}g^{\#}cl(\mathcal{M}_j{}^c)$. $\implies$ $\mathbf{1}_{\mathcal{N}} \supset \mathcal{N}g^{\#}cl(\mathcal{M}_j{}^c)$. Then $\mathcal{N}g^{\#}cl(\mathcal{M}_j{}^c) \neq \mathbf{1}_{\mathcal{N}}$. Also $\mathcal{M}_j \supset \mathcal{M}_i{}^c$, then $\mathcal{N}g^{\#}cl(\mathcal{M}_j) \supset \mathcal{N}g^{\#}cl(\mathcal{M}_i{}^c)$. $\implies$ $\mathbf{1}_{\mathcal{N}} \supset \mathcal{N}g^{\#}cl(\mathcal{M}_i{}^c)$. Therfore, $\mathcal{N}g^{\#}cl(\mathcal{M}_i{}^c) \neq \mathbf{1}_{\mathcal{N}}$. Since $\mathcal{M}_i$ is $\mathcal{N}g^{\#}-$ dense set and $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is $\mathcal{N}g^{\#}-$ resolvable, $\mathcal{N}g^{\#}cl(\mathcal{M}_i) = \mathbf{1}_{\mathcal{N}}$, but we got, $\mathcal{N}g^{\#}cl(\mathcal{M}_i{}^c) \neq \mathbf{1}_{\mathcal{N}}$ for all Neutrosophic set $\mathcal{M}_i$ in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ which is a contradiction. Hence $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ has a pair of $\mathcal{N}g^{\#}-$ dense sets $\mathcal{M}_1$ and $\mathcal{M}_2$ such that $\mathcal{M}_1 \subseteq \mathcal{M}_2{}^c$.

Conversly, Suppose that the $\mathcal{NTS}$ $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ has a pair of $\mathcal{N}g^{\#}-$ dense sets $\mathcal{M}_1$ and $\mathcal{M}_2$ such that $\mathcal{M}_1 \subseteq \mathcal{M}_2{}^c$. Now Assume that $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is a $\mathcal{N}g^{\#}-$ irresolvable space. Then for all $\mathcal{N}g^{\#}-$ dense sets $\mathcal{M}_1$ and $\mathcal{M}_2$, we have $\mathcal{N}g^{\#}cl(\mathcal{M}_1{}^c) \neq \mathbf{1}_{\mathcal{N}}$ and $\mathcal{N}g^{\#}cl(\mathcal{M}_2{}^c) \neq \mathbf{1}_{\mathcal{N}}$. Which implies that there exists a $\mathcal{N}g^{\#}CS$ $\mathcal{B}_{\mathcal{N}}$ in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ such that $\mathcal{M}_2{}^c \subset \mathcal{B}_{\mathcal{N}} \subset \mathbf{1}_{\mathcal{N}}$. Now $\mathcal{M}_1 \subseteq \mathcal{M}_2{}^c \subset \mathcal{B}_{\mathcal{N}} \subset \mathbf{1}_{\mathcal{N}} \implies \mathcal{M}_1 \subset \mathcal{B}_{\mathcal{N}} \subset \mathbf{1}_{\mathcal{N}}$. That is $\mathcal{M}_1$ is not a $\mathcal{N}g^{\#}-$ dense set. Which is a contradiction. Therefore, $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is a $\mathcal{N}g^{\#}-$ resolvable space. □

Theorem 3.5. *A Neutrosophic topological space $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is a $\mathcal{N}g^{\#}-$ resolvable space if $\mathbf{1}_{\mathcal{N}} = \bigcup_{i=1}^{n} \mathcal{A}_i$ where $\mathcal{N}g^{\#}int(\mathcal{A}_i) = \mathbf{0}_{\mathcal{N}}$.*

Proof. Assume that $\mathbf{1}_{\mathcal{N}} = \bigcup_{i=1}^{n} \mathcal{A}_i$ where $\mathcal{N}g^{\#}int(\mathcal{A}_i) = \mathbf{0}_{\mathcal{N}}$. Which implies that $(\bigcup \mathcal{A}_i)^c = \mathbf{0}_{\mathcal{N}}$. Then $\bigcap_{i=1}^{n}(\mathcal{A}_i^c) = \mathbf{0}_{\mathcal{N}}$. Then there exists atleast two non- zero disjoint Neutrosophic sets $\mathcal{A}_i^c$ and $\mathcal{A}_j^c$ in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$. Hence $\mathcal{A}_i^c \cup \mathcal{A}_j^c \subseteq \mathbf{1}_{\mathcal{N}}$. Therefore, $\mathcal{A}_i^c \subseteq \mathcal{A}_j$ which implies that $\mathcal{N}g^{\#}cl(\mathcal{A}_i^c) \subseteq \mathcal{N}g^{\#}cl(\mathcal{A}_j)$. But $\mathcal{N}g^{\#}int(\mathcal{A}_i) = \mathbf{0}_{\mathcal{N}}$ implies that $\mathcal{N}g^{\#}cl(\mathcal{A}_i^c) = \mathbf{1}_{\mathcal{N}}$. Hence $\mathbf{1}_{\mathcal{N}} \subseteq \mathcal{N}g^{\#}cl(\mathcal{A}_j)$ which implies that $\mathcal{N}g^{\#}cl(\mathcal{A}_j) = \mathbf{1}_{\mathcal{N}}$. Also $\mathcal{N}g^{\#}int(\mathcal{A}_j) = \mathbf{0}_{\mathcal{N}}$ imples that $\mathcal{N}g^{\#}cl(\mathcal{A}_j^c) = \mathbf{1}_{\mathcal{N}}$. Therefore, $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ has a $\mathcal{N}g^{\#}-$ dense set $\mathcal{A}_j$ such that $\mathcal{N}g^{\#}cl(\mathcal{A}_j^c) = \mathbf{1}_{\mathcal{N}}$. Hence $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is a $\mathcal{N}g^{\#}-$ resolvable space. □

Theorem 3.6. *A Neutrosophic topological space $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is a $\mathcal{N}g^{\#}-$ irresolvable space if and only if $\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}}) \neq \mathbf{0}_{\mathcal{N}}$ for all $\mathcal{N}g^{\#}-$ dense set in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$.*

Proof. Since $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is a $\mathcal{N}g^{\#}-$ irresolvable space, for all $\mathcal{N}g^{\#}-$ dense set $\mathcal{A}_{\mathcal{N}}$ in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$,$\mathcal{N}g^{\#}cl(\mathcal{A}_{\mathcal{N}}{}^c) \neq \mathbf{1}_{\mathcal{N}}$. Then $(\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}}))^c \neq \mathbf{1}_{\mathcal{N}}$ which implies $\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}}) \neq \mathbf{0}_{\mathcal{N}}$.

Conversely, $\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}}) \neq \mathbf{0}_{\mathcal{N}}$, for all $\mathcal{N}g^{\#}-$ dense set in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$. Suppose that $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is $\mathcal{N}g^{\#}-$ resolvable space. Then there exists a $\mathcal{N}g^{\#}-$ dense set $\mathcal{A}_{\mathcal{N}}$ in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ such that $\mathcal{N}g^{\#}cl(\mathcal{A}_{\mathcal{N}}{}^c) = \mathbf{1}_{\mathcal{N}}$. This implies that $(\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}}))^c = \mathbf{1}_{\mathcal{N}} \implies \mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}}) = \mathbf{0}_{\mathcal{N}}$. Which is a contradiction. Therefore, $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is $\mathcal{N}g^{\#}-$ irresolvable space. □

4. Strongly $\mathcal{N}g^{\#}-$ Irresolvable Spaces

Definition 4.1. A Neutrosophic topological space $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is called strongly $\mathcal{N}g^{\#}-$ irresolvable space if for any $\mathcal{N}g^{\#}-$ dense set $\mathcal{A}_{\mathcal{N}}$ in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$, $\mathcal{N}g^{\#}cl(\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}})) = \mathbf{1}_{\mathcal{N}}$.

Example

4.2. Let $\mathcal{X}_{\mathcal{N}} = \{p, q\}$. Consider the Neutrosophic sets $\mathcal{M}_{\mathcal{N}_1} = \langle(0.6, 0.6, 0.3), (0.6, 0.6, 0.3)\rangle$, $\mathcal{M}_{\mathcal{N}_2} = \langle(0.3, 0.4, 0.6), (0.3, 0.4, 0.6)\rangle$, $\mathcal{M}_{\mathcal{N}_3} = \langle(0.3, 0.3, 0.7), (0.3, 0.3, 0.7)\rangle$, $\mathcal{M}_{\mathcal{N}_4} = \langle(0.7, 0.7, 0.3), (0.7, 0.7, 0.3)\rangle$. Now $(\mathcal{X}_{\mathcal{N}}, \tau_N) = \{\mathbf{0}_{\mathcal{N}}, \mathcal{M}_{\mathcal{N}_1}, \mathcal{M}_{\mathcal{N}_2}, \mathcal{M}_{\mathcal{N}_3}, \mathcal{M}_{\mathcal{N}_4}, \mathbf{1}_{\mathcal{N}}\}$ is Neutrosophic topological space. Then, $\tau_N = \{\mathbf{0}_{\mathcal{N}}, \mathcal{M}_{\mathcal{N}_1}, \mathbf{1}_{\mathcal{N}}\}$ is Neutrosophic topology on $\mathcal{X}_{\mathcal{N}}$. Here, $\mathcal{N}g^{\#}CS(\mathcal{X}_{\mathcal{N}}) = \{\mathbf{0}_{\mathcal{N}}, \mathcal{M}_{\mathcal{N}_2}, \mathbf{1}_{\mathcal{N}}\}, \mathcal{N}g^{\#}DS(\mathcal{X}_{\mathcal{N}}) = \{\mathcal{M}_{\mathcal{N}_1}, \mathcal{M}_{\mathcal{N}_4}, \mathbf{1}_{\mathcal{N}}\}$. Now $\mathcal{N}g^{\#}cl(\mathcal{N}g^{\#}int(\mathcal{M}_{\mathcal{N}_1})) = \mathbf{1}_{\mathcal{N}}$ and $\mathcal{N}g^{\#}cl(\mathcal{N}g^{\#}int(\mathcal{M}_{\mathcal{N}_4})) = \mathbf{1}_{\mathcal{N}}$ Therefore, $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is a strongly $\mathcal{N}g^{\#}-$ irresolvable space.

Theorem 4.3. *If $\mathcal{N}g^{\#}cl(\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}})) \neq \mathbf{1}_{\mathcal{N}}$, for every Neutrosophic set $\mathcal{A}_{\mathcal{N}}$ in a strongly $\mathcal{N}g^{\#}-$ irresolvable space $(\mathcal{X}_{\mathcal{N}}, \tau_N)$, then $\mathcal{N}g^{\#}cl(\mathcal{A}_{\mathcal{N}}) \neq \mathbf{1}_{\mathcal{N}}$ in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$.*

Proof. Let $\mathcal{A}_{\mathcal{N}}$ be a $\mathcal{NS}$ in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ such that $\mathcal{N}g^{\#}cl(\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}})) \neq \mathbf{1}_{\mathcal{N}}$. Suppose that $\mathcal{N}g^{\#}cl(\mathcal{A}_{\mathcal{N}}) = \mathbf{1}_{\mathcal{N}}$ in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$. Since $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is strongly $\mathcal{N}g^{\#}-$ irresolvable space, $\mathcal{N}g^{\#}cl(\mathcal{A}_{\mathcal{N}}) = \mathbf{1}_{\mathcal{N}}$ implies that $\mathcal{N}g^{\#}cl(\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}})) = \mathbf{1}_{\mathcal{N}}$. This is a contradiction. Then it must be that $\mathcal{N}g^{\#}cl(\mathcal{A}_{\mathcal{N}}) \neq \mathbf{1}_{\mathcal{N}}$ in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$. □

Theorem 4.4. *Let $(\mathcal{A}_{\mathcal{N}})$ be a $\mathcal{NS}$ in a strongly $\mathcal{N}g^{\#}-$ irresolvable space $(\mathcal{X}_{\mathcal{N}}, \tau_N)$. If $\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}}) = \mathbf{0}_{\mathcal{N}}$ then $\mathcal{A}_{\mathcal{N}}$ is $\mathcal{N}g^{\#}-$ nowhere dense set.*

Proof. Let $\mathcal{A}_{\mathcal{N}}$ be a $\mathcal{NS}$ in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ such that $\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}}) = \mathbf{0}_{\mathcal{N}}$. Then, $(\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}}))^c = \mathbf{1}_{\mathcal{N}}$ which implies that $\mathcal{N}g^{\#}cl(\mathcal{A}_{\mathcal{N}}{}^c) = \mathbf{1}_{\mathcal{N}}$. Since $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is a strongly $\mathcal{N}g^{\#}-$ irresolvable space, $\mathcal{N}g^{\#}cl(\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}}{}^c)) = \mathbf{1}_{\mathcal{N}}. \implies \mathcal{N}g^{\#}cl(\mathcal{N}g^{\#}cl(\mathcal{A}_{\mathcal{N}}))^c = \mathbf{1}_{\mathcal{N}}. \implies (\mathcal{N}g^{\#}int(\mathcal{N}g^{\#}cl(\mathcal{A}_{\mathcal{N}})))^c = \mathbf{1}_{\mathcal{N}}$. Which implies that, $\mathcal{N}g^{\#}int(\mathcal{N}g^{\#}cl(\mathcal{A}_{\mathcal{N}})) = \mathbf{0}_{\mathcal{N}}$. Hence proved. □

Theorem 4.5. *If $\mathcal{A}_{\mathcal{N}}$ is a $\mathcal{NS}$ in a strongly $\mathcal{N}g^{\#}-$ irresolvable space $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ such that $\mathcal{N}g^{\#}int(\mathcal{N}g^{\#}cl(\mathcal{A}_{\mathcal{N}})) \neq \mathbf{0}_{\mathcal{N}}$ then $\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}}) \neq \mathbf{0}_{\mathcal{N}}$ in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$.*

Proof. Let $\mathcal{A}_{\mathcal{N}}$ be a $\mathcal{NS}$ in a strongly $\mathcal{N}g^{\#}-$ irresolvable space $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ such that $\mathcal{N}g^{\#}int(\mathcal{N}g^{\#}cl(\mathcal{A}_{\mathcal{N}})) \neq \mathbf{0}_{\mathcal{N}}$. Suppose that $\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}}) = \mathbf{0}_{\mathcal{N}}$ in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$. Now, $\mathcal{N}g^{\#}cl(\mathcal{A}_{\mathcal{N}}{}^c) = (\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}}))^c = \mathbf{1}_{\mathcal{N}}$.

$$\implies \mathcal{N}g^{\#}cl(\mathcal{A}_{\mathcal{N}}{}^c) = \mathbf{1}_{\mathcal{N}} \text{ (since } (\mathcal{X}_{\mathcal{N}}, \tau_N) \text{ is strongly } \mathcal{N}g^{\#} - \text{irresolvable space)}$$
$$\implies \mathcal{N}g^{\#}cl(\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}}{}^c)) = \mathbf{1}_{\mathcal{N}}$$
$$\implies \mathcal{N}g^{\#}cl(\mathcal{N}g^{\#}cl(\mathcal{A}_{\mathcal{N}}))^c = \mathbf{1}_{\mathcal{N}}$$
$$\implies (\mathcal{N}g^{\#}int(\mathcal{N}g^{\#}cl(\mathcal{A}_{\mathcal{N}})))^c = \mathbf{1}_{\mathcal{N}}$$
$$\implies \mathcal{N}g^{\#}int(\mathcal{N}g^{\#}cl(\mathcal{A}_{\mathcal{N}})) = \mathbf{0}_{\mathcal{N}}$$

Which is contradiction to the hypothesis. Then it must be that $\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}}) \neq \mathbf{0}_{\mathcal{N}}$ in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$. Hence proved. □

Theorem 4.6. *If $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is a strongly $\mathcal{N}g^{\#}-$ irresolvable space then $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is a $\mathcal{N}g^{\#}-$ irresolvable space.*

Proof. Let $\mathcal{A}_{\mathcal{N}}$ be a $\mathcal{NS}$ in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ such that $\mathcal{N}g^{\#}cl(\mathcal{A}_{\mathcal{N}}) = \mathbf{1}_{\mathcal{N}}$. We claim that $\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}}) \neq \mathbf{0}_{\mathcal{N}}$. Suppose that $\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}}) = \mathbf{0}_{\mathcal{N}}$. Then $(\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}}))^c = \mathbf{1}_{\mathcal{N}}$

$$\implies \mathcal{N}g^{\#}cl(\mathcal{A}_{\mathcal{N}}{}^c) = \mathbf{1}_{\mathcal{N}}$$
$$\implies \mathcal{N}g^{\#}int(\mathcal{N}g^{\#}cl(\mathcal{A}_{\mathcal{N}}{}^c)) = \mathcal{N}g^{\#}int(\mathbf{1}_{\mathcal{N}}) = \mathbf{1}_{\mathcal{N}}$$
$$\implies \mathcal{N}g^{\#}int((\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}}))^c) = \mathbf{1}_{\mathcal{N}}$$
$$\implies (\mathcal{N}g^{\#}cl(\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}})))^c = \mathbf{1}_{\mathcal{N}}$$
$$\implies \mathcal{N}g^{\#}cl(\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}})) = \mathbf{0}_{\mathcal{N}}$$

This is contradiction to $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ being a strongly $\mathcal{N}g^{\#}-$ irresolvable space. Hence our assumption that $\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}}) = \mathbf{0}_{\mathcal{N}}$ is wrong. Hence we must have $\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}}) \neq \mathbf{0}_{\mathcal{N}}$ for all $\mathcal{N}g^{\#}-$ dense sets $\mathcal{A}_{\mathcal{N}}$ in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$. Therefore, $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is $\mathcal{N}g^{\#}-$ irresolvable space. □

Remark 4.7. The reverse implication of the above theorem is not true that is shown in the following example.

Example 4.8. Let $\mathcal{X}_{\mathcal{N}} = \{p, q\}$. Consider the Neutrosophic sets $\mathcal{M}_{\mathcal{N}_1} = \langle(0.4, 0.5, 0.6), (0.4, 0.3, 0.5)\rangle$, $\mathcal{M}_{\mathcal{N}_2} = \langle(0.6, 0.5, 0.4), (0.5, 0.7, 0.4)\rangle$, $\mathcal{M}_{\mathcal{N}_3} = \langle(0.6, 0.5, 0.3), (0.6, 0.7, 0.3)\rangle$, $\mathcal{M}_{\mathcal{N}_4} = \langle(0.3, 0.5, 0.6), (0.3, 0.3, 0.6)\rangle$, $\mathcal{M}_{\mathcal{N}_5} = \langle(0.7, 0.5, 0.4), (0.6, 0.8, 0.4)\rangle$, $\mathcal{M}_{\mathcal{N}_6} = \langle(0.4, 0.5, 0.7), (0.4, 0.2, 0.6)\rangle$. Now, $(\mathcal{X}_{\mathcal{N}}, \tau_N) = \{\mathbf{0}_{\mathcal{N}}, \mathcal{M}_{\mathcal{N}_1}, \mathcal{M}_{\mathcal{N}_2}, \mathcal{M}_{\mathcal{N}_3}, \mathcal{M}_{\mathcal{N}_4}, \mathcal{M}_{\mathcal{N}_5}, \mathcal{M}_{\mathcal{N}_6}, \mathbf{1}_{\mathcal{N}}\}$ is Neutrosophic topological space. Then, $\tau_N = \{\mathbf{0}_{\mathcal{N}}, \mathcal{M}_{\mathcal{N}_1}, \mathbf{1}_{\mathcal{N}}\}$ is $\mathcal{NT}$ on $\mathcal{X}_{\mathcal{N}}$. Here $\mathcal{N}g^{\#}CS(\mathcal{X}_{\mathcal{N}}) =$

$\{\mathbf{0}_{\mathcal{N}}, \mathcal{M}_{\mathcal{N}_2}, \mathcal{M}_{\mathcal{N}_3}, \mathcal{M}_{\mathcal{N}_5}, \mathbf{1}_{\mathcal{N}}\}$, $\mathcal{N}g^{\#}OS(\mathcal{X}_{\mathcal{N}}) = \{\mathbf{0}_{\mathcal{N}}, \mathcal{M}_{\mathcal{N}_1}, \mathcal{M}_{\mathcal{N}_4}, \mathcal{M}_{\mathcal{N}_6}, \mathbf{1}_{\mathcal{N}}\}$,$\mathcal{N}g^{\#}DS(\mathcal{X}_{\mathcal{N}}) = \{\mathcal{M}_{\mathcal{N}_3}, \mathcal{M}_{\mathcal{N}_5}, \mathbf{1}_{\mathcal{N}}\}$. Here $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is $\mathcal{N}g^{\#}-$ irresolvable space. But $\mathcal{N}g^{\#}cl(\mathcal{N}g^{\#}int(\mathcal{M}_{\mathcal{N}_3})) \neq \mathbf{1}_{\mathcal{N}}$ and $\mathcal{N}g^{\#}cl(\mathcal{N}g^{\#}int(\mathcal{M}_{\mathcal{N}_5})) \neq \mathbf{1}_{\mathcal{N}}$. Therefore, $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is not strongly $\mathcal{N}g^{\#}-$ irresolvable space.

Definition 4.9. A Neutrosophic topological space $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is called maximal $\mathcal{N}g^{\#}-$ irresolvable if $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is $\mathcal{N}g^{\#}-$ irresolvable and every $\mathcal{N}g^{\#}-$ dense set of $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is $\mathcal{N}g^{\#}-$ open set.

Example

4.10. Let $\mathcal{X}_{\mathcal{N}} = \{p, q\}$. Consider the Neutrosophic sets $\mathcal{M}_{\mathcal{N}_1} = \langle(0.4, 0.4, 0.5), (0.4, 0.5, 0.5)\rangle$, $\mathcal{M}_{\mathcal{N}_2} = \langle(0.5, 0.6, 0.4), (0.5, 0.5, 0.4)\rangle$, $\mathcal{M}_{\mathcal{N}_3} = \langle(0.7, 0.7, 0.3), (0.8, 0.8, 0.2)\rangle$, $\mathcal{M}_{\mathcal{N}_4} = \langle(0.3, 0.3, 0.7), (0.2, 0.2, 0.8)\rangle$. Now $(\mathcal{X}_{\mathcal{N}}, \tau_N) = \{\mathbf{0}_{\mathcal{N}}, \mathcal{M}_{\mathcal{N}_1}, \mathcal{M}_{\mathcal{N}_2}, \mathcal{M}_{\mathcal{N}_3}, \mathcal{M}_{\mathcal{N}_4}, \mathbf{1}_{\mathcal{N}}\}$ is Neutrosophic topological space. Then $\tau_N = \{\mathbf{0}_{\mathcal{N}}, \mathcal{M}_{\mathcal{N}_1}, \mathcal{M}_{\mathcal{N}_2}, \mathbf{1}_{\mathcal{N}}\}$ is $\mathcal{NT}$ on $\mathcal{X}_{\mathcal{N}}$. Here, $\mathcal{N}g^{\#}CS(\mathcal{X}_{\mathcal{N}}) = \{\mathbf{0}_{\mathcal{N}}, \mathcal{M}_{\mathcal{N}_2}, \mathcal{M}_{\mathcal{N}_4}, \mathbf{1}_{\mathcal{N}}\}$, $\mathcal{N}g^{\#}OS(\mathcal{X}_{\mathcal{N}}) = \{\mathbf{0}_{\mathcal{N}}, \mathcal{M}_{\mathcal{N}_1}, \mathcal{M}_{\mathcal{N}_3}, \mathbf{1}_{\mathcal{N}}\}$, $\mathcal{N}g^{\#}DS(\mathcal{X}_{\mathcal{N}}) = \{\mathcal{M}_{\mathcal{N}_1}, \mathcal{M}_{\mathcal{N}_3}, \mathbf{1}_{\mathcal{N}}\}$. Here, $\mathcal{N}g^{\#}cl(\mathcal{M}_{\mathcal{N}_1}^c) \neq \mathbf{1}_{\mathcal{N}}$ and $\mathcal{N}g^{\#}cl(\mathcal{M}_{\mathcal{N}_3}^c) \neq \mathbf{1}_{\mathcal{N}}$. Therefore, $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is $\mathcal{N}g^{\#}-$ irresolvable space. Also, every $\mathcal{N}g^{\#}-$ dense set in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is $\mathcal{N}g^{\#}OS$. Therefore, $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is a maximal $\mathcal{N}g^{\#}-$ irresolvable space.

Theorem 4.11. *If $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is a maximal $\mathcal{N}g^{\#}-$ irresolvable space then $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is a strongly $\mathcal{N}g^{\#}-$ irresolvable space.*

Proof. Let $\mathcal{A}_{\mathcal{N}}$ be a $\mathcal{N}g^{\#}-$ dense set in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$. Then $\mathcal{N}g^{\#}cl(\mathcal{A}_{\mathcal{N}}) = \mathbf{1}_{\mathcal{N}}$. Since $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is a maximal $\mathcal{N}g^{\#}-$ irresolvable space, $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is $\mathcal{N}g^{\#}-$ irresolvable space and every $\mathcal{N}g^{\#}-$ dense set of $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is $\mathcal{N}g^{\#}-$ open set. Then $\mathcal{A}_{\mathcal{N}}$ is $\mathcal{N}g^{\#}OS$ which implies that $\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}}) = \mathcal{A}_{\mathcal{N}}$. Now $\mathcal{N}g^{\#}cl(\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}})) = \mathcal{N}g^{\#}cl(\mathcal{A}_{\mathcal{N}}) = \mathbf{1}_{\mathcal{N}}$. Hence $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is a strongly $\mathcal{N}g^{\#}-$ irresolvable space. □

Remark 4.12. Converse of the above theorem need not be true.

Example

4.13. Let $\mathcal{X}_{\mathcal{N}} = \{p, q\}$. Consider the Neutrosophic sets $\mathcal{M}_{\mathcal{N}_1} = \langle(0.6, 0.6, 0.3), (0.6, 0.6, 0.3)\rangle$, $\mathcal{M}_{\mathcal{N}_2} = \langle(0.3, 0.4, 0.6), (0.3, 0.4, 0.6)\rangle$, $\mathcal{M}_{\mathcal{N}_3} = \langle(0.3, 0.3, 0.7), (0.3, 0.3, 0.7)\rangle$, $\mathcal{M}_{\mathcal{N}_4} = \langle(0.7, 0.7, 0.3), (0.7, 0.7, 0.3)\rangle$. Now $(\mathcal{X}_{\mathcal{N}}, \tau_N) = \{\mathbf{0}_{\mathcal{N}}, \mathcal{M}_{\mathcal{N}_1}, \mathcal{M}_{\mathcal{N}_2}, \mathcal{M}_{\mathcal{N}_3}, \mathcal{M}_{\mathcal{N}_4}, \mathbf{1}_{\mathcal{N}}\}$ is Neutrosophic topological space. Then $\tau_N = \{\mathbf{0}_{\mathcal{N}}, \mathcal{M}_{\mathcal{N}_1}, \mathbf{1}_{\mathcal{N}}\}$ is $\mathcal{NT}$ on $\mathcal{X}_{\mathcal{N}}$. Here $\mathcal{N}g^{\#}OS(\mathcal{X}_{\mathcal{N}}) = \{\mathbf{0}_{\mathcal{N}}, \mathcal{M}_{\mathcal{N}_1}, \mathbf{1}_{\mathcal{N}}\}$, $\mathcal{N}g^{\#}DS(\mathcal{X}_{\mathcal{N}}) = \{\mathcal{M}_{\mathcal{N}_1}, \mathcal{M}_{\mathcal{N}_4}, \mathbf{1}_{\mathcal{N}}\}$. Then $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is a strongly $\mathcal{N}g^{\#}-$ irresolvable space. Here, $\mathcal{M}_{\mathcal{N}_4}$ is a $\mathcal{N}g^{\#}DS$ in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ but not a $\mathcal{N}g^{\#}OS$. Therefore, $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is not a maximal $\mathcal{N}g^{\#}-$irresolvable space.

Theorem 4.14. *Let $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ be a strongly $\mathcal{N}g^{\#}-$ irresolvable space. Then $\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}}) \subseteq (\mathcal{N}g^{\#}int(\mathcal{B}_{\mathcal{N}}))^c$ for any two $\mathcal{N}g^{\#}-$ dense sets $\mathcal{A}_{\mathcal{N}}, \mathcal{B}_{\mathcal{N}}$ in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$.*

Proof. Let $\mathcal{A}_{\mathcal{N}}$ and $\mathcal{B}_{\mathcal{N}}$ be any two $\mathcal{N}g^{\#}-$ dense sets in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$. Then $\mathcal{N}g^{\#}cl(\mathcal{A}_{\mathcal{N}}) = \mathbf{1}_{\mathcal{N}}$ and $\mathcal{N}g^{\#}cl(\mathcal{B}_{\mathcal{N}}) = \mathbf{1}_{\mathcal{N}}$ which implies that $\mathcal{N}g^{\#}int(\mathcal{N}g^{\#}cl(\mathcal{A}_{\mathcal{N}})) = \mathbf{1}_{\mathcal{N}} \neq \mathbf{0}_{\mathcal{N}}$ and $\mathcal{N}g^{\#}int(\mathcal{N}g^{\#}cl(\mathcal{B}_{\mathcal{N}})) = \mathbf{1}_{\mathcal{N}} \neq \mathbf{0}_{\mathcal{N}}$. Since $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is a strongly $\mathcal{N}g^{\#}-$ irresolvable space, by Theorem **??**, we have $\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}}) \neq \mathbf{0}_{\mathcal{N}}$ and $\mathcal{N}g^{\#}int(\mathcal{B}_{\mathcal{N}}) \neq \mathbf{0}_{\mathcal{N}}$. We know that every strongly $\mathcal{N}g^{\#}-$ irresolvable space is $\mathcal{N}g^{\#}-$ irresolvable space. Therefore, $(\mathcal{X}_{\mathcal{N}}, \tau_N)$ is a $\mathcal{N}g^{\#}-$ irresolvable space. Then by Theorem **??**, we get $\mathcal{A}_{\mathcal{N}} \subseteq \mathcal{B}_{\mathcal{N}}{}^c$. Now $\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}}) \subseteq \mathcal{A}_{\mathcal{N}} \subseteq \mathcal{B}_{\mathcal{N}}{}^c \subseteq (\mathcal{N}g^{\#}int(\mathcal{B}_{\mathcal{N}}))^c$. Therefore, $\mathcal{N}g^{\#}int(\mathcal{A}_{\mathcal{N}}) \subseteq (\mathcal{N}g^{\#}int(\mathcal{B}_{\mathcal{N}}))^c$ for any two $\mathcal{N}g^{\#}-$ dense sets $\mathcal{A}_{\mathcal{N}}, \mathcal{B}_{\mathcal{N}}$ in $(\mathcal{X}_{\mathcal{N}}, \tau_N)$. □

Conclusion:

Our work combines the concepts of topology and neutrosophy, resulting in the development of $\mathcal{N}g^{\#}-$ resolvable and irresolvable spaces. We derived the maximal $\mathcal{N}g^{\#}-$ irresolvable space , a stronger form of $\mathcal{N}g^{\#}-$ irresolvable space which is also $\mathcal{N}g^{\#}-$ irresolvable space with the property that every $\mathcal{N}g^{\#}-$ dense set is $\mathcal{N}g^{\#}-$ open.Further, we proved that a Strongly $\mathcal{N}g^{\#}-$ irresolvable space is open heriditarily irresolvable.This article allows researchers to probe the decomposability of topological spaces and uncover their underlying structure,which promises to reveal new insights and theoretical applications in Neutrosophic topological spaces.

Conflicts of Interest: The authors declare no conflict of interest.

References

[1] Atanassov K. T., Intuitionistic fuzzy sets, Fuzzy Sets and Systems,1986, 20, 87-96.

[2] Babisha Julit R L, Pious Missier S, On Slightly $Ng\#$- Continuous Functions in Neutrosophic Topological Space, International Journal of Mathematics Trends and Technology, 2022, Vol. 68, No. 2, pp. 61-65.

[3] Chang C. L., Fuzzy topological spaces, J.Math.Anal.Appl., **24**,1968, 182- 190.

[4] Dogan Coker, An introduction to intuitionistic fuzzy topological spaces', Fuzzy Sets and Systems, 1997, 88, 81-89.

[5] Dhavaseelan R., Narmada Devi R. and S. Jafari, Characterization Of Neutrosophic Nowhere Dense Sets, International Journal of Mathematical Archive, 2018, 9, 1-5.

[6] Dhavaseelan R. and Jafari S., Generalized Neutrosophic closed sets, New trends in Neutrosophic theory and applications, 2018, 2, 261-273.

[7] Floretin Smarandache, Neutrosophic Set:- A Generalization of Intuitionistic Fuzzy set, Journal of Defense Resources Management, 2010, 1, 107–116.

[8] Florentin Smarandache, New Types of Topologies and Neutrosophic Topologies, Neutrosophic Systems with Applications, vol.1, (2023): pp. 1–3. ($Doi : https : //doi.org/10.5281/zenodo.8166164$)

[9] Jayanthi D., On α Generalized closed sets in Neutrosophic topological spaces, International Conference on Recent Trends in Mathematics and Information Technology, 2018, 88-91.

[10] N.Levine, Generalized closed set in topology, Rend.Circ.Mat Palermo, (1970), 19, 89-96.

[11] Pious Missier S, Babisha Julit R. L,J,Martina Jency., On Quasi $Ng\#$ - Open and Quasi $Ng\#$ - Closed Mappings in Neutrosophic Topological Spaces,International journal of Neutrosophic Science, 2022, Vol. 18, No. 3, pp. 41- 47.

[12] Pious Missier S, Babisha Julit R. L.,On $Ng\#$- Contra Continuous Functions in Neutrosophic Topological Space, Design Engineering, 2021, Vol. 6, pp. 8606-8614.

[13] Pious Missier S, Babisha Julit R. L,J,Martina Jency.,$Ng\#$- Homeomorphism in Neutrosophic Topological Spaces, International Journal of Mechanical Engineering, 2021, Vol. 6, No. 3, pp. 2801- 2805.

[14] Pious Missier S., Babisha Julit R. L., On Neutrosophic $g^{\#}-$ Continuous Functions and Neurtosophic $g^{\#}-$ Irresolute Functions, Abstract Proceedings of 24th FAI-ICDBSMD 2021, Vol. 6(i), 49

[15] Pious Missier S., Babisha Julit R. L., On $Ng^{\#}-$ Interior and $Ng^{\#}-$ Closure in Neutrosophic Topological Space,Conference Proceedings of NCAGT- 2021, 122- 133.

[16] Pious Missier S, Babisha Julit R. L,J,Martina Jency.,On Totally $Ng\#$- Continuous Functions in Neutrosophic Topological Space, IOSR Journal of Mathematics, 2022, Vol. 18, No. 1, pp. 64-68.

[17] Pious Missier S, Babisha Julit R. L,Sucila.E., On Almost $Ng\#$- Contra Continuous Functions,pp. 1- 5, National Seminar on Recent Trends in Graph Theory and its Applications, G. Venkataswamy Naidu College, Kovilpatti, Volume 1, 2021,

[18] Runu Dhar, Compactness and Neutrosophic Topological Space via Grills, Neutrosophic Systems with Applications, vol.2, (2023): pp. 1–7. ($Doi : https://doi.org/10.5281/zenodo.8179373$)

[19] Salama A. A. and Alblowi S. A., Neutrosophic set and Neutrosophic topological spaces, IOSR Jour. of Mathematics, 2012, 31-35.

[20] Salama A. A., Florentin Smarandache and Valeri Kroumov, Neutrosophic Closed set and Neutrosophic Continuous Function, Neutrosophic Sets and Systems, 2014, 4, 4–8.

[21] Sudeep Dey, Gautam Chandra Ray, Separation Axioms in Neutrosophic Topological Spaces, Neutrosophic Systems with Applications, vol.2, (2023): pp. 38–54. ($Doi : https://doi.org/10.5281/zenodo.8195851$)

[22] Wadei Al-Omeri and Saeid Jafari, On Generalized Closed Sets and Generalized Pre-Closed in Neutrosophic Topological Spaces, Mathematics MDPI, 2018, 7, 01-12.

[23] Zadeh L. A., Fuzzy sets, Information and control, 1965, 8, 338-353.

Received: June 27, 2024. Accepted: Oct 14, 2024

NSS
Neutrosophic Sets and Systems, Vol. 75, 2025
University of New Mexico

Modeling Uncertainties Associated with Single-Valued Neutrosophic Multi-Attribute Decision-Making for Performance Evaluation of Risk Investment in Small and Medium-Sized High-Technology Venture Enterprises

Huihui Shi[1], Ting Li[1*], Fan Yang[2], Lei Qiao[1]
[1]Anhui Wenda University of Information Engineering, Hefei, 230000, Anhui, China
[2]Anhui Academy of Governance, Hefei, 230000, Anhui, China
*Corresponding author, E-mail:18715010668@163.com

Abstract: In recent years, a group of rapidly growing technology-based entrepreneurial enterprises have emerged in China, playing an increasingly important role in the national economy. However, small and medium-sized technology startups have low profitability and high risk, making it difficult to obtain sufficient funds from traditional financing channels. At present, research on venture capital evaluation mainly focuses on the evaluation of projects before venture capital, and empirical methods are mostly used to study the promoting effect of venture capital on performance after investment, ignoring the issue of continuous evaluation after investment. The performance evaluation of risk investment in small and medium-sized high-technology venture enterprises is MADM. In such study, the generalized weighted geometric Bonferroni mean (GWGBM) technique is illustrated for MADM under single-valued neutrosophic sets (SVNSs). We begin by constructing the single-valued neutrosophic number GWGBM (SVNNGWGBM) operator, followed by the introduction of the MADM based on this technique. Subsequently, we provide an illustrative example that evaluates the performance of risk investment in small and medium-sized high-technology venture enterprises, supplemented by some comparative analyses to demonstrate the SVNNGWGBM technique. The principal contributions of this research are highlighted: (1) The maximizing deviation method is employed to derive the weight; (2)The GSVNNWBM technique is developed, setting the stage for the introduction of MADM methods based on GSVNNWBM technique; (3) Numerical example is presented that focuses on the performance evaluation of risk investment in small and medium-sized high-technology venture enterprises, along with a comparative decision analysis.

Keywords: Multiple-attribute decision-making (MADM); SVNSs; GWGBM operator; medium-sized high-technology venture enterprises

1. Introduction

In the era of knowledge economy, science and technology have become the decisive force for economic and social development[1]. To remain invincible in fierce international competition, a country must possess strong technological innovation capabilities. To this end, various countries have introduced various policies to support the development of science and technology[2]. China has also pointed out that China should improve its independent innovation capacity, build an innovative country, actively

encourage and guide innovation, and make science and technology a strong support for economic development. In this context, technology startups have become a bridge for transforming science and technology into real productivity, driving continuous technology development, promoting the continuous optimization and upgrading of industrial structure, and creating a large number of employment opportunities, becoming the important driving force for social development. In 2012, Global Entrepreneurship Watch released the "Global Entrepreneurship Watch China Report", which showed that compared to 2010, China's entrepreneurial activity index significantly increased in 2011, ranking first among 54 member states. In recent years, a group of rapidly growing technology-based entrepreneurial enterprises have emerged in China, which have rapidly grown in the short term and become industry leaders[3, 4]. For example, the fourth-generation laser typesetting system of Founder Group directly leapfrogged the second and third-generation technologies, bidding farewell to the era of type printing in China's typesetting and printing industry; Dalian Luming Group has broken the monopoly of the international lighting industry and driven the development of China's semiconductor lighting industry by utilizing its pioneering rare earth light storage self-luminescent materials and luminescent chip technology. The application of flash memory technology by Shenzhen Langke Company gives it an unparalleled competitive advantage in this field. From this, it can be seen that China's technology-based entrepreneurial enterprises are very active in technological innovation activities, which can effectively promote the transformation of scientific and technological achievements [5]. Therefore, the development of technology-based entrepreneurial enterprises has obtained widespread attention from the theoretical community and the government[6]. Technology-based startups are in the entrepreneurial stage, where their main task is to develop new products with low profitability and high technological and market risks, which can easily lead to business failure. Studies conducted abroad reveal that in the United States, a significant 68% of small and medium-sized technology enterprises fail within their first five years of operation. Only 19% manage to survive between 6 to 10 years, and a mere 13% endure beyond the 10-year mark. In contrast, data from China indicate that 44.31% of small and medium-sized enterprises fail within the first 3 years, 25.66% survive for 3 to 5 years, 23.08% persist for 5 to 10 years, and just 6.95% manage to survive longer than a decade [7, 8]. The high-risk characteristics of small and medium-sized technology startups make it difficult for them to raise the necessary funds from traditional financing

channels[9, 10]. In addition to the high-risk characteristics mentioned above, small and medium-sized technology startups also have the characteristic of high returns: once the developed technology is completed and successfully enters the market, they will obtain high profits. This high-risk and high return characteristic precisely meets the requirements of venture capital, and the intervention of venture capital can solve the financial problems in its development [11, 12]. Venture capital, as a new financing method, provides an effective financing channel for small and medium-sized technology startups.

Since the 1950s, with rapid development of science and technology, scientific decision-making has gradually replaced traditional empirical decision-making and played a crucial role in the development of world politics, military, economy, and science and technology[13-16]. Therefore, the difficulty of making scientific decisions gradually increases, and the decision-making results often deviate from expectations. To reduce decision-making bias and improve decision-making level, scientific and effective quantitative decision-making methods are indispensable[17-21]. Decision-making challenges involving the expression of uncertain information have garnered widespread attention globally, as highlighted in numerous studies [22, 23]. The field of uncertain MADM has particularly drawn significant interest from international scholars in recent years [24-26]. Key contributions to this field include Zadeh's introduction of fuzzy sets (FSs) [27], Atanassov's development of intuitionistic fuzzy sets (IFSs) [28], and Ye's proposal of SVNSs [29]. Further advancements were made by Liu and Wang [30] who illustrated the SVNNWBM, and Yager [31] illustrated the GBM specifically for MADM. Zhu et al. [32] illustrated geometric BM (GBM) operator. Xia, Xu and Zhu [33] Illustrated the generalized WBM operator. Despite these theoretical advancements, real-world MADM applications often encounter complex interdependencies among input variables, which are not adequately addressed by previously studied operators [34, 35]. To mitigate this limitation, the GWGBM operator was adopted, noted for its capability to reflect the nuances of data interrelationships [33]. This operator was further tailored to SVNSs, resulting in the development of the GSVNNWBM operator. Based on this new operator, innovative MADM methods were formulated. The practical application of these methods was demonstrated through a case study focused on the performance evaluation of risk investment in small and medium-sized high-technology venture enterprises, supplemented by comparative decision analysis. The primary objectives of this research were clearly illustrated: (1) Utilizing the maximizing deviation method (MDM) to derive

weight information; (2) Developing the GSVNNWBE technique and subsequently proposing MADM based on this enhanced technique; (3) Illustrating these methodologies through a detailed case study on risk investment in high-tech ventures, including some comparative decision analysis to highlight the effectiveness of GSVNNWBE technique.

The remainder of this study is illustrated: Section 2 provides a concise review of SVNSs. Section 3 details the construction of GSVNNWBM operator. In Section 4, we develop the MADM that utilize the GSVNNWBM operator. Section 5 presents a practical application, showcasing the performance evaluation of risk investment in small and medium-sized high-technology venture enterprises. Finally, the conclusions are summarized in Section 6.

2. **Preliminaries**

Wang et al. [36] illustrated the SVNSs

Definition 1 [36]. The SVNSs PA in Θ is illustrated:

$$PA=\left\{\left(\theta,PT_A(\theta),PI_A(\theta),PF_A(\theta)\right)\middle|\theta\in\Theta\right\} \tag{1}$$

where $PT_A(\theta),PI_A(\theta),PF_A(\theta)$ is truth-membership, indeterminacy-membership and falsity-membership, $PT_A(\theta),PI_A(\theta),PF_A(\theta)\in[0,1]$, $0\le PT_A(\theta)+PI_A(\theta)+PF_A(\theta)\le 3$.

The SVNN is illustrated as $PA=(PT_A,PI_A,PF_A)$, where $PT_A,PI_A,PF_A\in[0,1]$, and $0\le PT_A+PI_A+PF_A\le 3$.

Definition 2 [37]. Let $PA=(PT_A,PI_A,PF_A)$, the score function is illustrated:

$$SF(PA)=\frac{(2+PT_A-PI_A-PF_A)}{3},\ SF(PA)\in[0,1]. \tag{2}$$

Definition 3[37]. Let $PA=(PT_A,PI_A,PF_A)$, the accuracy function is illustrated:

$$AF(PA)=\frac{PT_A-PF_A+1}{2},\ AF(PA)\in[0,1]\ . \tag{3}$$

Peng et al.[37] illustrated the order structure.

Definition 4[37]. Let $PA=\left(PT_A, PI_A, PF_A\right)$, $PB=\left(PT_B, PI_B, PF_B\right)$, let $SF\left(PA\right)=\frac{\left(2+PT_A-PI_A-PF_A\right)}{3}$ and $SF\left(PB\right)=\frac{\left(2+PT_B-PI_B-PF_B\right)}{3}$, and let $AF\left(PA\right)=\frac{PT_A-PF_A+1}{2}$ and $AF\left(PB\right)=\frac{PT_B-PF_B+1}{2}$, if $SF\left(PA\right)<SF\left(PB\right)$, $PA<PB$; if $SF\left(PA\right)=SF\left(PB\right)$, (1)if $AF\left(PA\right)=AF\left(PB\right)$, $PA=PB$; (2) if $AF\left(PA\right)<AF\left(PB\right)$, $PA<PB$.

Definition 5[36]. Let $PA=\left(PT_A, PI_A, PF_A\right)$ and $PB=\left(PT_B, PI_B, PF_B\right)$, the illustrated operations are:

(1) $PA\oplus PB=\left(PT_A+PT_B-PT_APT_B, PI_API_B, PF_APF_B\right)$;
(2) $PA\otimes PB=\left(PT_APT_B, PI_A+PI_B-PI_API_B, PF_A+PF_B-PF_APF_B\right)$;
(3) $\lambda PA=\left(1-\left(1-PT_A\right)^{\lambda}, \left(PI_A\right)^{\lambda}, \left(PF_A\right)^{\lambda}\right), \lambda>0$;
(4) $\left(PA\right)^{\lambda}=\left(\left(PT_A\right)^{\lambda}, \left(PI_A\right)^{\lambda}, 1-\left(1-PF_A\right)^{\lambda}\right), \lambda>0.$

Definition 6[33]. Let $ps, pt, pr>0$, $\beta_i (i=1,2,3,\ldots,n)$ be positive values along with weight $pw=(pw_1, pw_2,\ldots, pw_n)^T$, $pw_i\in[0,1]$, $\sum_{i=1}^{n} pw_i=1$. The GWGBM technique is illustrated:

$$\begin{aligned}&\mathrm{GWGBM}^{ps,pt,pr>0}(\beta_1,\beta_2,\ldots,\beta_n)\\&=\frac{1}{ps+pt+pr}\prod_{i,j,k=1}^{n}(ps\beta_i+pt\beta_j+pr\beta_k)^{pw_ipw_jpw_k}\end{aligned}\tag{4}$$

3. SVNNGWGBM operator

The SVNNGWGBM technique is illustrated on the GWGBM technique.

Definition 7. Let $ps, pt, pr>0$, $PA_i=(PT_i, PI_i, PF_i)$ be SVNNs with weight $pw_i=(pw_1, pw_2,\ldots, pw_n)^T$, $\sum_{i=1}^{n} pw_i=1$. If

$$\begin{aligned}&SVNNGWGBM_{pw}^{ps,pt,pr}(PA_1,PA_2,\cdots,PA_n)\\&=\frac{1}{ps+pt+pr}\bigotimes_{i,j,k=1}^{n}(psPA_i\oplus ptPA_j\oplus prPA_k)^{pw_ipw_jpw_k}\end{aligned}\tag{5}$$

then $\mathrm{SVNNGWGBM}_{pw}^{ps,pt,pr}$ is called SVNNGWGBM operator.

The Theorem 1 is illustrated.

Theorem 1. Let $ps, pt, pr > 0$ and $PA_i = (PT_i, PI_i, PF_i)(i = 1, 2, \ldots, n)$ be SVNNs. The fused result of SVNNGWGBM is SVNN and

$$\begin{aligned} &SVNNGWGBM_{pw}^{ps,pt,pr}(PA_1, PA_2, \cdots, PA_n) \\ &= \frac{1}{ps+pt+pr} \overset{n}{\underset{i,j,k=1}{\otimes}} (psPA_i \oplus ptPA_j \oplus prPA_k)^{pw_i pw_j pw_k} \\ &= \begin{pmatrix} 1-(1-\prod\limits_{i,j,k=1}^{n}(1-(1-PT_i)^{ps}(1-PT_j)^{pt}(1-PT_k)^{pr})^{pw_i pw_j pw_k})^{1/(ps+pt+pr)}, \\ (1-\prod\limits_{i,j,k=1}^{n}(1-PI_i^{ps}PI_j^{pt}PI_k^{pr})^{pw_i pw_j pw_k})^{1/(ps+pt+pr)}, \\ (1-\prod\limits_{i,j,k=1}^{n}(1-PF_i^{ps}PF_j^{pt}PF_k^{pr})^{pw_i pw_j pw_k})^{1/(ps+pt+pr)}. \end{pmatrix} \end{aligned} \tag{6}$$

Proof: From Definition 5, it could be obtained:

$$\begin{aligned} psPA_i &= (1-(1-PT_i)^{ps}, PI_i^{ps}, PF_i^{ps}), \\ ptPA_j &= (1-(1-PT_j)^{pt}, PI_j^{pt}, PF_j^{pt}), \\ prPA_k &= (1-(1-PT_k)^{pr}, PI_k^{pr}, PF_k^{pr}). \end{aligned} \tag{7}$$

$$\begin{aligned} &psPA_i \oplus ptPA_j \oplus prPA_k \\ &= \begin{pmatrix} 1-(1-PT_i)^{ps}(1-PT_j)^{pt}(1-PT_k)^{pr}, \\ PI_i^{ps}PI_j^{pt}PI_k^{pr}, PF_i^{ps}PF_j^{pt}PF_k^{pr} \end{pmatrix} \end{aligned} \tag{8}$$

Thereafter,

$$\begin{aligned} &\left(psPA_i \oplus ptPA_j \oplus prPA_k\right)^{pw_i pw_j pw_k} \\ &= \begin{pmatrix} (1-(1-PT_i)^{ps}(1-PT_j)^{pt}(1-PT_k)^{pr})^{pw_i pw_j pw_k}, \\ 1-(1-PI_i^{ps}PI_j^{pt}PI_k^{pr})^{pw_i pw_j pw_k}, \\ 1-(1-PF_i^{ps}PF_j^{pt}PF_k^{pr})^{pw_i pw_j pw_k}. \end{pmatrix} \end{aligned} \tag{9}$$

Therefore,

$$\begin{aligned} &\overset{n}{\underset{i,j,k=1}{\otimes}}\left(psPA_i \oplus ptPA_j \oplus prPA_k\right)^{pw_i pw_j pw_k} \\ &= \begin{pmatrix} \prod\limits_{i,j,k=1}^{n}(1-(1-PT_i)^{ps}(1-PT_j)^{pt}(1-PT_k)^{pr})^{pw_i pw_j pw_k}, \\ 1-\prod\limits_{i,j,k=1}^{n}(1-PI_i^{ps}PI_j^{pt}PI_k^{pr})^{pw_i pw_j pw_k}, \\ 1-\prod\limits_{i,j,k=1}^{n}(1-PF_i^{ps}PF_j^{pt}PF_k^{pr})^{pw_i pw_j pw_k}. \end{pmatrix} \end{aligned} \tag{10}$$

Thus,

$$\begin{aligned}&\frac{1}{ps+pt+pr}\overset{n}{\underset{i,j,k=1}{\otimes}}\left(psPA_i \oplus ptPA_j \oplus prPA_k\right)^{pw_i pw_j pw_k}\\&=\begin{pmatrix}1-(1-\prod_{i,j,k=1}^{n}(1-(1-PT_i)^{ps}(1-PT_j)^{pt}(1-PT_k)^{pr})^{pw_i pw_j pw_k})^{1/(ps+pt+pr)},\\(1-\prod_{i,j,k=1}^{n}(1-PI_i^{ps}PI_j^{pt}PI_k^{pr})^{pw_i pw_j pw_k})^{1/(ps+pt+pr)},\\(1-\prod_{i,j,k=1}^{n}(1-PF_i^{ps}PF_j^{pt}PF_k^{pr})^{pw_i pw_j pw_k})^{1/(ps+pt+pr)}.\end{pmatrix}\end{aligned}\tag{11}$$

Thus, (6) is obtained.

Then,

$$\begin{aligned}&0\le 1-(1-\prod_{i,j,k=1}^{n}(1-(1-PT_i)^{ps}(1-PT_j)^{pt}(1-PT_k)^{pr})^{pw_i pw_j pw_k})^{1/(ps+pt+pr)}\le 1,\\&0\le (1-\prod_{i,j,k=1}^{n}(1-PI_i^{ps}PI_j^{pt}PI_k^{pr})^{pw_i pw_j pw_k})^{1/(ps+pt+pr)}\le 1,\\&0\le (1-\prod_{i,j,k=1}^{n}(1-PF_i^{ps}PF_j^{pt}PF_k^{pr})^{pw_i pw_j pw_k})^{1/(ps+pt+pr)}\le 1.\end{aligned}\tag{12}$$

Therefore,

$$\begin{aligned}0\le{}&1-(1-\prod_{i,j,k=1}^{n}(1-(1-PT_i)^{ps}(1-PT_j)^{pt}(1-PT_k)^{pr})^{pw_i pw_j pw_k})^{1/(ps+pt+pr)}\\&+(1-\prod_{i,j,k=1}^{n}(1-PI_i^{ps}PI_j^{pt}PI_k^{pr})^{pw_i pw_j pw_k})^{1/(ps+pt+pr)}\\&+(1-\prod_{i,j,k=1}^{n}(1-PF_i^{ps}PF_j^{pt}PF_k^{pr})^{pw_i pw_j pw_k})^{1/(ps+pt+pr)}\le 3.\end{aligned}\tag{13}$$

Thereby finishing the proof.

The SVNNGWGBM technique has some properties.

Property 4. Let $ps, pt, pr > 0$ and $PA_i = (PT_i, PI_i, PF_i)(i = 1, 2, \ldots, n)$ be a set of SVNNs.

(1) (Idempotency). If $PA_i = (PT_i, PI_i, PF_i)(i = 1, 2, \ldots, n)$ are equal, that is, $PA_i = PA = (PT, PI, PF)$, then

$$SVNNGWGBM_{pw}^{ps,pt,pr}(PA_1, PA_2, \cdots, PA_n) = PA \tag{14}$$

(2) (Monotonicity). Let $PA_i = (PT_{A_i}, PI_{A_i}, PF_{A_i})(i = 1, 2, 3, \ldots, n)$ and $PB_i = (PT_{B_i}, PI_{B_i}, PF_{B_i})$ $(i = 1, 2, 3, \ldots, n)$ be two sets of SVNNs. If $PT_{A_i} \le PT_{B_i}$, $PI_{A_i} \ge PI_{B_i}$, $PF_{A_i} \ge PF_{B_i}$ holds for

all i, then

$$SVNNGWGBM_{pw}^{ps,pt,pr}(PA_1, PA_2, \cdots, PA_n) \leq SVNNGWGBM_{pw}^{ps,pt,pr}(PB_1, PB_2, \cdots, PB_n) \tag{15}$$

(3) (Boundedness). Let $PA_i = (PT_{A_i}, PI_{A_i}, PF_{A_i})(i = 1, 2, 3, \ldots, n)$ be SVNNS. If $PA^+ = (\max_i(PT_i), \min_i(PI_i), \min_i(PF_i))$ and $PA^- = (\min_i(PT_i), \max_i(PI_i), \max_i(PF_i))$ then

$$PA^- \leq SVNNGWGBM_{pw}^{ps,pt,pr}(PA_1, PA_2, \cdots, PA_n) \leq PA^+. \tag{16}$$

4. MADM technique based on SVNNGWGBM technique under SVNNs

The SVNNGWGBM technique is illustrated for MADM for SVNSs. Let $PX = \{PX_1, PX_2, \ldots, PX_n\}$ be attributes, $pw = \{pw_1, pw_2, \ldots, pw_n\}$ be weight of PX_j. Let $PA = \{PA_1, PA_2, \ldots, PA_m\}$ be alternatives. $PP = (pp_{ij})_{m\times n} = (PT_{ij}, PI_{ij}, PF_{ij})_{m\times n}$ is SVNN-matrix.

The SVNNGWGBM technique for MADM is illustrated (See Figure 1).

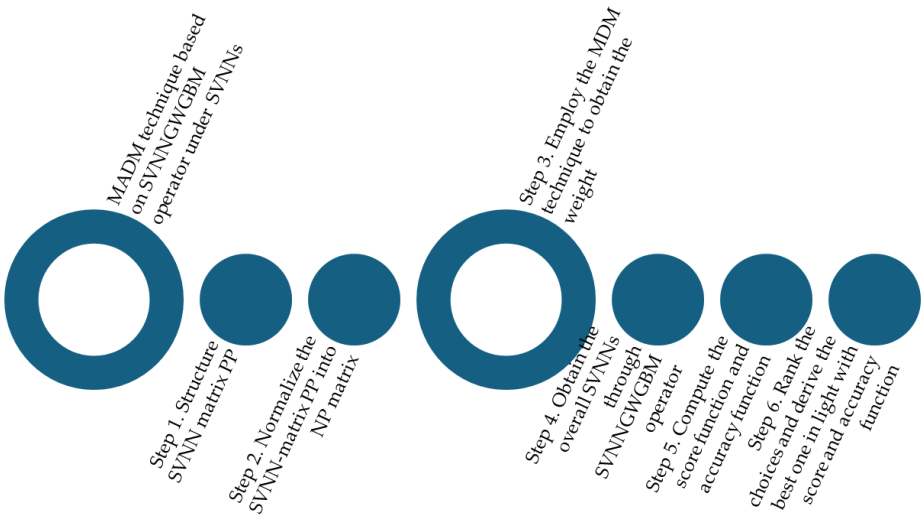

MADM technique based on SVNNGWGBM operator under SVNNs
Step 1. Structure SVNN matrix PP
Step 2. Normalize the SVNN-matrix PP into NP matrix
Step 3. Employ the MDM technique to obtain the weight
Step 4. Obtain the overall SVNNs through SVNNGWGBM operator
Step 5. Compute the score function and accuracy function
Step 6. Rank the choices and derive the best one in light with score and accuracy function

Figure 1. MADM technique based on SVNNGWGBM operator under SVNNs

Step 1. Illustrate SVNN matrix $PP = (pp_{ij})_{m\times n} = (PT_{ij}, PI_{ij}, PF_{ij})_{m\times n}$.

$$PP=\left(PP_{ij}\right)_{m\times n}=\begin{bmatrix} PP_{11} & PP_{12} & \cdots & PP_{1n} \\ PP_{21} & PP_{22} & \cdots & PP_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ PP_{m1} & PP_{m2} & \cdots & PP_{mn} \end{bmatrix}$$

$$PP=\left(PP_{ij}\right)_{m\times n}=\left(PT_{ij},PI_{ij},PF_{ij}\right)_{m\times n} \tag{17}$$

$$=\begin{bmatrix} \left(PT_{11},PI_{11},PF_{11}\right) & \left(PT_{12},PI_{1},PF_{12}\right) & \cdots & \left(PT_{1n},PI_{1n},PF_{1n}\right) \\ \left(PT_{21},PI_{21},PF_{21}\right) & \left(PT_{22},PI_{22},PF_{22}\right) & \cdots & \left(PT_{2n},PI_{2n},PF_{2n}\right) \\ \vdots & \vdots & \vdots & \vdots \\ \left(PT_{m1},PI_{m1},PF_{m1}\right) & \left(PT_{m2},PI_{m2},PF_{m2}\right) & \cdots & \left(PT_{mn},PI_{mn},PF_{mn}\right) \end{bmatrix}$$

Step 2. Normalize the $PP=\left[PP_{ij}\right]_{m\times n}$ to $NP=\left[NP_{ij}\right]_{m\times n}$.

$$\begin{aligned} NP_{ij} &=\left(NPT_{ij},NPI_{ij},NPF_{ij}\right) \\ &=\begin{cases} \left(PT_{ij},PI_{ij},PF_{ij}\right), & PX_j \text{ is benefit attribute} \\ \left(PF_{ij},1-PI_{ij},PT_{ij}\right), & PX_j \text{ is benefit attribute} \end{cases} \end{aligned} \tag{18}$$

Step 3. Employ the maximizing deviation method (MDM) to ascertain the weights.

The MDM [38] under SVNSs to ascertain the weight.

(1) Depending on $NP=\left[NP_{ij}\right]_{m\times n}$, the deviation degree PA_i to others is illustrated.

$$SVNNDD_{ij}=\sum_{t=1}^{m}pw_j\cdot\left|AF\left(NP_{ij}\right)-AF\left(NP_{tj}\right)\right| \tag{19}$$

(2) Illustrate the total weighted deviation.

$$\begin{aligned} SVNNDD_j\left(pw\right) &=\sum_{i=1}^{m}SVNNDD_{ij}\left(pw\right) \\ &=\sum_{i=1}^{m}\sum_{t=1}^{m}pw_j\left|AF\left(NP_{ij}\right)-AF\left(NP_{tj}\right)\right| \end{aligned} \tag{20}$$

(3) Illustrate non-linear programming technique.

$$\left(M-1\right)\begin{cases} \max SVNNDD\left(pw\right)=\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{t=1}^{m}pw_j\left|AF\left(NP_{ij}\right)-AF\left(NP_{tj}\right)\right| \\ s.t.\ \ \mathrm{pw}_j\geq 0,\ \ \sum_{j=1}^{n}pw_j^2=1 \end{cases} \tag{21}$$

The Lagrange decision function is illustrated for this model.

$$LF\left(pw,\xi\right)=\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{t=1}^{m}pw_j\left|AF\left(NP_{ij}\right)-AF\left(NP_{tj}\right)\right|+\frac{\xi}{2}\left(\sum_{j=1}^{n}pw_j^2-1\right)$$

where ξ is Lagrange multiplier. The given partial derivatives are illustrated.

$$\begin{cases} \dfrac{\partial LF}{\partial pw_j} = \sum_{i=1}^{m}\sum_{t=1}^{m}\left|AF\left(NP_{ij}\right)-AF\left(NP_{tj}\right)\right|+\xi pw_j = 0 \\ \dfrac{\partial LF}{\partial \xi} = \dfrac{1}{2}\left(\sum_{j=1}^{n} pw_j^2 - 1\right) = 0 \end{cases}$$

The weight is illustrated: $pw_j^* = \dfrac{\sum_{i=1}^{m}\sum_{t=1}^{m}\left|AF\left(NP_{ij}\right)-AF\left(NP_{tj}\right)\right|}{\sqrt{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{t=1}^{m}\left|AF\left(NP_{ij}\right)-AF\left(NP_{tj}\right)\right|\right)^2}}$

Finally, the normalized weights are:

$$pw_j = \frac{\sum_{i=1}^{m}\sum_{t=1}^{m} AF\left(NP_{ij}\right)-AF\left(NP_{tj}\right)}{\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{t=1}^{m} AF\left(NP_{ij}\right)-AF\left(NP_{tj}\right)} \tag{22}$$

Step 4. In light with $NP=\left[NP_{ij}\right]_{m\times n}$, we could fuse the $NP=\left[NP_{ij}\right]_{m\times n}$ through SVNNGWGBM technique to obtain the SVNN $NP_i=\left(NPT_i, NPI_i, NPF_i\right)$:

$$\begin{aligned} NP_i &= \left(NPT_i, NPI_i, NPF_i\right) \\ &= SVNNGWGBM_{pw}^{ps,pt,pr}\left(NP_{i1}, NP_{i2}, \cdots, NP_{in}\right) \\ &= \frac{1}{ps+pt+pr}\overset{n}{\underset{i,j,k=1}{\otimes}}\left(psNP_{if}\oplus ptNP_{ij}\oplus prNP_{ik}\right)^{pw_i pw_j pw_k} \\ &= \begin{pmatrix} 1-\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-NPT_{if}\right)^{ps}\left(1-NPT_{ij}\right)^{pt}\left(1-NPT_{ik}\right)^{pr}\right)^{pw_i pw_j pw_k}\right)^{1/(ps+pt+pr)}, \\ \left(1-\prod_{i,j,k=1}^{n}\left(1-NPI_{if}^{ps}NPI_{ij}^{pt}NPI_{ik}^{pr}\right)^{pw_i pw_j pw_k}\right)^{1/(ps+pt+pr)}, \\ \left(1-\prod_{i,j,k=1}^{n}\left(1-NPF_{if}^{ps}NPF_{ij}^{pt}NPF_{ik}^{pr}\right)^{pw_i pw_j pw_k}\right)^{1/(ps+pt+pr)}. \end{pmatrix} \end{aligned} \tag{23}$$

Step 5. Illustrate the $SF\left(NP_i\right), AF\left(NP_i\right)\left(i=1,2,\cdots,m\right)$.

$$SF\left(nq_i\right)=\frac{\left(2+NPT_i-NPI_i-NPF_i\right)}{3}, AF\left(NP_i\right)=\frac{NPT_i-NPF_i+1}{2} \tag{24}$$

Step 6. Rank the $PA_i\left(i=1,2,\cdots,m\right)$ through $SF\left(NP_i\right), AF\left(NP_i\right)\left(i=1,2,\cdots,m\right)$.

5. Case Study and Comparative Evaluation

In this section, we will present a case study followed by a comparative evaluation to explore the key findings and implications of our research. By analyzing specific examples, we aim to highlight differences and similarities that can inform us of our understanding of the subject matter.

5.1 Case Study

In recent years, scholars have believed that the sustained growth of enterprises is related to their entrepreneurial behavior, such as innovation, initiative, and risk-taking. That is to say, enterprises should support new concepts, actively take action, and create competitive advantages. With the foresight of the enterprise, proactive enterprises can develop emerging opportunities. However, not all small and medium-sized enterprises can quickly identify opportunities and take immediate action under high market uncertainty, which means that the process of entrepreneurs discovering opportunities has costs and is not an "epiphany". Therefore, whether start-up small and medium-sized enterprises can overcome the danger period or achieve sustained and stable growth, and truly transform into growth performance, the identification and development of opportunities are key. The proposal of entrepreneurial orientation precisely solves this problem. Entrepreneurship orientation is a fusion of innovation, initiative, and risk-taking, which is more reflected in the entrepreneurial attitude and willingness of entrepreneurs. It is a mechanism for entrepreneurship and innovation, so the foresight or entrepreneurial spirit of entrepreneurs plays a greater guiding role throughout the entire process. Entrepreneurial orientation is not simply equivalent to behavioral orientation with the fundamental goal of creating a new enterprise. It also encompasses the understanding of new entry opportunities and the attitude and behavioral tendencies towards taking action that enterprises exhibit during their successful entrepreneurial process in complex and uncertain environments. It is a strategy of achieving success through targeted actions. Therefore, from the perspective of entrepreneurial orientation, resource pooling can enable small and medium-sized enterprises to demonstrate stronger risk resistance and achieve sustained growth. Firstly, entrepreneurship utilizes the resources at hand rather than the resources illustrated through search. Secondly, individuals or organizations that adopt entrepreneurial patchwork do not wait for the "right" resources. They use the resources at hand, take positive actions to address problems, seize opportunities, and do not hesitate about

whether the resources at hand are feasible. They believe that "can" is more important than "should", and this process includes the experimental process of restructuring, restructuring, and resource acquisition. Finally, entrepreneurial patchwork requires examining the resources at hand from different perspectives and perspectives and reconsidering the way resources are utilized, which is a "creative reengineering" behavior. Organizations that are good at entrepreneurial patchwork can identify new attributes of the resources at hand and use them to implement innovation, generating differentiated value from homogeneous resources. Entrepreneurship patchwork is an ability for organizations to achieve sustained competitive advantages. The performance evaluation of risk investment in small and medium-sized high-technology venture enterprises is MADM. In this study, an empirical application of performance evaluation of risk investment in small and medium-sized high-technology venture enterprises is given through SVNNGWGBM method. There are five medium-sized high-technology venture enterprises are evaluated their performance evaluation of risk investment. Five medium-sized high-technology venture enterprises are assessed fairly under four attributes:

PX_1 is the personnel input. The forefront of technology is a crucial driver for the growth and continuous innovation of high-tech enterprises. Yet, advancing technologically not only demands substantial capital but also a significant investment in top-notch personnel. When high-tech enterprises secure funding from entrepreneurial venture capital, the venture capitalists often leverage their extensive experience and networks to assist in recruiting key talents essential for the enterprise's development. These key talents include not only scientists and engineers directly involved in technological innovation but also managerial talents who foster an environment conducive to innovation. Additionally, similar to the role investment banks play in underwriting corporate IPOs, entrepreneurial venture capital can act as a "reputation medium" for high-tech enterprises. Before these enterprises receive investment, a significant information asymmetry exists between them and potential employees due to the lack of publicly available information. The news of an investment signals to top talents that the enterprise likely has better success prospects and a more promising future than those without such backing, thus attracting high-quality candidates. This helps mitigate the initial information gaps and positions the enterprise more favorably in the competitive market.

PX_2 is the capital investment. Technological innovation is the cornerstone of growth for high-tech enterprises, which, unlike traditional enterprises, require significantly more funding for research and development. Despite this need, securing funding is often more challenging for high-tech enterprises than for their traditional counterparts. A pervasive shortage of funds acts as a critical barrier to technological innovation across the globe, including in China. This is primarily due to the high-risk nature and lack of sufficient collateral assets typical of high-tech enterprises, making traditional financing avenues like bank loans less viable. However, entrepreneurial venture capital, which inherently seeks out high-risk, high-return opportunities, aligns well with the financial needs of high-tech enterprises. This type of funding not only addresses the difficulties these enterprises face in accessing traditional financial support but also becomes a principal mode of financing for their technological innovations. Moreover, venture capital does more than just supply direct financial assistance; it also facilitates access to additional external funds. Venture capitalists are well-informed and experienced, and their investment in a company often serves as a testament to the company's quality. Consequently, the reputational capital of venture capitalists sends a strong, positive signal to other potential investors. In essence, entrepreneurial venture capital helps high-tech enterprises overcome traditional financing barriers and expand their funding avenues, playing a crucial role in their ongoing innovation and development.

PX_3 is the sales revenue and profit. The primary objective of high-tech enterprises in technological innovation is to commercialize and industrialize their innovations, while also enabling entrepreneurial venture capitalists to achieve substantial capital returns through these commercial activities. Consequently, the sales revenue and profits generated by high-tech enterprises serve as critical indicators for assessing the impact of entrepreneurial venture capital on their technological advancements. Entrepreneurial venture capitalists bring invaluable experience gained from investing in various enterprises, offering crucial insights and guidance that are often lacking within high-tech enterprises. For instance, these investors leverage their extensive industry-specific investment experience to preemptively address potential challenges, thereby reducing the likelihood of market failures for new products. Moreover, venture capitalists facilitate connections between high-tech companies and potential customers and suppliers through their extensive social networks and portfolio companies. Additionally, the enhanced reputation

that comes with association with entrepreneurial venture capital can broaden a high-tech company's network of potential customers and suppliers, further boosting its market presence.

PX_4 is the innovation capability. Technology forms the bedrock of small and medium-sized technology startups. Venture capital plays a crucial role in supporting and enhancing their capacity for technological innovation. Consequently, when evaluating the performance of these startups, it is essential to specifically assess their capabilities in technological innovation.

Clearly, PX_2 represents the cost, while the other elements are considered benefits. Subsequently, the SVNNGWGBM method is utilized in MADM to assess the performance of risk investment in small and medium-sized high-technology venture enterprises using SVNNs. The decision-making process using the SVNNGWGBM method includes the following steps:

Step 1. Illustrate the SVNN-matrix $PP = \left(PP_{ij} \right)_{5\times4}$ (Table 1).

Table 1. SVNN matrix

	PX_1	PX_2	PX_3	PX_4
PA_1	(0.39, 0.28, 0.37)	(0.29, 0.27, 0.28)	(0.32, 0.55, 0.34)	(0.25, 0.32, 0.43)
PA_2	(0.36, 0.24, 0.51)	(0.54, 0.26, 0.24)	(0.39, 0.34, 0.63)	(0.37, 0.43, 0.38)
PA_3	(0.59, 0.36, 0.53)	(0.54, 0.27, 0.68)	(0.32, 0.28, 0.56)	(0.56, 0.27, 0.46)
PA_4	(0.52, 0.23, 0.59)	(0.58, 0.37, 0.35)	(0.53, 0.45, 0.36)	(0.58, 0.36, 0.61)
PA_5	(0.51, 0.26, 0.53)	(0.56, 0.29, 0.54)	(0.57, 0.24, 0.53)	(0.39, 0.43, 0.56)

Step 2. Normalize $PP = \left(PP_{ij} \right)_{5\times4}$ to $NP = \left(NP_{ij} \right)_{5\times4}$ (See the Table 2).

Table 2. The NP matrix

	PX_1	PX_2	PX_3	PX_4
PA_1	(0.39, 0.28, 0.37)	(0.28, 0.73, 0.29)	(0.32, 0.55, 0.34)	(0.25, 0.32, 0.43)
PA_2	(0.36, 0.24, 0.51)	(0.24, 0.74, 0.54)	(0.39, 0.34, 0.63)	(0.37, 0.43, 0.38)
PA_3	(0.59, 0.36, 0.53)	(0.68, 0.73, 0.54)	(0.32, 0.28, 0.56)	(0.56, 0.27, 0.46)
PA_4	(0.52, 0.23, 0.59)	(0.35, 0.63, 0.58)	(0.53, 0.45, 0.36)	(0.58, 0.36, 0.61)

PA_5	(0.51, 0.26, 0.53)	(0.54, 0.71, 0.56)	(0.57, 0.24, 0.53)	(0.39, 0.43, 0.56)

Step 3. Illustrate the weights (Table 3).

Table 3. The weight information

	PX_1	PX_2	PX_3	PX_4
pw_j	0.2477	0.1796	0.2548	0.3179

Step 4. Obtain the $NP_i\left(i=1,2,\cdots,5\right)$ through utilizing SVNNGWGBM technique (Table 4).

Table 4. The $NP_i\left(i=1,2,3,\cdots,5\right)$ by SVNNGWGBM technique ($s=t=r=1$)

Techniques	$NP_i\left(i=1,2,\cdots,5\right)$
PA_1	(0.5223,0.4796,0.3453)
PA_2	(0.6976,0.2957,0.3889)
PA_3	(0.5637,0.6493,0.3199)
PA_4	(0.5865,0.5728,0.3326)
PA_5	(0.5875,0.6457,0.1847)

Step 5. Obtain the $SV\left(NP_i\right)\left(i=1,2,3,\cdots,5\right)$ (Table 5).

Table 5. $SV\left(NP_i\right)\left(i=1,2,3,\cdots,5\right)$

Techniques	$SV\left(NP_i\right)\left(i=1,2,\cdots,5\right)$	Order
PA_1	0.5850	5
PA_2	0.7374	1

PA_3	0.6170	4
PA_4	0.6293	3
PA_5	0.6441	2

Step 6. From Table 5, the order is illustrated: $PA_2 > PA_5 > PA_4 > PA_3 > PA_1$, and the optimal small and medium-sized high-technology venture enterprise is PA_2.

5.2. Influence analysis

To demonstrate how varying decision parameters of the SVNNGWGBM method influence the ranking outcomes, the results are presented in Tables 6 and 7.

Table 6. Different parameters for SVNNGWGBM

(s,t,r)	$SF(PA_1)$	$SF(PA_2)$	$SF(PA_3)$	$SF(PA_4)$	$SF(PA_5)$
(1,1,1)	0.5850	0.7374	0.6170	0.6293	0.6441
(2,2,2)	0.8287	0.9320	0.8595	0.8551	0.8600
(3,3,3)	0.8957	0.9677	0.9213	0.9022	0.9245
(4,4,4)	0.9213	0.9775	0.9447	0.9165	0.9523
(5,5,5)	0.9337	0.9813	0.9567	0.9224	0.9674
(6,6,6)	0.9410	0.9833	0.9644	0.9256	0.9766
(7,7,7)	0.9457	0.9847	0.9697	0.9277	0.9828
(8,8,8)	0.9492	0.9859	0.9739	0.9295	0.9873
(9,9,9)	0.9517	0.9870	0.9773	0.9309	0.9907
(10,10,10)	0.9539	0.9881	0.9801	0.9321	0.9934

Table 7. Order for SVNNGWGBM

(s,t,r)	Order
(1,1,1)	$PA_2 > PA_5 > PA_4 > PA_3 > PA_1$
(2,2,2)	$PA_2 > PA_5 > PA_3 > PA_4 > PA_1$
(3,3,3)	$PA_2 > PA_5 > PA_3 > PA_4 > PA_1$
(4,4,4)	$PA_2 > PA_5 > PA_3 > PA_1 > PA_4$
(5,5,5)	$PA_2 > PA_5 > PA_3 > PA_1 > PA_4$
(6,6,6)	$PA_2 > PA_5 > PA_3 > PA_1 > PA_4$
(7,7,7)	$PA_2 > PA_5 > PA_3 > PA_1 > PA_4$
(8,8,8)	$PA_2 > PA_5 > PA_3 > PA_1 > PA_4$
(9,9,9)	$PA_2 > PA_5 > PA_3 > PA_1 > PA_4$
(10,10,10)	$PA_2 > PA_5 > PA_3 > PA_1 > PA_4$

The information presented in Tables 6 and 7 illustrates that the prioritization of advantages and disadvantages shifts slightly when different parameter values are applied in the SVNNGWGBM process. This variability underscores the flexibility of parameter selection, which can be adjusted to align with the decision-maker's subjective perspective. On one hand, DMs can manipulate these parameter values to generate varying decision outcomes. This ability to modify results based on parameter adjustments allows decision-makers to explore different scenarios and their potential impacts thoroughly. On the other hand, the choice of parameter also serves as an indicator of the decision-maker's risk tolerance. By selecting specific parameters, decision-makers can express their propensity for risk, whether they are risk-averse, risk-neutral, or risk-seeking. This aspect of parameter selection not only influences the decision outcome but also provides insight into the decision-maker's strategic approach and risk management preferences.

5.3 Comparative analysis

The SVNNGWGBM approach is compared with SVNNWA [37] , SVNNWG [37], SVNN-CODAS [39], SVNN-WASPAS [40], SVNN-TOPSIS [41] and SVNN-EDAS [42]. The final results are recorded in Table 8.

Table 8. Results of different techniques

Techniques	Order
SVNNWA technique [37]	$PA_2 > PA_5 > PA_4 > PA_3 > PA_1$
SVNNWG technique [37]	$PA_2 > PA_5 > PA_3 > PA_4 > PA_1$
SVNN-CODAS technique [39]	$PA_2 > PA_5 > PA_4 > PA_3 > PA_1$
SVNN-WASPAS technique [40]	$PA_2 > PA_5 > PA_4 > PA_3 > PA_1$
SVNN-TOPSIS technique [41]	$PA_2 > PA_5 > PA_4 > PA_3 > PA_1$
SVNN-EDAS technique [42]	$PA_2 > PA_5 > PA_3 > PA_4 > PA_1$
The proposed SVNNGWGBM technique	$PA_2 > PA_5 > PA_4 > PA_3 > PA_1$

The above analysis indicates that the ranking outcomes derived from the proposed SVNNGWGBM technique are consistent with those obtained using the SVNNWA technique [37] , SVNN-CODAS technique [39], SVNN-WASPAS [40] and SVNN-TOPSIS [41]. However, there are subtle differences when compared to the results from the SVNNWG technique [37]. The major reason for these differences is that the SVNNWG technique [37] emphasizes the influence of individual criteria, whereas the SVNN-EDAS technique [42] focuses on measuring distances from the average solution. Upon comparing the decision results of SVNNGWGBM technique [42] with those from the SVNNWA, SVNNWG, SVNN-CODAS, SVNN-WASPAS, SVNN-TOPSIS and SVNN-EDAS, it is observed that although there are minor variations, the identification of the best and worst small and medium-sized high-technology venture enterprises remains consistent across methods. This consistency validates the effectiveness of the SVNNGWGBM technique as a relatively straightforward and reliable option for MADM.

6. Conclusion

In the era of "innovation and entrepreneurship," technology-based small and medium-sized entrepreneurial enterprises are encountering extraordinary opportunities. However, the high degree of market uncertainty also presents formidable challenges for their sustained growth. Particularly for these

entrepreneurial ventures, the rapid shifts in market conditions and environmental uncertainties pose significant risks. Additionally, technological innovation demands frequent product updates, and the ever-changing customer demands further escalate the market's unpredictability and risk. Confronting this dynamic and complex market environment requires these enterprises to swiftly identify opportunities, creatively overcome resource limitations, proactively establish lasting competitive advantages, enhance resilience, and secure sustainable growth. These challenges are particularly acute for startup technology-based small and medium-sized enterprises. The performance evaluation of risk investment in such enterprises is a classic issue in MADM. In this context, the WBM technique is adapted for addressing MADM challenges with SVNSs. The SVNNGWGBM technique is detailed, and MADM methodologies based on this operator are subsequently introduced. An illustrative example concerning the performance evaluation of risk investment in small and medium-sized high-technology venture enterprises, along with comparative analysis, are provided to verify the efficacy of SVNNGWGBM technique.

Acknowledgment

The work was supported by the Project of Philosophy and Social Science of Scientific Research Planning in Anhui Province under Grant No.NO.2022AH040350.

References

[1] M. Benaroch, Managing information technology investment risk: A real options perspective, Journal of Management Information Systems, 19 (2002) 43-84.

[2] G.J. Zhu, Z.J. Zhang, Z.Z. Peng, The research and assessment of the creation investment risk in the high and new technology enterprise, in: 5th Wuhan International Conference on E-Business, Alfred Univ, Wuhan, PEOPLES R CHINA, 2006, pp. 1692-1696.

[3] M. Benaroch, M. Jeffery, R.J. Kauffman, S. Shah, Option-based risk management: A field study of sequential information technology investment decisions, Journal of Management Information Systems, 24 (2007) 103-140.

[4] S. Dewan, C. Shi, V. Gurbaxani, Investigating the risk-return relationship of information technology investment: Firm-level empirical analysis, Management Science, 53 (2007) 1829-1842.

[5] S. Otim, K.E. Dow, V. Grover, J.A. Wong, The impact of information technology investments on downside risk of the firm: Alternative measurement of the business value of it, Journal of Management Information Systems, 29 (2012) 159-193.

[6] E. Garnier, R. Madlener, The influence of policy regime risks on investments in innovative energy technology, Energy Journal, 37 (2016) 145-160.

[7] C. Mayer, P. Breun, F. Schultmann, Considering risks in early stage investment planning for emission abatement technologies in large combustion plants, Journal of Cleaner Production, 142 (2017) 133-144.

[8] A. Golub, O. Lugovoy, V. Potashnikov, Quantifying barriers to decarbonization of the russian economy: Real

options analysis of investment risks in low-carbon technologies, Climate Policy, 19 (2019) 716-724.

[9] E. Omri, N. Chtourou, D. Bazin, Risk management and policy implications for concentrating solar power technology investments in tunisia, Journal of Environmental Management, 237 (2019) 504-518.

[10] Y. Shao, Y.C. Hu, V.M. Zavala, Mitigating investment risk using modular technologies, Computers & Chemical Engineering, 153 (2021) 12.

[11] M. Sharp, M. Dadfarnia, T. Sprock, D. Thomas, Procedural guide for system-level impact evaluation of industrial artificial intelligence-driven technologies: Application to risk-based investment analysis for condition monitoring systems in manufacturing, Journal of Manufacturing Science and Engineering-Transactions of the Asme, 144 (2022) 14.

[12] T. To, M. Heleno, A. Valenzuela, Risk-constrained multi-period investment model for distributed energy resources considering technology costs and regulatory uncertainties, Applied Energy, 319 (2022) 11.

[13] Y.G. Xue, Y. Deng, H. Garg, Uncertain database retrieval with measure - based belief function attribute values under intuitionistic fuzzy set, Information Sciences, 546 (2021) 436-447.

[14] Z.L. Yang, H. Garg, J.Q. Li, G. Srivastava, Z.H. Cao, Investigation of multiple heterogeneous relationships using a q-rung orthopair fuzzy multi-criteria decision algorithm, Neural Computing & Applications, 33 (2021) 10771-10786.

[15] U. Cali, M. Deveci, S.S. Saha, U. Halden, F. Smarandache, Prioritizing energy blockchain use cases using type-2 neutrosophic number-based edas, Ieee Access, 10 (2022) 34260-34276.

[16] X.D. Peng, F. Smarandache, A decision-making framework for china's rare earth industry security evaluation by neutrosophic soft cocoso method, Journal of Intelligent & Fuzzy Systems, 39 (2020) 7571-7585.

[17] M.R. Hashmi, M. Riaz, F. Smarandache, M-polar neutrosophic topology with applications to multi-criteria decision-making in medical diagnosis and clustering analysis, International Journal of Fuzzy Systems, 22 (2020) 273-292.

[18] X.D. Peng, F. Smarandache, New multiparametric similarity measure for neutrosophic set with big data industry evaluation, Artificial Intelligence Review, 53 (2020) 3089-3125.

[19] W. Yang, L.L. Cai, S.A. Edalatpanah, F. Smarandache, Triangular single valued neutrosophic data envelopment analysis: Application to hospital performance measurement, Symmetry-Basel, 12 (2020) 14.

[20] F. Ahmad, A.Y. Adhami, F. Smarandache, Neutrosophic optimization model and computational algorithm for optimal shale gas water management under uncertainty, Symmetry-Basel, 11 (2019) 34.

[21] Q. Khan, P.D. Liu, T. Mahmood, F. Smarandache, K. Ullah, Some interval neutrosophic dombi power bonferroni mean operators and their application in multi-attribute decision-making, Symmetry-Basel, 10 (2018) 32.

[22] M. Arshad, M. Saeed, A.U. Rahman, S.A. Bajri, H. Alqahtani, H.A. Khalifa, Modeling uncertainties associated with multi-attribute decision-making based evaluation of cooling system using interval-valued complex intuitionistic fuzzy hypersoft settings, Aims Mathematics, 9 (2024) 11396-11422.

[23] P.D. Liu, Q. Khan, T. Mahmood, Multiple-attribute decision making based on single-valued neutrosophic

schweizer-sklar prioritized aggregation operator, Cognitive Systems Research, 57 (2019) 175-196.

[24] A. Saha, H. Garg, D. Dutta, Generalized dombi and bonferroni mean operators based group decision-making under probabilistic linguistic q-rung orthopair fuzzy environment, International Journal of Intelligent Systems, 36 (2021) 7770-7804.

[25] A.R. Mishra, P. Rani, A. Saha, Single - valued neutrosophic similarity measure - based additive ratio assessment framework for optimal site selection of electric vehicle charging station, International journal of intelligent systems, 36 (2021) 5573-5604.

[26] I.M. Hezam, A.R. Mishra, P. Rani, A. Saha, F. Smarandache, D. Pamucar, An integrated decision support framework using single-valued neutrosophic-maswip-copras for sustainability assessment of bioenergy production technologies, Expert Systems with Applications, 211 (2023) 118674.

[27] L.A. Zadeh, Fuzzy sets, in: Information and Control, 1965, pp. 338-356.

[28] K.T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20 (1986) 87-96.

[29] J. Ye, A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets, Journal of Intelligent & Fuzzy Systems, 26 (2014) 2459-2466.

[30] P.D. Liu, Y.M. Wang, Multiple attribute decision-making method based on single-valued neutrosophic normalized weighted bonferroni mean, Neural Computing & Applications, 25 (2014) 2001-2010.

[31] R.R. Yager, On generalized bonferroni mean operators for multi-criteria aggregation, International Journal of Approximate Reasoning, 50 (2009) 1279-1286.

[32] B. Zhu, Z.S. Xu, M.M. Xia, Hesitant fuzzy geometric bonferroni means, Information Sciences, 205 (2012) 72-85.

[33] M.M. Xia, Z.S. Xu, B. Zhu, Generalized intuitionistic fuzzy bonferroni means, International Journal of Intelligent Systems, 27 (2012) 23-47.

[34] G. Beliakov, S. James, On extending generalized bonferroni means to atanassov orthopairs in decision making contexts, Fuzzy Sets and Systems, 211 (2013) 84-98.

[35] D.J. Yu, Intuitionistic fuzzy geometric heronian mean aggregation operators, Applied Soft Computing, 13 (2013) 1235-1246.

[36] H. Wang, F. Smarandache, Y.Q. Zhang, R. Sunderraman, Single valued neutrosophic sets, Multispace Multistruct, (2010) 410-413.

[37] J.J. Peng, J.Q. Wang, J. Wang, H.Y. Zhang, X.H. Chen, Simplified neutrosophic sets and their applications in multi-criteria group decision-making problems, International Journal of Systems Science, 47 (2016) 2342-2358.

[38] Y. Wang, Using the method of maximizing deviation to make decision for multiindices, Journal of Systems Engineering & Electronics, 8 (1997) 21-26.

[39] E. Bolturk, A. Karasan, Prioritization of investment alternatives for a hospital by using neutrosophic codas method, Journal of Multiple-Valued Logic and Soft Computing, 33 (2019) 381-396.

[40] E.K. Zavadskas, R. Bausys, D. Stanujkic, M. Magdalinovic-Kalinovic, Selection of lead-zinc flotation circuit design by applying waspas method with single-valued neutrosophic set, Acta Montanistica Slovaca, 21 (2016)

85-92.

[41] G. Selvachandran, S.G. Quek, F. Smarandache, S. Broumi, An extended technique for order preference by similarity to an ideal solution (topsis) with maximizing deviation method based on integrated weight measure for single-valued neutrosophic sets, Symmetry-Basel, 10 (2018) 17.

[42] D. Stanujkic, D. Karabasevic, G. Popovic, D. Pamucar, Z. Stevic, E.K. Zavadskas, F. Smarandache, A single-valued neutrosophic extension of the edas method, Axioms, 10 (2021) 13.

Received: June 28, 2024. Accepted: Oct 12, 2024

NSS

Neutrosophic Sets and Systems, Vol. 75, 2025

University of New Mexico

The Theoretical Framework on Approximation of Neutrosophic Numbers and Their Application

Augus Kurian[1], Sumathi I R[2,*] and Omaima Al-Shanqiti[3]

[1,2,*]Department of Mathematics, Amrita School of Physical Sciences, Coimbatore, Amrita Vishwa Vidyapeetham, India [1]auguskurian@gmail.com

[3] Umm Al-Qura University, Department of Mathematics, Mekkah, Saudia Arabia; omshanqiti@uqu.edu.sa

[*]Correspondence: ir_sumathi@cb.amrita.edu

Abstract. Neutrosophic sets are effectual logic represented to understand ambiguous and inconsistent information. They are frequently used to explain many types of partial or incomplete information. Researchers have given much attention to the decision-making theory and its associated methodologies based on uncertain linguistic factors. This article emphasizes the novel neutrosophic number approximations to handle linguistic variables and their application in multiple-attribute decision-making. Different approximation techniques are introduced in neutrosophic sets, but substantial data loss may occur. Hence, a hexagonal neutrosophic number was proposed to deal with information loss during approximation. Also, the comparison study with existing techniques is explored to show the effectiveness of the proposed approximation. The expected interval criterion was retained although an approximation was made to give more desirable features. An MCDM (Multi-Criteria Decision Making) problem is presented to demonstrate efficiency and simplicity with uncertain parameters.

Keywords: Neutrosophic Number; Generalized Nonlinear Hexagonal Neutrosophic Number with asymmetry; Approximation; Expected Interval; Values; MCDM.

1. Introduction

Fuzzy logic enables greater flexibility when dealing with imprecise or uncertain data. Zadeh [1] initiated the fuzzy logic to deal with ambiguity and uncertainty in a flexible and enhanced way of reasoning, which allows the truth value to range between 0 and 1, that helps to model imprecise data in various fields like artificial intelligence, control systems, and decision-making. Fuzzy numbers have been introduced to deal with imprecise numerical quantities in decision analysis, risk assessment, finance, etc., In [2–4] various fuzzy numbers, such as triangular, trapezoidal, pentagonal, and hexagonal, were discussed along with their arithmetic operators.

A new concept of generalized 'n' gonal linear fuzzy numbers, which encompasses triangular, trapezoidal, hexagonal, octagonal, and decagonal fuzzy numbers was introduced in [5], and a novel total ordering method was presented. Initially, interval approximations [6] were provided. Later in [7], an interval approximation operator that preserves this nonspecificity measure is provided, along with an uncertainty measure known as entropy-like nonspecificity. Various forms of approximation of fuzzy numbers were extensively discussed in [8–19]. Later, using the Karush Kuhn Tucker theorem [20–22], LR fuzzy representations and the L fuzzy rough set were more thoroughly analyzed, and their approximations were described.

Compared to traditional fuzzy sets, Intuitionistic fuzzy sets (IFS) [23] provide a more comprehensive representation of uncertainty, which can be helpful in decision-making where a lack of information or conflicting evidence is present. IFS captures not only an element's degree of membership but also its degree of non-membership. Similarly, various Intuitionistic Fuzzy Numbers (IFN) were developed for MCDM, and their approximation operators [24, 25] were determined for multiple polygonal IFNs.

Neutrosophic sets [26, 27] are extensions of fuzzy sets and IFS to address even more complex aspects of uncertainty. The importance of neutrosophic sets lies in their ability to handle both the degrees of truth, falsehood, and indeterminacy. The added dimension allows for better representation in areas where the boundary between membership and non-membership is poorly defined. Similarly, different neutrosophic numbers [28–40] were developed to approach MCDM problems systematically. To systematically deal with decision-making problems, various arithmetic operations are required. The trapezoidal approximation [41] of neutrosophic numbers is defined to deal with transportation problems. Since neutrosophic is an effective tool to represent indeterminacy, and inspired by the numerous aspects of neutrosophic set, in this paper, we describe the approximations of neutrosophic numbers using GNHNNA. In many problem-solving scenarios, particularly those involving uncertainty and imprecision, it can be challenging to find accurate arithmetic operations. Traditional approximation methods frequently give rise to data loss and sometimes insufficient generalization of the (α, β, γ)-cuts. Due to these limitations, we investigated new approximation methods, and the hexagonal neutrosophic approximation method can be a more suitable alternative. An example is given to show the comparison with other existing approximations. We provided various theorems pertaining to the approximation points and their values. The Karush Kuhn Tucker (KKT) theorem and the expected interval criterion are applied in approximation. The result obtained approximates efficiently any linear and nonlinear neutrosophic numbers to generalized hexagonal neutrosophic numbers.

Contributions

- Using Lagrange's method, a novel approach for the approximation of neutrosophic numbers using hexagonal neutrosophic numbers is introduced.
- The paper significantly contributes by employing advanced distance measures and expected interval criteria in the approximation process. This enhances the accuracy and reliability of the neutrosophic number approximation, addressing practical concerns in real-world applications.
- Providing a solid theoretical foundation, the paper establishes and discusses theorems related to the values, ambiguity, and approximation of neutrosophic numbers. This contributes to the understanding of the mathematical principles underlying the proposed approximation method.
- A numerical example is given to show the effectiveness of the proposed method in solving real-world problems. This application demonstrates the applicability and efficiency of the hexagonal neutrosophic number approximation in addressing complex scenarios.
- The paper makes a noteworthy contribution by conducting a comparative analysis, demonstrating that Hexagonal neutrosophic numbers yield the least error compared to other neutrosophic numbers.

The systematic framework of this article is as follows: The preliminaries provide the basic definitions relevant to this paper. In the next section, the distance measure and expected interval are discussed in the case of neutrosophic sets. Then, by satisfying the Karush Kuhn Tucker theorem and the expected interval criterion, the distance measure is minimized. In section 3, the approximation of any neutrosophic number to GNHNNA is explored along with graphical interpretation. As one of the particular cases of the previous section, the approximation of linear hexagonal neutrosophic numbers with symmetry is explored. The comparison with an approximation of various neutrosophic numbers was given to show that the proposed approximation has minimal data loss. In Quantitative Analysis of Decision-Making Research, a theoretical framework was given to deal with MCDM problems effectively. Finally, a decision-making problem was given in the stock market with vague criteria, and the future scope is given in the conclusion.

2. **Preliminaries**

Definition 2.1. [36] Generalized Nonlinear Hexagonal Neutrosophic Numbers with Asymmetry (GNHNNA) is defined as, $A_{GNHNNA} = \Big\{ T\left(a_1, a_2, a_3, a_4, a_5, a_6; r, s; \omega\right)_{(n_1,n_2,n_3,n_4)}, I\left(b_1, b_2, b_3, b_4, b_5, b_6; r_1, s_1; \rho\right)_{(m_1,m_2,m_3,m_4)}, F\left(c_1, c_2, c_3, c_4, c_5, c_6; r_2, s_2; \delta\right)_{(p_1,p_2,p_3,p_4)} \Big\}$.
where the membership function is defined as,

$$T_{A_{GNHNNA}} = \begin{cases} r\left(\frac{x-a_1}{a_2-a_1}\right)^{n_1} & , \text{ if } a_1 \le x \le a_2 \\ r+(\omega-r)\left(\frac{x-a_2}{a_3-a_2}\right)^{n_2} & , \text{ if } a_2 \le x \le a_3 \\ \omega & , \text{ if } a_3 \le x \le a_4 \\ s+(\omega-s)\left(\frac{x-a_5}{a_4-a_5}\right)^{n_3} & , \text{ if } a_4 \le x \le a_5 \\ s\left(\frac{x-a_6}{a_5-a_6}\right)^{n_4} & , \text{ if } a_5 \le x \le a_6 \\ 0 & , \text{ otherwise.} \end{cases}$$

The indeterminacy function can be described as,

$$I_{A_{GNHNNA}} = \begin{cases} 1-r_1\left(\frac{x-b_1}{b_2-b_1}\right)^{m_1} & , \text{ if } b_1 \le x \le b_2 \\ 1-r_1+(r_1-\rho)\left(\frac{x-b_2}{b_3-b_2}\right)^{m_2} & , \text{ if } b_2 \le x \le b_3 \\ 1-\rho & , \text{ if } b_3 \le x \le b_4 \\ 1-s_1+(s_1-\rho)\left(\frac{x-b_5}{b_4-b_5}\right)^{m_3} & , \text{ if } b_4 \le x \le b_5 \\ 1-s_1\left(\frac{x-b_6}{b_5-b_6}\right)^{m_4} & , \text{ if } b_5 \le x \le b_6 \\ 1 & , \text{ otherwise.} \end{cases}$$

Non-membership function can be described as,

$$F_{A_{GNHNNA}} = \begin{cases} 1-r_2\left(\frac{x-c_1}{c_2-c_1}\right)^{p_1} & , \text{ if } c_1 \le x \le c_2 \\ 1-r_2+(r_2-\delta)\left(\frac{x-c_2}{c_3-c_2}\right)^{p_2} & , \text{ if } c_2 \le x \le c_3 \\ 1-\delta & , \text{ if } c_3 \le x \le c_4 \\ 1-s_2+(s_2-\delta)\left(\frac{x-c_5}{c_4-c_5}\right)^{p_3} & , \text{ if } c_4 \le x \le c_5 \\ 1-s_2\left(\frac{x-c_6}{c_5-c_6}\right)^{p_4} & , \text{ if } c_5 \le x \le c_6 \\ 1 & , \text{ otherwise.} \end{cases}$$

where $a_1 < a_2 < a_3 < a_4 < a_5 < a_6$, $b_1 < b_2 < b_3 < b_4 < b_5 < b_6$ and $c_1 < c_2 < c_3 < c_4 < c_5 < c_6 \ \forall \ a_i,\ b_i$ and $c_i\,(i = 1, ..., 6)$ are real constants and $0 < r, s < \omega$, $1-\rho < r_1, s_1 < 1$ and $1-\delta < r_2, s_2 < 1$, $\omega,\ \rho,\ \delta \in [0, 1]$.

Definition 2.2. [36] The (α, β, γ)-cut form of GNHNNA is as follows, $A_{(\alpha,\beta,\gamma)} = \{x \in X/\ T_{A_{GNHNNA}} \ge \alpha,\ I_{A_{GNHNNA}} \le \beta,\ F_{A_{GNHNNA}} \le \gamma\}$. Let $T_\alpha = \{x \in X/\ T_{A_{GNHNNA}} \ge \alpha\}$ where $\alpha \in (0, \omega]$.

If $r \le s$ then, $T_\alpha = \begin{cases} \left[a_1+\left(\frac{\alpha}{r}\right)^{\frac{1}{n_1}}(a_2-a_1), a_6+\left(\frac{\alpha}{s}\right)^{\frac{1}{n_4}}(a_5-a_6)\right] & , \text{ if } 0 < \alpha \le r \\ \left[a_2+\left(\frac{\alpha-r}{\omega-r}\right)^{\frac{1}{n_2}}(a_3-a_2), a_6+\left(\frac{\alpha}{s}\right)^{\frac{1}{n_4}}(a_5-a_6)\right] & , \text{ if } r \le \alpha \le s \\ \left[a_2+\left(\frac{\alpha-r}{\omega-r}\right)^{\frac{1}{n_2}}(a_3-a_2), a_5+\left(\frac{\alpha-s}{\omega-s}\right)^{\frac{1}{n_3}}(a_4-a_5)\right] & , \text{ if } s \le \alpha \le \omega \\ [a_3, a_4] & , \text{ if } \alpha = \omega. \end{cases}$

$$\text{If } s \le r, \text{ then, } T_\alpha = \begin{cases} \left[a_1 + \left(\frac{\alpha}{r}\right)^{\frac{1}{n_1}}(a_2 - a_1), a_6 + \left(\frac{\alpha}{s}\right)^{\frac{1}{n_4}}(a_5 - a_6)\right] & , \text{ if } 0 < \alpha \le s \\ \left[a_1 + \left(\frac{\alpha}{r}\right)^{\frac{1}{n_1}}(a_2 - a_1), a_5 + \left(\frac{\alpha - s}{\omega - s}\right)^{\frac{1}{n_3}}(a_4 - a_5)\right] & , \text{ if } s \le \alpha \le r \\ \left[a_2 + \left(\frac{\alpha - r}{\omega - r}\right)^{\frac{1}{n_2}}(a_3 - a_2), a_5 + \left(\frac{\alpha - s}{\omega - s}\right)^{\frac{1}{n_3}}(a_4 - a_5)\right] & , \text{ if } r \le \alpha \le \omega \\ [a_3, a_4] & , \text{ if } \alpha = \omega. \end{cases}$$

Let $I_\beta = \{x \in X/\ I_{A_{GNHNNA}} \le \beta\}$, where $\beta \in [1-\rho, 1)$. If $r_1 \le s_1$,

$$I_\beta = \begin{cases} [b_3, b_4] & , \text{ if } \beta = 1 - \rho \\ \left[b_2 + \left(\frac{1-\beta-r_1}{\rho - r_1}\right)^{\frac{1}{m_2}}(b_3 - b_2), b_5 + \left(\frac{1-\beta-s_1}{\rho - s_1}\right)^{\frac{1}{m_3}}(b_4 - b_5)\right] & , \text{ if } 1-\rho \le \beta \le 1 - s_1 \\ \left[b_2 + \left(\frac{1-\beta-r_1}{\rho - r_1}\right)^{\frac{1}{m_2}}(b_3 - b_2), b_6 + \left(\frac{1-\beta}{s_1}\right)^{\frac{1}{m_4}}(b_5 - b_6)\right] & , \text{ if } 1-s_1 \le \beta \le 1 - r_1 \\ \left[b_1 + \left(\frac{1-\beta}{r_1}\right)^{\frac{1}{m_1}}(b_2 - b_1), b_6 + \left(\frac{1-\beta}{s_1}\right)^{\frac{1}{m_4}}(b_5 - b_6)\right] & , \text{ if } 1-r_1 \le \beta < 1. \end{cases}$$

If $s_1 \le r_1$, then,

$$I_\beta = \begin{cases} [b_3, b_4] & , \text{ if } \beta = 1 - \rho \\ \left[b_2 + \left(\frac{1-\beta-r_1}{\rho - r_1}\right)^{\frac{1}{m_2}}(b_3 - b_2), b_5 + \left(\frac{1-\beta-s_1}{\rho - s_1}\right)^{\frac{1}{m_3}}(b_4 - b_5)\right] & , \text{ if } 1-\rho \le \beta \le 1 - r_1 \\ \left[b_1 + \left(\frac{1-\beta}{r_1}\right)^{\frac{1}{m_1}}(b_2 - b_1), b_5 + \left(\frac{1-\beta-s_1}{\rho - s_1}\right)^{\frac{1}{m_3}}(b_4 - b_5)\right] & , \text{ if } 1-r_1 \le \beta \le 1 - s_1 \\ \left[b_1 + \left(\frac{1-\beta}{r_1}\right)^{\frac{1}{m_1}}(b_2 - b_1), b_6 + \left(\frac{1-\beta}{s_1}\right)^{\frac{1}{m_4}}(b_5 - b_6)\right] & , \text{ if } 1-s_1 \le \beta < 1. \end{cases}$$

Let $F_\gamma = \{x \in X/F_{A_{GNHNNA}} \le \gamma\}$, where $\gamma \in [1-\delta, 1)$. If $r_2 \le s_2$, then,

$$F_\gamma = \begin{cases} [c_3, c_4] & , \text{ if } \gamma = 1 - \delta \\ \left[c_2 + \left(\frac{1-\gamma-r_2}{\delta - r_2}\right)^{\frac{1}{p_2}}(c_3 - c_2), c_5 + \left(\frac{1-\gamma-s_2}{\delta - s_2}\right)^{\frac{1}{p_3}}(c_4 - c_5)\right] & , \text{ if } 1-\delta \le \gamma \le 1 - s_2 \\ \left[c_2 + \left(\frac{1-\gamma-r_2}{\delta - r_2}\right)^{\frac{1}{p_2}}(c_3 - c_2), c_6 + \left(\frac{1-\gamma}{s_2}\right)^{\frac{1}{p_4}}(c_5 - c_6)\right] & , \text{ if } 1-s_2 \le \gamma \le 1 - r_2 \\ \left[c_1 + \left(\frac{1-\gamma}{r_2}\right)^{\frac{1}{p_1}}(c_2 - c_1), c_6 + \left(\frac{1-\gamma}{s_2}\right)^{\frac{1}{p_4}}(c_5 - c_6)\right] & , \text{ if } 1-r_2 \le \gamma < 1. \end{cases}$$

If $s_2 \le r_2$, then

$$F_\gamma = \begin{cases} [c_3, c_4] & , \text{ if } \gamma = 1 - \delta \\ \left[c_2 + \left(\frac{1-\gamma-r_2}{\delta - r_2}\right)^{\frac{1}{p_2}}(c_3 - c_2), c_5 + \left(\frac{1-\gamma-s_2}{\delta - s_2}\right)^{\frac{1}{p_3}}(c_4 - c_5)\right] & , \text{ if } 1-\delta \le \gamma \le 1 - r_2 \\ \left[c_1 + \left(\frac{1-\gamma}{r_2}\right)^{\frac{1}{p_1}}(c_2 - c_1), c_5 + \left(\frac{1-\gamma-s_2}{\delta - s_2}\right)^{\frac{1}{p_3}}(c_4 - c_5)\right] & , \text{ if } 1-r_2 \le \gamma \le 1 - s_2 \\ \left[c_1 + \left(\frac{1-\gamma}{r_2}\right)^{\frac{1}{p_1}}(c_2 - c_1), c_6 + \left(\frac{1-\gamma}{s_2}\right)^{\frac{1}{p_4}}(c_5 - c_6)\right] & , \text{ if } 1-s_2 \le \gamma < 1. \end{cases}$$

Definition 2.3. [36] Let $A_{GNHNNA} = \Big\{T\left(a_1, a_2, a_3, a_4, a_5, a_6; r, s; \omega\right)_{(n_1, n_2, n_3, n_4)}, I\left(b_1, b_2, b_3, b_4, b_5, b_6; r_1, s_1; \rho\right)_{(m_1, m_2, m_3, m_4)}, F\left(c_1, c_2, c_3, c_4, c_5, c_6; r_2, s_2; \delta\right)_{(p_1, p_2, p_3, p_4)}\Big\}$ be the GNHNNA and $T_\alpha = [L(\alpha), R(\alpha)]$, $I_\beta = [L_1(\beta), R_1(\beta)]$ and $F_\gamma = [L_2(\gamma), R_2(\gamma)]$ be the α, β and γ - cut respectively.

(1) The values of the A_{GNHNNA} corresponding to α - cut set, denoted by $V_T(A_{GNHNNA})$, is defined as, $V_T(A_{GNHNNA}) = \int_0^{\omega} [L(\alpha) + R(\alpha)] f_1(\alpha)\, d\alpha$. where $f_1(\alpha) \in [0,1]$ $(\alpha \in [0,\omega])$, $f_1(0) = 0$ and $f_1(\alpha)$ is increasing and monotonic in $\alpha \in [0,\omega]$.

(2) The values of the A_{GNHNNA} corresponding to β - cut set, denoted by $V_I(A_{GNHNNA})$, is defined as, $V_I(A_{GNHNNA}) = \int_{1-\rho}^{1} [L_1(\beta) + R_1(\beta)] f_2(\beta)\, d\beta$. where $f_2(\beta) \in [0,1]$ $(\beta \in [\rho,1])$, $f_2(1) = 0$ and $f_2(\beta)$ is decreasing and monotonic in $\beta \in [\rho,1]$.

(3) The values of the A_{GNHNNA} corresponding to γ - cut set, denoted by $V_F(A_{GNHNNA})$, is defined as, $V_F(A_{GNHNNA}) = \int_{1-\delta}^{1} [L_2(\gamma) + R_2(\gamma)] f_3(\gamma)\, d\gamma$. where $f_3(\gamma) \in [0,1]$ $(\gamma \in [\delta,1])$, $f_3(1) = 0$ and $f_3(\gamma)$ is decreasing and monotonic in $\gamma \in [\delta,1]$.

Without loss of generality, we choose $f_1(\alpha) = \alpha$ $(\alpha \in [0,\omega])$, $f_2(\beta) = 1-\beta$ $(\beta \in [1-\rho,1])$ and $f_3(\gamma) = 1-\gamma$ $(\gamma \in [1-\delta,1])$.

Theorem 2.4. *[36] Let* $A_{GNHNNA} = \Big\{ T(a_1,a_2,a_3,a_4,a_5,a_6;r,s;\omega)_{(n_1,n_2,n_3,n_4)}, I(b_1,b_2,b_3,b_4,b_5,b_6;r_1,s_1;\rho)_{(m_1,m_2,m_3,m_4)}, F(c_1,c_2,c_3,c_4,c_5,c_6;r_2,s_2;\delta)_{(p_1,p_2,p_3,p_4)} \Big\}$. *be the GNHNNA. Then,*

(1) *The α-cut set of the A_{GNHNNA} for the truth-membership is computed as $T_\alpha = [L(\alpha), R(\alpha)]$ where $\alpha \in [0,\omega]$. If $f_1(\alpha) = \alpha$, we obtain the value and ambiguity of the GNHNNA number A_{GNHNNA} as,*

$$\begin{aligned} V_T(A_{GNHNNA}) = {} & \left[\frac{1}{2} - \frac{n_1}{1+2n_1}\right] r^2 a_1 + \left[\frac{\omega^2 - r^2}{2} - n_2 \frac{(\omega - r)^2}{1+2n_2} - r n_2 \frac{\omega - r}{1+n_2}\right] a_2 \\ & + \left[\frac{n_2 (\omega - r)^2}{1+2n_2} + r n_2 \frac{\omega - r}{1+n_2}\right] a_3 + \left[\frac{n_3 (\omega - s)^2}{1+2n_3} + s n_3 \frac{\omega - s}{1+n_3}\right] a_4 \\ & + \left[\frac{\omega^2 - s^2}{2} - n_3 \frac{(\omega - s)^2}{1+2n_3} - s n_3 \frac{\omega - s}{1+n_3}\right] a_5 + \left[\frac{1}{2} - \frac{n_4}{1+2n_4}\right] s^2 a_6 \end{aligned}$$

(2) *The β-cut set of A_{GNHNNA} for indeterminacy -membership is calculated as $I_\beta = [L_1(\beta), R_1(\beta)]$ where $\beta \in [1-\rho,1]$. When $f_2(\beta) = 1-\beta$, we obtain the value and ambiguity of A_{GNHNNA}, respectively as,*

$$\begin{aligned} V_I(A_{GNHNNA}) = {} & \left[\frac{1}{2} - \frac{m_1}{1+2m_1}\right] r_1^2 b_1 + \left[\frac{\rho^2 - r_1^2}{2} - m_2 \frac{(\rho - r_1)^2}{1+2m_2} - r_1 m_2 \frac{\rho - r_1}{1+m_2}\right] b_2 \\ & + \left[\frac{m_2 (\rho - r_1)^2}{1+2m_2} + r_1 m_2 \frac{\rho - r_1}{1+m_2}\right] b_3 + \left[\frac{m_3 (\rho - s_1)^2}{1+2m_3} + s_1 m_3 \frac{\rho - s_1}{1+m_3}\right] b_4 \\ & + \left[\frac{\rho^2 - s_1^2}{2} - m_3 \frac{(\rho - s_1)^2}{1+2m_3} - s_1 m_3 \frac{\rho - s_1}{1+m_3}\right] b_5 + \left[\frac{1}{2} - \frac{m_4}{1+2m_4}\right] s_1^2 b_6 \end{aligned}$$

(3) *The γ-cut set of A_{GNHNNA} for falsity-membership is computed as $F_\gamma = [L_2(\gamma), R_2(\gamma)]$ where $\gamma \in [1-\delta,1]$. If $f_3(\gamma) = 1-\beta$, we obtain the value and ambiguity of A_{GNHNNA},*

by the following steps,

$$V_F\left(A_{GNHNNA}\right) = \left[\frac{1}{2} - \frac{p_1}{1+2p_1}\right] r_2^2 c_1 + \left[\frac{\delta^2 - r_2^2}{2} - p_2\frac{(\delta - r_2)^2}{1+2p_2} - r_2 p_2 \frac{\delta - r_2}{1+p_2}\right] c_2 + \left[\frac{p_2\left(\delta - r_2\right)^2}{1+2p_2} + r_2 p_2 \frac{\delta - r_2}{1+p_2}\right] c_3 + \left[\frac{p_3\left(\delta - s_2\right)^2}{1+2p_3} + s_2 p_3 \frac{\delta - s_2}{1+p_3}\right] c_4 + \left[\frac{\delta^2 - s_2^2}{2} - p_3\frac{(\delta - s_2)^2}{1+2p_3} - s_2 p_3 \frac{\delta - s_2}{1+p_3}\right] c_5 + \left[\frac{1}{2} - \frac{p_4}{1+2p_4}\right] s_2^2 c_6$$

Definition 2.5 ([38])**.** The score function of a SVNS($\mathscr{N}$) is defined as $\tilde{S}(S) = \theta(x) - \iota(x) - \phi(x)$.

3. Hexagonal Approximation of a Neutrosophic Number

The first two definitions will provide the structure to approximate any curve to a GNHNNA.

Definition 3.1. Let A and B be two neutrosophic numbers with (α, β, γ)-cuts ($[A_{T-}, A_{T+}]$, $[A_{I-}, A_{I+}]$, $[A_{F-}, A_{F+}]$) and ($[B_{T-}, B_{T+}]$, $[B_{I-}, B_{I+}]$, $[B_{F-}, B_{F+}]$), then the distance measure between A and B is defined as,

$$d(A,B) = \sqrt{\frac{1}{3}(M+N+O)} \quad \text{where,} \tag{1}$$

$M = \int_0^\omega (A_{T+} - B_{T+})^2 d\alpha + \int^\omega (A_{T-} - B_{T-})^2 d\alpha$, $N = \int_{1-\rho}^1 (A_{I+} - B_{I+})^2 d\beta + \int_{1-\rho}^1 (A_{I-} - B_{I-})^2 d\beta$ and $O = \int_{1-\delta}^1 (A_{F+} - B_{F+})^2 d\gamma + \int_{1-\delta}^1 (A_{F-} - B_{F-})^2 d\gamma$

Definition 3.2. Let A be a neutrosophic number with (α, β, γ)-cut ($[A_{T-}, A_{T+}]$, $[A_{I-}, A_{I+}]$, $[A_{F-}, A_{F+}]$), then the expected interval of A is defined as,

$$EI(A) = [X, Y, Z] \quad \text{where,} \tag{2}$$

$X = [\int_0^\omega A_{T-} d\alpha, \int_0^\omega A_{T+} \mathrm{d}\alpha]$, $Y = [\int_0^\omega A_{I-} d\beta, \int_0^\omega A_{I+} d\beta]$ and $Z = [\int_0^\omega A_{F-} d\gamma, \int_0^\omega A_{F+} d\gamma]$.

3.1. *Approximation of GNHNNA*

In this subsection, we approximate any neutrosophic number by preserving expected intervals, and the distance measure defined in definition 3.1 to a GNHNNA by using an approximation operator H: $\mathscr{N}(\mathscr{R}) \to \mathscr{N}^H(\mathscr{R})$ where $\mathscr{N}(\mathscr{R})$ represent the set of all neutrosophic numbers defined on the real domain and $\mathscr{N}^H(\mathscr{R})$ is the set of all GNHNNA.

Suppose A be any neutrosophic number with (α, β, γ)-cut ($[A_{T-}, A_{T+}]$, $[A_{I-}, A_{I+}]$, $[A_{F-}, A_{F+}]$). Now we will try to find the nearest GNHNNA H(A), the closest to A that satisfies the distance measure defined in definition 3.1. Let (α, β, γ)-cut of H(A) be ($[H_{T-}, H_{T+}]$, $[H_{I-}, H_{I+}]$, $[H_{F-}, H_{F+}]$), then the distance measure between A and $B = H(A)$ is defined as,

$$d(A,B) = \sqrt{\frac{1}{3}(M+N+O)} \quad \text{where,} \tag{3}$$

$M = \int_0^{\omega}(A_{T+} - H_{T+})^2 d\alpha + \int_0^{\omega}(A_{T-} - H_{T-})^2 d\alpha$, $N = \int_{1-\rho}^{1}(A_{I+} - H_{I+})^2 d\beta + \int_{1-\rho}^{1}(A_{I-} - H_{I-})^2 d\beta$ and $O = \int_{1-\delta}^{1}(A_{F+} - H_{F+})^2 d\gamma + \int_{1-\delta}^{1}(A_{F-} - H_{F-})^2 d\gamma$

By using definition 2.2, equation 3 reduces to,

$$d(A,B) = \sqrt{\frac{1}{3}(M+N+O)} \quad \text{where,} \tag{4}$$

$$M = \int_0^{s}(A_{T+} - (a_6 + \left(\frac{\alpha}{s}\right)^{\frac{1}{n_4}}(a_5 - a_6))^2 d\alpha + \int_s^{\omega}(A_{T+} - (a_5 + \left(\frac{\alpha - s}{\omega - s}\right)^{\frac{1}{n_3}}(a_4 - a_5)))^2 d\alpha$$
$$+ \int_0^{r}(A_{T-} - (a_1 + \left(\frac{\alpha}{r}\right)^{\frac{1}{n_1}}(a_2 - a_1)))^2 d\alpha + \int_r^{\omega}(A_{T-} - (a_2 + \left(\frac{\alpha - r}{\omega - r}\right)^{\frac{1}{n_2}}(a_3 - a_2)))^2 d\alpha$$

$N = \int_{1-\rho}^{1-s_1}(A_{I+} - (b_5 + \left(\frac{1-\beta-s_1}{\rho - s_1}\right)^{\frac{1}{m_3}}(b_4 - b_5)))^2 d\beta + \int_{1-s_1}^{1}(A_{I+} - (b_6 + \left(\frac{1-\beta}{s_1}\right)^{\frac{1}{m_4}}(b_5 - b_6)))^2 d\beta +$
$\int_{1-\rho}^{1-r_1}(A_{I-} - (b_2 + \left(\frac{1-\beta-r_1}{\rho - r_1}\right)^{\frac{1}{m_2}}(b_3 - b_2)))^2 d\beta + \int_{1-r_1}^{1}(A_{I-} - (b_1 + \left(\frac{1-\beta}{r_1}\right)^{\frac{1}{m_1}}(b_2 - b_1)))^2 d\beta$ and
$O = \int_{1-\delta}^{1-s_2}(A_{F+} - (c_5 + \left(\frac{1-\gamma-s_2}{\delta - s_2}\right)^{\frac{1}{p_3}}(c_4 - c_5)))^2 d\gamma + \int_{1-s_2}^{1}(A_{F+} - (c_6 + \left(\frac{1-\gamma}{s_2}\right)^{\frac{1}{p_4}}(c_5 - c_6)))^2 d\gamma +$
$\int_{1-\delta}^{1-r_2}(A_{F-} - (c_2 + \left(\frac{1-\gamma-r_2}{\delta - r_2}\right)^{\frac{1}{p_2}}(c_3 - c_2)))^2 d\gamma + \int_{1-r_2}^{1}(A_{F-} - (c_1 + \left(\frac{1-\gamma}{r_2}\right)^{\frac{1}{p_1}}(c_2 - c_1)))^2 d\gamma$

For GNHNNA, $\int_0^r (A_{T-}) =$

$$\begin{cases} r\frac{a_1 + a_2 n_1}{n_1 + 1} & , \text{ if } 0 < n_1 \\ a_1 r - \lim_{\alpha \to 0} a_1 \alpha - \dfrac{\alpha n_1 (a_1 - a_2)(\frac{\alpha}{r})^{\frac{1}{n_1}}}{n_1 + 1} & , \text{ if } n_1 \leq 0 \end{cases} \tag{5}$$

In a similar way we can compute the other integrals, $\int_0^{\omega} A_{T+} \mathrm{d}\alpha$, $\int_0^{\omega} A_{I-} d\beta$, $\int_0^{\omega} A_{I+} d\beta$, $\int_0^{\omega} A_{F-} d\gamma$, $\int_0^{\omega} A_{F+} d\gamma$ given in Definition 3.2.

Minimizing $\mathrm{D}(a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2, c_3, c_4, c_5, c_6) = d^2(A, H(A))$ in considering the following constraints is sufficient for finding the approximation.

$$\int_0^{\omega} H_{T-}\, d\alpha - \int_0^{\omega} A_{T-}\, d\alpha = 0 \qquad \int_0^{\omega} H_{T+}\, d\alpha - \int_0^{\omega} A_{T+}\, d\alpha = 0 \tag{6}$$

$$\int_0^{\omega} H_{I-}\, d\beta - \int_0^{\omega} A_{I-}\, d\beta = 0 \qquad \int_0^{\omega} H_{I+}\, d\beta - \int_0^{\omega} A_{I+}\, d\beta = 0 \tag{7}$$

$$\int_0^{\omega} H_{F-}\, d\gamma - \int_0^{\omega} A_{F-}\, d\gamma = 0 \qquad \int_0^{\omega} H_{F+}\, d\gamma - \int_0^{\omega} A_{F+}\, d\gamma = 0 \tag{8}$$

Using the Lagrangian multiplier method, we can find the value of real numbers $a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6$ such that $a_1 \leq a_2 \leq ... \leq a_6, b_1 \leq b_2 \leq ... \leq b_6$ and

$c_1 \leq c_2 \leq ... \leq c_6$ that minimize the function,

$$L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6) = d^2(A, H(A)) + \lambda_1(\int_0^\omega H_{T-}d\alpha - \int_0^\omega A_{T-}d\alpha)$$
$$+ \lambda_2(\int_0^\omega H_{T+}d\alpha - \int_0^\omega A_{T+}d\alpha) + \lambda_3(\int_0^\omega H_{I-}d\beta - \int_0^\omega A_{I-}d\beta) + \lambda_4(\int_0^\omega HI+d\beta - \int_0^\omega A_{I+}d\beta)$$
$$+ \lambda_5(\int_0^\omega H_{F-}d\gamma - \int_0^\omega A_{F-}d\gamma) + \lambda_6(\int_0^\omega H_{F+}d\gamma - \int_0^\omega A_{F+}d\gamma) \quad (9)$$

where λ_1, λ_2,..., λ_6 are Lagrangian multipliers.

Now we have to find the partial derivatives. The minimization problem is rewritten as follows using the KKT theorem.

$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial a_1} = 0 \quad (10)$$
$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial a_2} = 0 \quad (11)$$
$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial a_3} = 0 \quad (12)$$
$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial a_4} = 0 \quad (13)$$
$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial a_5} = 0 \quad (14)$$
$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial a_6} = 0 \quad (15)$$
$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial b_1} = 0 \quad (16)$$
$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial b_2} = 0 \quad (17)$$
$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial b_3} = 0 \quad (18)$$
$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial b_4} = 0 \quad (19)$$
$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial b_5} = 0 \quad (20)$$
$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial b_6} = 0 \quad (21)$$
$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial c_1} = 0 \quad (22)$$
$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial c_2} = 0 \quad (23)$$
$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial c_3} = 0 \quad (24)$$
$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial c_4} = 0 \quad (25)$$
$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial c_5} = 0 \quad (26)$$

$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial c_6} = 0 \tag{27}$$

By solving 10, 11 and 12 we get

$$a_1 = \frac{\begin{array}{l} -(2X^-\omega - 2X^-r + 5X^-n_1\omega - 4X^-n_1r - Y^-n_1r + Z^-n_1r + 4X^-n_1^2\omega + X^-n_1^3\omega - 2X^-n_1^2r - Y^-n_1^2r + \\ Z^-n_1^2r + 2X^-n_1n_2r - 2Y^-n_1n_2r + Z^-n_1n_2r + X^-n_1n_2^2r + 4X^-n_1^2n_2r + 2X^-n_1^3n_2r - Y^-n_1n_2^2r - \\ 2Y^-n_1^2n_2r + Z^-n_1^2n_2r + 2X^-n_1^2n_2^2r + X^-n_1^3n_2^2r - Y^-n_1^2n_2^2r) \end{array}}{2r(n_1rn_2^2 + 2n_1rn_2 + 2\omega - 2r + 2n_1\omega - n_1r)}$$

$$a_2 = \frac{\begin{array}{l} 2Z^- - 2Y^- + X^-n_1 - 2Y^-n_1 - 4Y^-n_2 + 2Z^-n_1 + 2Z^-n_2 + X^-n_1^2 - 2Y^-n_2^2 + X^-n_1^2n_2^2 + \\ 2X^-n_1n_2 - 4Y^-n_1n_2 + 2Z^-n_1n_2 + X^-n_1n_2^2 + 2X^-n_1^2n_2 - 2Y^-n_1n_2^2 \end{array}}{2n_1rn_2^2 + 4n_1rn_2 + 4\omega - 4r + 4n_1\omega - 2n_1r}$$

$$a_3 = \frac{\begin{array}{l} -(4Z^-\omega - 4Z^-r - 2Y^-n_2\omega + 4Z^-n_1\omega + 4Z^-n_2\omega + 2Y^-n_2r - 2Z^-n_1r - 4Z^-n_2r - 2Y^-n_2^2\omega + 2Y^-n_2^2r + \\ X^-n_1n_2\omega - 2Y^-n_1n_2\omega + 4Z^-n_1n_2\omega - X^-n_1n_2r + 2Y^-n_1n_2r + Z^-n_1n_2r + X^-n_1n_2^2\omega + X^-n_1^2n_2\omega - \\ 2Y^-n_1n_2^2\omega - X^-n_1n_2^2r - X^-n_1^2n_2r + 2Y^-n_1n_2^2r + 4Z^-n_1n_2^2r + Z^-n_1n_2^3r + X^-n_1^2n_2^2\omega - X^-n_1^2n_2^2r) \end{array}}{2n_2(\omega - r)(n_1rn_2^2 + 2n_1rn_2 + 2\omega - 2r + 2n_1\omega - n_1r)}$$

By solving 13, 14 and 15 we get

$$a_4 = \frac{\begin{array}{l} -(4Z^+\omega - 4Z^+s - 2Y^+n_3\omega + 4Z^+n_4\omega + 4Z^+n_3\omega + 2Y^+n_3s - 2Z^+n_4s - 4Z^+n_3s - 2Y^+n_3^2\omega + 2Y^+n_3^2s + \\ X^+n_4n_3\omega - 2Y^+n_4n_3\omega + 4Z^+n_4n_3\omega - X^+n_4n_3s + 2Y^+n_4n_3s + Z^+n_4n_3s + X^+n_4n_3^2\omega + X^+n_4^2n_3\omega - \\ 2Y^+n_4n_3^2\omega - X^+n_4n_3^2s - X^+n_4^2n_3s + 2Y^+n_4n_3^2s + 4Z^+n_4n_3^2s + Z^+n_4n_3^3s + X^+n_4^2n_3^2\omega - X^+n_4^2n_3^2s) \end{array}}{2n_3(\omega - s)(n_4sn_3^2 + 2n_4sn_3 + 2\omega - 2s + 2n_4\omega - n_4s)}$$

$$a_5 = \frac{\begin{array}{l} 2Z^+ - 2Y^+ + X^+n_4 - 2Y^+n_4 - 4Y^+n_3 + 2Z^+n_4 + 2Z^+n_3 + X^+n_4^2 - 2Y^+n_3^2 + X^+n_4^2n_3^2 + \\ 2X^+n_4n_3 - 4Y^+n_4n_3 + 2Z^+n_4n_3 + X^+n_4n_3^2 + 2X^+n_4^2n_3 - 2Y^+n_4n_3^2 \end{array}}{2n_4sn_3^2 + 4n_4sn_3 + 4\omega - 4s + 4n_4\omega - 2n_4s}$$

$$a_6 = \frac{\begin{array}{l} -(2X^+\omega - 2X^+s + 5X^+n_4\omega - 4X^+n_4s - Y^+n_4s + Z^+n_4s + 4X^+n_4^2\omega + X^+n_4^3\omega - 2X^+n_4^2s - \\ Y^+n_4^2s + Z^+n_4^2s + 2X^+n_4n_3s - 2Y^+n_4n_3s + Z^+n_4n_3s + X^+n_4n_3^2s + 4X^+n_4^2n_3s + 2X^+n_4^3n_3s - \\ Y^+n_4n_3^2s - 2Y^+n_4^2n_3s + Z^+n_4^2n_3s + 2X^+n_4^2n_3^2s + X^+n_4^3n_3^2s - Y^+n_4^2n_3^2s) \end{array}}{2s(n_4sn_3^2 + 2n_4sn_3 + 2\omega - 2s + 2n_4\omega - n_4s)},$$

where

$$X^- = 2\int_0^r \left[\left(\frac{\alpha}{r}\right)^{\frac{1}{n_1}} - 1\right] A_{T-}\, d\alpha,$$

$$Y^- = 2\int_r^\omega \left[\left(\frac{\alpha - r}{\omega - r}\right)^{\frac{1}{n_2}} - 1\right] A_{T-}\, d\alpha - 2\int_0^r \left(\frac{\alpha}{r}\right)^{\frac{1}{n_1}} A_{T-}\, d\alpha,$$

$$Z^- = -2\int_r^\omega \left(\frac{\alpha - r}{\omega - r}\right)^{\frac{1}{n_2}} A_{T-}\, d\alpha, X^+ = 2\int_0^s \left[\left(\frac{\alpha}{s}\right)^{\frac{1}{n_4}} - 1\right] A_{T+}\, d\alpha,$$

$$Y^+ = 2\int_s^\omega \left[\left(\frac{\alpha - s}{\omega - s}\right)^{\frac{1}{n_3}} - 1\right] A_{T+}\, d\alpha - 2\int_0^s \left(\frac{\alpha}{s}\right)^{\frac{1}{n_4}} A_{T+}\, d\alpha,$$

$$Z^+ = -2\int_s^\omega \left(\frac{\alpha - s}{\omega - s}\right)^{\frac{1}{n_3}} A_{T+}\, d\alpha.$$

By solving 16, 17 and 18 we get

$$b_1 = \frac{\begin{array}{l} -(2X^-\rho - 2X^-r_1 - 4X^-m_1r_1 + 5X^-m_1\rho - Y^-m_1r_1 + Z^-m_1r_1 - 2X^-m_1^2r_1 + 4X^-m_1^2\rho + X^-m_1^3\rho) \\ -(Y^-m_1^2r_1 - Z^-m_1^2r_1 + 2X^-m_1m_2r_1 - 2Y^-m_1m_2r_1 + Z^-m_1m_2r_1 + X^-m_1m_2^2r_1) - (4X^-m_1^2m_2r_1 + \\ 2X^-m_1^3m_2r_1 - Y^-m_1m_2^2r_1 - 2Y^-m_1^2m_2r_1 + Z^-m_1^2m_2r_1 + 2X^-m_1^2m_2^2r_1 + X^-m_1^3m_2^2r_1 - Y^-m_1^2m_2^2r_1) \end{array}}{2r_1(m_1r_1m_2^2 + 2m_1r_1m_2 - 2r_1 + 2\rho - m_1r_1 + 2m_1\rho)}$$

$$b_2 = \frac{\begin{array}{l} 2Z^- - 2Y^- + X^-m_1 - 2Y^-m_1 - 4Y^-m_2 + 2Z^-m_1 + 2Z^-m_2 + X^-m_1^2 - 2Y^-m_2^2 \\ + X^-m_1^2m_2^2 + 2X^-m_1m_2 - 4Y^-m_1m_2 + 2Z^-m_1m_2 + X^-m_1m_2^2 + 2X^-m_1^2m_2 - 2Y^-m_1m_2^2 \end{array}}{2m_1r_1m_2^2 + 4m_1r_1m_2 - 4r_1 + 4\rho - 2m_1r_1 + 4m_1\rho}$$

$$b_3 = \frac{\begin{array}{l} 4Z^-\rho - 4Z^-r_1 + 2Y^-m_2r_1 - 2Y^-m_2\rho - 2Z^-m_1r_1 - 4Z^-m_2r_1 + 4Z^-m_1\rho + 4Z^-m_2\rho \\ + 2Y^-m_2^2r_1 - 2Y^-m_2^2\rho - X^-m_1m_2r_1 + X^-m_1m_2\rho + 2Y^-m_1m_2r_1 - 2Y^-m_1m_2\rho \\ + Z^-m_1m_2r_1 + 4Z^-m_1m_2\rho - X^-m_1m_2^2r_1 - X^-m_1^2m_2r_1 + X^-m_1m_2^2\rho + X^-m_1^2m_2\rho \\ + 2Y^-m_1m_2^2r_1 - 2Y^-m_1m_2^2\rho + 4Z^-m_1m_2^2r_1 + Z^-m_1m_2^3r_1 - X^-m_1^2m_2^2r_1 + X^-m_1^2m_2^2\rho \end{array}}{2m_2(r_1 - \rho)(m_1r_1m_2^2 + 2m_1r_1m_2 - 2r_1 + 2\rho - m_1r_1 + 2m_1\rho)}$$

By solving 19, 20 and 21 we get

$$b_4 = \frac{\begin{array}{l} 4Z^+\rho - 4Z^+s_1 + 2Y^+m_3s_1 - 2Y^+m_3\rho - 2Z^+m_4s_1 - 4Z^+m_3s_1 + 4Z^+m_4\rho + 4Z^+m_3\rho \\ + 2Y^+m_3^2s_1 - 2Y^+m_3^2\rho - X^+m_4m_3s_1 + X^+m_4m_3\rho + 2Y^+m_4m_3s_1 - 2Y^+m_4m_3\rho \\ + Z^+m_4m_3s_1 + 4Z^+m_4m_3\rho - X^+m_4m_3^2s_1 - X^+m_4^2m_3s_1 + X^+m_4m_3^2\rho + X^+m_4^2m_3\rho \\ + 2Y^+m_4m_3^2s_1 - 2Y^+m_4m_3^2\rho + 4Z^+m_4m_3^2s_1 + Z^+m_4m_3^3s_1 - X^+m_4^2m_3^2s_1 + X^+m_4^2m_3^2\rho \end{array}}{2m_3(s_1 - \rho)(m_4s_1m_3^2 + 2m_4s_1m_3 - 2s_1 + 2\rho - m_4s_1 + 2m_4\rho)}$$

$$b_5 = \frac{\begin{array}{l} 2Z^+ - 2Y^+ + X^+m_4 - 2Y^+m_4 - 4Y^+m_3 + 2Z^+m_4 + 2Z^+m_3 + X^+m_4^2 - 2Y^+m_3^2 \\ + X^+m_4^2m_3^2 + 2X^+m_4m_3 - 4Y^+m_4m_3 + 2Z^+m_4m_3 + X^+m_4m_3^2 + 2X^+m_4^2m_3 - 2Y^+m_4m_3^2 \end{array}}{2m_4s_1m_3^2 + 4m_4s_1m_3 - 4s_1 + 4\rho - 2m_4s_1 + 4m_4\rho}$$

$$b_6 = \frac{\begin{array}{l} -(2X^+\rho - 2X^+s_1 - 4X^+m_4s_1 + 5X^+m_4\rho - Y^+m_4s_1 + Z^+m_4s_1 - 2X^+m_4^2s_1 + 4X^+m_4^2\rho + X^+m_4^3\rho) \\ -(Y^+m_4^2s_1 - Z^+m_4^2s_1 + 2X^+m_4m_3s_1 - 2Y^+m_4m_3s_1 + Z^+m_4m_3s_1 + X^+m_4m_3^2s_1) - (4X^+m_4^2m_3s_1 + \\ 2X^+m_4^3m_3s_1 - Y^+m_4m_3^2s_1 - 2Y^+m_4^2m_3s_1 + Z^+m_4^2m_3s_1 + 2X^+m_4^2m_3^2s_1 + X^+m_4^3m_3^2s_1 - Y^+m_4^2m_3^2s_1) \end{array}}{2s_1(m_4s_1m_3^2 + 2m_4s_1m_3 - 2s_1 + 2\rho - m_4s_1 + 2m_4\rho)}$$

where

$$\begin{aligned}
X^- &= 2\int_{1-r_1}^{1}\left[\left(\frac{1-\beta}{r_1}\right)^{\frac{1}{m_1}} - 1\right] A_{I-}\, d\beta, \\
Y^- &= -2\int_{1-r_1}^{1}\left[\left(\frac{1-\beta}{r_1}\right)^{\frac{1}{m_1}}\right] A_{I-}\, d\beta - 2\int_{1-\rho}^{1-r_1}\left[\left(\frac{1-\beta-r_1}{\rho-r_1}\right)^{\frac{1}{m_2}} - 1\right] A_{I-}\, d\beta, \\
Z^- &= -2\int_{1-\rho}^{1-r_1}\left(\frac{1-\beta-r_1}{\rho-r_1}\right)^{\frac{1}{m_2}} A_{I-}\, d\beta, X^+ = 2\int_{1-s_1}^{1}\left[\left(\frac{1-\beta}{s_1}\right)^{\frac{1}{m_4}} - 1\right] A_{I+}\, d\beta, \\
Y^+ &= -2\int_{1-s_1}^{1}\left[\left(\frac{1-\beta}{s_1}\right)^{\frac{1}{m_4}}\right] A_{I+}\, d\beta - 2\int_{1-\rho}^{1-s_1}\left[\left(\frac{1-\beta-s_1}{\rho-s_1}\right)^{\frac{1}{m_3}} - 1\right] A_{I+}\, d\beta, \\
Z^+ &= -2\int_{1-\rho}^{1-s_1}\left(\frac{1-\beta-s_1}{\rho-s_1}\right)^{\frac{1}{m_3}} A_{I+}\, d\beta.
\end{aligned}$$

By solving 20, 21 and 22 we get

$$c_1 = \frac{\begin{array}{l} -(2X^-\delta - 2X^-r_2 - 4X^-p_4r_2 + 5X^-p_4\delta - Y^-p_4r_2 + Z^-p_4r_2 - 2X^-p_4^2r_2 + 4X^-p_4^2\delta + X^-p_4^3\delta) \\ -(Y^-p_4^2r_2 - Z^-p_4^2r_2 + 2X^-p_4p_2r_2 - 2Y^-p_4p_2r_2 + Z^-p_4p_2r_2 + X^-p_4p_2^2r_2) - (4X^-p_4^2p_2r_2 + \\ 2X^-p_4^3p_2r_2 - Y^-p_4p_2^2r_2 - 2Y^-p_4^2p_2r_2 + Z^-p_4^2p_2r_2 + 2X^-p_4^2p_2^2r_2 + X^-p_4^3p_2^2r_2 - Y^-p_4^2p_2^2r_2) \end{array}}{2r_2(p_4r_2p_2^2 + 2p_4r_2p_2 - 2r_2 + 2\delta - p_4r_2 + 2p_4\delta)}$$

$$c_2 = \frac{\begin{array}{l} 2Z^- - 2Y^- + X^-p_4 - 2Y^-p_4 - 4Y^-p_2 + 2Z^-p_4 + 2Z^-p_2 + X^-p_4^2 - 2Y^-p_2^2 \\ + X^-p_4^2p_2^2 + 2X^-p_4p_2 - 4Y^-p_4p_2 + 2Z^-p_4p_2 + X^-p_4p_2^2 + 2X^-p_4^2p_2 - 2Y^-p_4p_2^2 \end{array}}{2p_4r_2p_2^2 + 4p_4r_2p_2 - 4r_2 + 4\delta - 2p_4r_2 + 4p_4\delta}$$

$$c_3 = \frac{\begin{array}{l} 4Z^-\delta - 4Z^-r_2 + 2Y^-p_2r_2 - 2Y^-p_2\delta - 2Z^-p_4r_2 - 4Z^-p_2r_2 + 4Z^-p_4\delta + 4Z^-p_2\delta \\ + 2Y^-p_2^2r_2 - 2Y^-p_2^2\delta - X^-p_4p_2r_2 + X^-p_4p_2\delta + 2Y^-p_4p_2r_2 - 2Y^-p_4p_2\delta \\ + Z^-p_4p_2r_2 + 4Z^-p_4p_2\delta - X^-p_4p_2^2r_2 - X^-p_4^2p_2r_2 + X^-p_4p_2^2\delta + X^-p_4^2p_2\delta \\ + 2Y^-p_4p_2^2r_2 - 2Y^-p_4p_2^2\delta + 4Z^-p_4p_2^2r_2 + Z^-p_4p_2^3r_2 - X^-p_4^2p_2^2r_2 + X^-p_4^2p_2^2\delta \end{array}}{2p_2(r_2 - \delta)(p_4r_2p_2^2 + 2p_4r_2p_2 - 2r_2 + 2\delta - p_4r_2 + 2p_4\delta)}$$

By solving 23, 24 and 25 we get

$$c_4 = \frac{\begin{array}{l} 4Z^+\delta - 4Z^+s_2 + 2Y^+p_3s_2 - 2Y^+p_3\delta - 2Z^+p_4s_2 - 4Z^+p_3s_2 + 4Z^+p_4\delta + 4Z^+p_3\delta \\ + 2Y^+p_3^2s_2 - 2Y^+p_3^2\delta - X^+p_4p_3s_2 + X^+p_4p_3\delta + 2Y^+p_4p_3s_2 - 2Y^+p_4p_3\delta \\ + Z^+p_4p_3s_2 + 4Z^+p_4p_3\delta - X^+p_4p_3^2s_2 - X^+p_4^2p_3s_2 + X^+p_4p_3^2\delta + X^+p_4^2p_3\delta \\ + 2Y^+p_4p_3^2s_2 - 2Y^+p_4p_3^2\delta + 4Z^+p_4p_3^2s_2 + Z^+p_4p_3^3s_2 - X^+p_4^2p_3^2s_2 + X^+p_4^2p_3^2\delta \end{array}}{2p_3(s_2-\delta)(p_4s_2p_3^2 + 2p_4s_2p_3 - 2s_2 + 2\delta - p_4s_2 + 2p_4\delta)}$$

$$c_5 = \frac{\begin{array}{l} 2Z^+ - 2Y^+ + X^+p_4 - 2Y^+p_4 - 4Y^+p_3 + 2Z^+p_4 + 2Z^+p_3 + X^+p_4^2 - 2Y^+p_3^2 \\ + X^+p_4^2p_3^2 + 2X^+p_4p_3 - 4Y^+p_4p_3 + 2Z^+p_4p_3 + X^+p_4p_3^2 + 2X^+p_4^2p_3 - 2Y^+p_4p_3^2 \end{array}}{2p_4s_2p_3^2 + 4p_4s_2p_3 - 4s_2 + 4\delta - 2p_4s_2 + 4p_4\delta}$$

$$c_6 = \frac{\begin{array}{l} -(2X^+\delta - 2X^+s_2 - 4X^+p_4s_2 + 5X^+p_4\delta - Y^+p_4s_2 + Z^+p_4s_2 - 2X^+p_4^2s_2 + 4X^+p_4^2\delta + X^+p_4^3\delta) \\ -(Y^+p_4^2s_2 - Z^+p_4^2s_2 + 2X^+p_4p_3s_2 - 2Y^+p_4p_3s_2 + Z^+p_4p_3s_2 + X^+p_4p_3^2s_2) - (4X^+p_4^2p_3s_2 + \\ 2X^+p_4^3p_3s_2 - Y^+p_4p_3^2s_2 - 2Y^+p_4^2p_3s_2 + Z^+p_4^2p_3s_2 + 2X^+p_4^2p_3^2s_2 + X^+p_4^3p_3^2s_2 - Y^+p_4^2p_3^2s_2) \end{array}}{2s_2(p_4s_2p_3^2 + 2p_4s_2p_3 - 2s_2 + 2\delta - p_4s_2 + 2p_4\delta)}$$

$$X^- = 2\int_{1-r_2}^{1}\left[\left(\frac{1-\gamma}{r_2}\right)^{\frac{1}{p_1}} - 1\right] A_{F-}\, d\gamma,$$

$$Y^- = -2\int_{1-r_2}^{1}\left[\left(\frac{1-\gamma}{r_2}\right)^{\frac{1}{p_1}}\right] A_{F-}\, d\gamma - 2\int_{1-\rho}^{1-r_2}\left[\left(\frac{1-\gamma-r_2}{\rho-r_2}\right)^{\frac{1}{p_2}} - 1\right] A_{F-}\, d\gamma,$$

$$Z^- = -2\int_{1-\rho}^{1-r_2}\left(\frac{1-\gamma-r_2}{\rho-r_2}\right)^{\frac{1}{p_2}} A_{F-}\, d\gamma, X^+ = 2\int_{1-s_2}^{1}\left[\left(\frac{1-\gamma}{s_2}\right)^{\frac{1}{p_4}} - 1\right] A_{F+}\, d\gamma,$$

$$Y^+ = -2\int_{1-s_2}^{1}\left[\left(\frac{1-\gamma}{s_2}\right)^{\frac{1}{p_4}}\right] A_{F+}\, d\gamma - 2\int_{1-\rho}^{1-s_2}\left[\left(\frac{1-\gamma-s_2}{\rho-s_2}\right)^{\frac{1}{p_3}} - 1\right] A_{F+}\, d\gamma,$$

$$Z^+ = -2\int_{1-\rho}^{1-s_2}\left(\frac{1-\gamma-s_2}{\rho-s_2}\right)^{\frac{1}{p_3}} A_{F+}\, d\gamma.$$

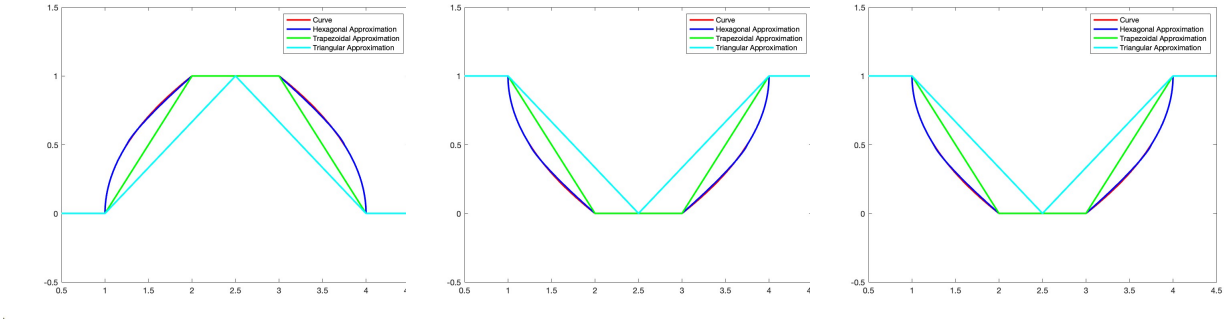
Suppose A be a neutrosophic number defined as given below.

Where the membership function is defined as, $T_A = \begin{cases} (x-1)^{\frac{1}{2}} & , \text{ if } 1 \le x \le 2 \\ 1 & , \text{ if } 2 \le x \le 3 \\ (4-x)^{\frac{1}{2}} & , 3 \le x \le 4 \\ 0 & , \text{ otherwise.} \end{cases}$

Indeterminacy function is described as, $I_A = \begin{cases} 1-(x-1)^{\frac{1}{2}} & , \text{ if } 1 \le x \le 2 \\ 0 & , \text{ if } 2 \le x \le 3 \\ 1-(4-x)^{\frac{1}{2}} & , 3 \le x \le 4 \\ 1 & , \text{ otherwise.} \end{cases}$

The non-membership function is described as, $F_A = \begin{cases} 1-(x-1)^{\frac{1}{2}} & , \text{ if } 1 \leq x \leq 2 \\ 0 & , \text{ if } 2 \leq x \leq 3 \\ 1-(4-x)^{\frac{1}{2}} & , 3 \leq x \leq 4 \\ 1 & , \text{ otherwise.} \end{cases}$

The comparison of the Membership,Indeterminacy and falsity function with GNHNNA, Trapezoidal and triangular neutrosophic numbers is given in FIGURE 1a,1b and 1c.



(a) Membership function (b) Indeterminacy function (c) Non-membership function

Figure 1. Approximation of Neutrosophic Numbers

4. **Approximation of Linear Hexagonal Neutrosophic Numbers with Symmetry**

In this section, the approximation of Linear Hexagonal Neutrosophic Numbers with Symmetry (LHNNS) is given by putting $r = s, n_1 = n_2 = n_3 = n_4 = 1, r_1 = s_1, m_1 = m_2 = m_3 = m_4 = 1, r_2 = s_2, p_1 = p_2 = p_3 = p_4 = 1$. The LHNNS is defined as, $A_{LHNNS} = \left\{T\left(a_1, a_2, a_3, a_4, a_5, a_6; r, r; 1\right)_{(1,1,1,1)}, I\left(b_1, b_2, b_3, b_4, b_5, b_6; r_1, r_1; 1\right)_{(1,1,1,1)}, F\left(c_1, c_2, c_3, c_4, c_5, c_6; r_2, r_2; 1\right)_{(1,1,1,1)}\right\} = \{T\left(a_1, a_2, a_3, a_4, a_5, a_6; r\right), I\left(b_1, b_2, b_3, b_4, b_5, b_6; r_1\right), F\left(c_1, c_2, c_3, c_4, c_5, c_6; r_2\right)\}$

Lemma 4.1. *Let* $\{T\left(a_1, a_2, a_3, a_4, a_5, a_6; r\right), I\left(b_1, b_2, b_3, b_4, b_5, b_6; r_1\right), F\left(c_1, c_2, c_3, c_4, c_5, c_6; r_2\right)\}$ *be a Linear Hexagonal Neutrosophic Number with Symmetry then approximation of the truth function of any curve A which preserves the expected interval criterion is given by,*

$$a_1 = -\frac{6A - 6B + 6Ar - 2Br - 6Cr + 4Dr}{2r} \tag{28}$$

$$a_2 = 6A - 2B - 6C + 4D \quad \text{and} \tag{29}$$

$$a_3 = \frac{6A - 2B - 12C + 4D - 6Ar + 2Br + 6Cr - 4Dr}{2(r-1)}, \tag{30}$$

$$a_4 = \frac{6A_1 - 2B_1 - 12C_1 + 4D_1 - 6A_1r + 2B_1r + 6C_1r - 4D_1r}{2(r-1)},$$

$$a_5 = 6A_1 - 2B_1 - 6C_1 + 4D_1 \quad \text{and } a_6 = -\frac{6A_1 - 6B_1 + 6Ar - 2B_1r - 6C_1r + 4D_1r}{2r}$$

where

$$A = \int_0^r [(\frac{\alpha}{r})A_{T-}]d\alpha, B = \int_0^r [A_{T-}]d\alpha, C = \int_r^1 [(\frac{\alpha - r}{1-r})A_{T-}]d\alpha, \text{ and } D = \int_r^1 [A_{T-}]d\alpha$$

$$A_1 = \int_0^r [(\frac{\alpha}{r})A_{T+}]d\alpha, B_1 = \int_0^r [A_{T+}]d\alpha, C_1 = \int_r^1 [(\frac{\alpha - r}{1-r})A_{T+}]d\alpha, \text{ and } D_1 = \int_r^1 [A_{T+}]d\alpha$$

Proof. By using equation 3.2 the lower limit of expected interval for truth value is given by

$$\int_0^\omega H_{T-}d\alpha = \frac{r(a_1 + a_2) - (a_2 + a_3)(r-1)}{2} \tag{31}$$

By equation 3 we get,

$$d(A, H(A)) = \sqrt{\frac{1}{3}(M + N + O)}$$

where

$$M = \int_0^r (A_{T+} - (a_6 + \left(\frac{\alpha}{r}\right)(a_5 - a_6))^2 d\alpha + \int_r^1 (A_{T+} - (a_5 + \left(\frac{\alpha - r}{1-r}\right)(a_4 - a_5)))^2 d\alpha$$
$$+ \int_0^r (A_{T-} - (a_1 + \left(\frac{\alpha}{r}\right)(a_2 - a_1)))^2 d\alpha + \int_r^1 (A_{T-} - (a_2 + \left(\frac{\alpha - r}{1-r}\right)(a_3 - a_2)))^2 d\alpha,$$

$N = N(b_1, b_2, b_3, b_4, b_5, b_6)$ and $O = O(c_1, c_2, c_3, c_4, c_5, c_6)$

By Equation 9 we get,

$$L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6) = d^2(A, H(A)) + P_1(a_4, a_5, a_6) +$$
$$\lambda_1(\frac{r(a_1 + a_2) - (a_2 + a_3)(r-1)}{2} - \int_0^1 A_{T-}d\alpha) + K(b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2, c_3, c_4, c_5, c_6)$$

Using Equations 10, 11, 12 and KKT theorem we form the partial derivatives corresponding to the Lagrangian multipliers

$$2A - 2B + \frac{\lambda_1 r}{2} + \frac{r(2a_1 + a_2)}{3} = 0 \tag{32}$$

$$2C - 2A - 2E + \frac{\lambda_1}{2} + \frac{r(a_1 + 2a_2)}{3} - \frac{(2a_2 + a_3)(r-1)}{3} = 0 \tag{33}$$

$$-2C - \frac{(a_2 + 2a_3)(r-1)}{3} - \lambda_1 \frac{r-1}{2} = 0 \tag{34}$$

By solving equations 31, 32 and 34 we will get

$$a_1 = -\frac{6A - 6B + 6Ar - 2Br - 6Cr + 4Dr}{2r}, a_2 = 6A - 2B - 6C + 4D \quad \text{and}$$

$$a_3 = \frac{6A - 2B - 12C + 4D - 6Ar + 2Br + 6Cr - 4Dr}{2(r-1)}$$

By using equation 3.2 the lower limit of expected interval for truth value is given by

$$\int_0^\omega H_{T+}d\alpha = \frac{r(a_6 + a_5) - (a_5 + a_4)(r-1)}{2} \tag{35}$$

By equation 3 we get,

$$d(A,H(A)) = \sqrt{\frac{1}{3}(M+N+O)} \quad \text{where,}$$

$$M = \int_0^r (A_{T+} - (a_6 + \left(\frac{\alpha}{r}\right)(a_5 - a_6))^2 d\alpha + \int_r^1 (A_{T+} - (a_5 + \left(\frac{\alpha - r}{1-r}\right)(a_4 - a_5)))^2 d\alpha$$
$$+ \int_0^r (A_{T-} - (a_1 + \left(\frac{\alpha}{r}\right)(a_2 - a_1)))^2 d\alpha + \int_r^1 (A_{T-} - (a_2 + \left(\frac{\alpha - r}{1-r}\right)(a_3 - a_2)))^2 d\alpha,$$

$N = N(b_1, b_2, b_3, b_4, b_5, b_6)$ and $O = O(c_1, c_2, c_3, c_4, c_5, c_6)$. By Equation 9 we get,

$$L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6) = d^2(A, H(A)) + P_2(a_1, a_2, a_3) +$$
$$\lambda_2(\frac{r(a_6 + a_5) - (a_5 + a_4)(r-1)}{2} - \int_0^1 A_{T+} d\alpha) + K(b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2, c_3, c_4, c_5, c_6)$$

Using Equations 13, 14, 15 and KKT theorem we form the partial derivatives corresponding to the Lagrangian multipliers

$$2A_1 - 2B_1 + \frac{\lambda_2 r}{2} + \frac{r(2a_6 + a_5)}{3} = 0 \tag{36}$$

$$2C_1 - 2A_1 - 2E_1 + \frac{\lambda_2}{2} + \frac{r(a_6 + 2a_5)}{3} - \frac{(2a_5 + a_4)(r-1)}{3} = 0 \tag{37}$$

$$-2C_1 - \frac{(a_5 + 2a_4)(r-1)}{3} - \lambda_1 \frac{r-1}{2} = 0 \tag{38}$$

By solving equations 35, 36 and 38 we will get

$$a_4 = \frac{6A_1 - 2B_1 - 12C_1 + 4D_1 - 6A_1 r + 2B_1 r + 6C_1 r - 4D_1 r}{2(r-1)},$$

$$a_5 = 6A_1 - 2B_1 - 6C_1 + 4D_1 \quad \text{and } a_6 = -\frac{6A_1 - 6B_1 + 6Ar - 2B_1 r - 6C_1 r + 4D_1 r)}{2r}$$

□

Lemma 4.2. *Let* $\{T(a_1, a_2, a_3, a_4, a_5, a_6; r), I(b_1, b_2, b_3, b_4, b_5, b_6; r_1),$
$F(c_1, c_2, c_3, c_4, c_5, c_6; r_2)\}$ *be a Linear Hexagonal Neutrosophic Number with Symmetry then approximation of the indeterminacy function of any curve A which preserves the expected interval criterion is given by,*

$$b_1 = -\frac{6CI - 6DI + 4AIr_1 - 6BIr_1 + 6CIr_1 - 2DIr_1}{2r_1}, b_2 = 4AI - 6BI + 6CI - 2DI \tag{39}$$

$$b_3 = \frac{4AI - 12BI + 6CI - 2DI - 4AIr_1 + 6BIr_1 - 6CIr_1 + 2DIr_1}{2(r_1 - 1)} \tag{40}$$

$$b_4 = \frac{4AI_1 - 12BI_1 + 6CI_1 - 2DI_1 - 4AI_1 r_1 + 6BI_1 r_1 - 6CI_1 r_1 + 2DI_1 r_1}{2(r_1 - 1)}, \tag{41}$$

$$b_5 = 4AI_1 - 6BI_1 + 6CI_1 - 2DI_1 \quad \text{and } b_6 = -\frac{6CI_1 - 6DI_1 + 4AI_1 r_1 - 6BI_1 r_1 + 6CI_1 r_1 - 2DI_1 r_1}{2r_1} \tag{42}$$

where $AI=\int_0^{1-r_1}[A_{I-}]d\beta$, $BI=\int_0^{1-r_1}[(\frac{1-\beta-r_1}{1-r_1})A_{I-}]d\beta$, $CI=\int_{1-r_1}^{1}[(\frac{1-\beta}{r_1})A_{I-}]d\beta$ *and* $DI=\int_{1-r_1}^{1}[A_{I-}]d\beta$. $AI_1=\int_0^{1-r_1}[A_{I+}]d\beta$, $BI_1=\int_0^{1-r_1}[(\frac{1-\beta-r_1}{1-r_1})A_{I+}]d\beta$, $CI_1=\int_{1-r_1}^{1}[(\frac{1-\beta}{r_1})A_{I+}]d\beta$ *and* $DI_1=\int_{1-r_1}^{1}[A_{I+}]d\beta$

Proof. By using equation 3.2 the upper limit of the expected interval for indeterminacy value is given by

$$\int_0^1 H_{I-}d\beta=\frac{r_1(b_1+b_2)-(b_2+b_3)(r_1-1)}{2} \tag{43}$$

By equation 3 we get,

$$d(A,H(A))=\sqrt{\frac{1}{3}(M+N+O)}$$

where $M=M(a_1,a_2,a_3,a_4,a_5,a_6)$,

$$N=\int_0^{1-r_1}(A_{I+}-(b_5+\left(\frac{1-\beta-r_1}{1-s_1}\right)(b_4-b_5)))^2d\beta+\int_{1-r_1}^{1}(A_{I+}-(b_6+\left(\frac{1-\beta}{r_1}\right)(b_5-b_6)))^2d\beta$$
$$+\int_0^{1-r_1}(A_{I-}-(b_2+\left(\frac{1-\beta-r_1}{1-r_1}\right)(b_3-b_2)))^2d\beta+\int_{1-r_1}^{1}(A_{I-}-(b_1+\left(\frac{1-\beta}{r_1}\right)(b_2-b_1)))^2d\beta$$

and $O=O(c_1,c_2,c_3,c_4,c_5,c_6)$. By Equation 9 we get,

$$L(a_1,a_2,...,a_6,b_1,b_2,...,b_6,c_1,c_2,...,c_6)=d^2(A,H(A))+P_3(b_4,b_5,b_6)+$$
$$\lambda_3(\frac{r_1(b_1+b_2)-(b_2+b_3)(r_1-1)}{2}-\int_0^1 A_{I-}d\alpha)+K(a_1,a_2,a_3,a_4,a_5,a_6,c_1,c_2,c_3,c_4,c_5,c_6)$$

Using Equations 16, 17, 18, and KKT theorem we form the partial derivatives corresponding to the Lagrangian multipliers

$$2CI-2EI+\frac{\lambda_3 r_1}{2}+\frac{r_1(2b_1+b_2)}{3}=0 \tag{44}$$

$$2BI-2AI-2CI+\frac{\lambda_3}{2}+\frac{r_1(b_1+2b_2)}{3}-\frac{(2b_2+b_3)(r_1-1)}{3}=0 \tag{45}$$

$$-2BI-\frac{(b2+2b3)(r1-1)}{3}-\lambda_3(\frac{r}{2}-\frac{1}{2})=0 \tag{46}$$

By solving equations 43, 44, 45 and 46 we will get

$$b_1=-\frac{6CI-6DI+4AIr_1-6BIr_1+6CIr_1-2DIr_1}{2r_1},$$

$$b_2=4AI-6BI+6CI-2DI \quad \text{and}$$

$$b_3=(4AI-12BI+6CI-2DI-4AIr_1+6BIr_1-6CIr_1+2DIr_1)/(2(r_1-1)).$$

By using equation 3.2 the upper limit of the expected interval for indeterminacy value is given by

$$\int_0^1 H_{I+}d\beta=\frac{r_1(b_6+b_5)-(b_5+b_4)(r_1-1)}{2} \tag{47}$$

By equation 3 we get,

$$d(A,H(A))=\sqrt{\frac{1}{3}(M+N+O)} \quad \text{where,}$$

$M = M(a_1, a_2, a_3, a_4, a_5, a_6)$,

$$N = \int_0^{1-r_1} (A_{I+} - (b_5 + \left(\frac{1-\beta-r_1}{1-s_1}\right)(b_4 - b_5)))^2 d\beta + \int_{1-r_1}^{1} (A_{I+} - (b_6 + \left(\frac{1-\beta}{r_1}\right)(b_5 - b_6)))^2 d\beta$$
$$+ \int_0^{1-r_1} (A_{I-} - (b_2 + \left(\frac{1-\beta-r_1}{1-r_1}\right)(b_3 - b_2)))^2 d\beta + \int_{1-r_1}^{1} (A_{I-} - (b_1 + \left(\frac{1-\beta}{r_1}\right)(b_2 - b_1)))^2 d\beta$$

By Equation 9 we get,

$$L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6) = d^2(A, H(A)) + P(b_1, b_2, b_3) +$$
$$\lambda_4(\frac{r_1(b_6 + b_5) - (b_5 + b_4)(r_1 - 1)}{2} - \int_0^1 A_{I+} d\beta) + K(a_1, a_2, a_3, a_4, a_5, a_6, c_1, c_2, c_3, c_4, c_5, c_6)$$

Using Equations 19, 20, 21, and KKT theorem we form the partial derivatives corresponding to the Lagrangian multipliers

$$2CI_1 - 2EI_1 + \frac{\lambda_4 r_1}{2} + \frac{r_1(2b_6 + b_5)}{3} = 0 \tag{48}$$

$$2BI_1 - 2AI_1 - 2CI_1 + \frac{\lambda_4}{2} + \frac{r_1(b_6 + 2b_5)}{3} - \frac{(2b_5 + b_4)(r_1 - 1)}{3} = 0 \tag{49}$$

$$-2BI_1 - \frac{(b_5 + 2b_4)(r1 - 1)}{3} - \lambda_4(\frac{r}{2} - \frac{1}{2}) = 0 \tag{50}$$

By solving equations 47, 48, 49 and 50 we will get

$$b_4 = \frac{4AI_1 - 12BI_1 + 6CI_1 - 2DI_1 - 4AI_1 r_1 + 6BI_1 r_1 - 6CI_1 r_1 + 2DI_1 r_1}{2(r_1 - 1)},$$

$b_5 = 4AI_1 - 6BI_1 + 6CI_1 - 2DI_1$ and $b_6 = -\dfrac{6CI_1 - 6DI_1 + 4AI_1 r_1 - 6BI_1 r_1 + 6CI_1 r_1 - 2DI_1 r_1}{2r_1}$

□

Lemma 4.3. *Let* $\{T(a_1, a_2, a_3, a_4, a_5, a_6; r), I(b_1, b_2, b_3, b_4, b_5, b_6; r_1),$
$F(c_1, c_2, c_3, c_4, c_5, c_6; r_2)\}$ *be a Linear Hexagonal Neutrosophic Number with Symmetry then approximation of the falsity function of any curve A which preserves the expected interval criterion is given by,*

$$c_1 = -\frac{6CF - 6DF + 4AFr_2 - 6BFr_2 + 6CFr_1 - 2DFr_2}{2r_2}, c_2 = 4AF - 6BF + 6CF - 2DF \tag{51}$$

$$c_3 = \frac{4AF - 12BF + 6CF - 2DF - 4AFr_2 + 6BFr_2 - 6CFr_2 + 2DFr_2}{2(r_2 - 1)} \tag{52}$$

$$c_4 = \frac{4AF_1 - 12BF_1 + 6CF_1 - 2DF_1 - 4AF_1 r_2 + 6BF_1 r_2 - 6CF_1 r_2 + 2DF_1 r_2}{2(r_2 - 1)}, \tag{53}$$

$$c_5 = 4AF_1 - 6BF_1 + 6CF_1 - 2DF_1 \quad \text{and} \tag{54}$$

$$c_6 = -\frac{6CF_1 - 6DF_1 + 4AF_1 r_2 - 6BF_1 r_2 + 6CF_1 r_1 - 2DF_1 r_2}{2r_2} \tag{55}$$

where $AF=\int_0^{1-r_2}[A_{F-}]d\gamma$, $BF=\int_0^{1-r_2}[(\frac{1-\gamma-r_2}{1-r_2})A_{F-}]d\gamma$, $CF=\int_{1-r_2}^{1}[(\frac{1-\gamma}{r_2})A_{F-}]d\gamma$ *and* $DF=\int_{1-r_2}^{1}[A_{F-}]d\gamma$. $AF_1=\int_0^{1-r_2}[A_{F+}]d\gamma$, $BF_1=\int_0^{1-r_2}[(\frac{1-\gamma-r_2}{1-r_2})A_{F+}]d\gamma$, $CF_1=\int_{1-r_2}^{1}[(\frac{1-\gamma}{r_2})A_{F+}]d\gamma$ *and* $DF_1=\int_{1-r_2}^{1}d\gamma$.

Proof. By using equation 3.2 the upper limit of expected interval for falsity value is given by

$$\int_0^1 H_{F-}d\gamma=\frac{r_2(c_1+c_2)-(c_2+c_3)(r_2-1)}{2} \tag{56}$$

By equation 3 we get,

$$d(A,H(A))=\sqrt{\frac{1}{3}(M+N+O)} \text{ where,}$$

$M=M(a_1,a_2,a_3,a_4,a_5,a_6)$, $N=N(b_1,b_2,b_3,b_4,b_5,b_6)$ and

$$O=\int_0^{1-r_2}(A_{F+}-(c_5+\left(\frac{1-\gamma-r_2}{1-r_2}\right)(c_4-c_5)))^2d\gamma+\int_{1-r_2}^{1}(A_{F+}-(c_6+\left(\frac{1-\gamma}{r_2}\right)(c_5-c_6)))^2d\gamma$$
$$+\int_0^{1-r_2}(A_{F-}-(c_2+\left(\frac{1-\gamma-r_2}{1-r_2}\right)(c_3-c_2)))^2d\gamma+\int_{1-r_2}^{1}(A_{F-}-(c_1+\left(\frac{1-\gamma}{r_2}\right)(c_2-c_1)))^2d\gamma$$

By Equation 9 we get,

$$L(a_1,a_2,...,a_6,b_1,b_2,...,b_6,c_1,c_2,...,c_6)=d^2(A,H(A))+P(c_4,c_5,c_6)+$$
$$\lambda_5(\frac{r_2(c_1+c_2)-(c_2+c_3)(r_2-1)}{2}-\int_0^1 A_{F-}d\gamma)+K(a_1,a_2,a_3,a_4,a_5,a_6,b_1,b_2,b_3,b_4,b_5,b_6)$$

Using Equations 22, 23, 24, and KKT theorem we form the partial derivatives corresponding to the Lagrangian multipliers

$$2CF-2EI+\frac{\lambda_4 r_2}{2}+\frac{r_2(2c_1+c_2)}{3}=0 \tag{57}$$

$$2BF-2AF-2CF+\frac{\lambda_4}{2}+\frac{r_2(c_1+2c_2)}{3}-\frac{(2c_2+c_3)(r_2-1)}{3}=0 \tag{58}$$

$$-2BI-\frac{(c_2+2c_3)(r1-1)}{3}-\lambda_4(\frac{r}{2}-\frac{1}{2})=0 \tag{59}$$

By solving equations 56, 57, 58 and 59 we will get

$$c_1=-\frac{6CF-6DF+4AFr_2-6BFr_2+6CFr_1-2DFr_2}{2r_2}, c_2=4AF-6BF+6CF-2DF$$

$$c_3=\frac{4AF-12BF+6CF-2DF-4AFr_2+6BFr_2-6CFr_2+2DFr_2}{2(r_2-1)}$$

. By using equation 3.2 the upper limit of expected interval for falsity value is given by

$$\int_0^1 H_{F+}d\alpha=\frac{r_2(c_6+c_5)-(c_5+c_4)(r_2-1)}{2} \tag{60}$$

By equation 3 we get,

$$d(A,H(A))=\sqrt{\frac{1}{3}(M+N+O)} \quad \text{where,}$$

$M = M(a_1, a_2, a_3, a_4, a_5, a_6)$, $N = N(b_1, b_2, b_3, b_4, b_5, b_6)$ and

$$O = \int_0^{1-r_2} (A_{F+} - (c_5 + \left(\frac{1-\gamma-r_2}{1-r_2}\right)(c_4 - c_5)))^2 d\gamma + \int_{1-r_2}^{1} (A_{F+} - (c_6 + \left(\frac{1-\gamma}{r_2}\right)(c_5 - c_6)))^2 d\gamma$$
$$+ \int_0^{1-r_2} (A_{F-} - (c_2 + \left(\frac{1-\gamma-r_2}{1-r_2}\right)(c_3 - c_2)))^2 d\gamma + \int_{1-r_2}^{1} (A_{F-} - (c_1 + \left(\frac{1-\gamma}{r_2}\right)(c_2 - c_1)))^2 d\gamma$$

By Equation 9 we get,

$$L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6) = d^2(A, H(A)) + P(c_1, c_2, c_3) +$$
$$\lambda_6(\frac{r_2(c_6 + c_5) - (c_5 + c_4)(r_2 - 1)}{2} - \int_0^1 A_{F+} d\gamma) + K(a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6)$$

Using Equations 25, 26, 27, and KKT theorem we form the partial derivatives corresponding to the Lagrangian multipliers

$$2CF_1 - 2DF_1 + \frac{\lambda_6 r_2}{2} + \frac{r_2(2c_6 + c_5)}{3} = 0 \tag{61}$$

$$2BF_1 - 2AF_1 - 2CF_1 + \frac{\lambda_6}{2} + \frac{r_2(c_6 + 2c_5)}{3} - \frac{(2c_5 + c_4)(r_2 - 1)}{3} = 0 \tag{62}$$

$$-2BF_1 - \frac{(c_5 + 2c_4)(r1 - 1)}{3} - \lambda_6(\frac{r}{2} - \frac{1}{2}) = 0 \tag{63}$$

By solving equations 47, 61, 62 and 63 we will get

$$c_4 = \frac{4AF_1 - 12BF_1 + 6CF_1 - 2DF_1 - 4AF_1 r_2 + 6BF_1 r_2 - 6CF_1 r_2 + 2DF_1 r_2}{2(r_2 - 1)}, \tag{64}$$

$$c_5 = 4AF_1 - 6BF_1 + 6CF_1 - 2DF_1 \quad \text{and} \tag{65}$$

$$c_6 = -\frac{6CF_1 - 6DF_1 + 4AF_1 r_2 - 6BF_1 r_2 + 6CF_1 r_1 - 2DF_1 r_2}{2r_2} \tag{66}$$

□

Note 4.4. *The solution of the equations in Lemma 4.1 to Lemma 4.3 are solved using Matlab.*

Example 4.5. Suppose A be a neutrosophic number defined as given below.

Where the membership function is defined as, $T_A = \begin{cases} (x-1)^{\frac{1}{2}} & , \text{ if } 1 \le x \le 2 \\ 1 & , \text{ if } 2 \le x \le 3 \\ (4-x)^{\frac{1}{2}} & , 3 \le x \le 4 \\ 0 & , \text{ otherwise.} \end{cases}$

Indeterminacy function is described as, $I_A = \begin{cases} 1-(x-1)^{\frac{1}{2}} & , \text{ if } 1 \le x \le 2 \\ 0 & , \text{ if } 2 \le x \le 3 \\ 1-(4-x)^{\frac{1}{2}} & , 3 \le x \le 4 \\ 1 & , \text{ otherwise.} \end{cases}$

The non-membership function is described as, $F_A = \begin{cases} 1-(x-1)^{\frac{1}{2}} & , \text{ if } 1 \le x \le 2 \\ 0 & , \text{ if } 2 \le x \le 3 \\ 1-(4-x)^{\frac{1}{2}} & , 3 \le x \le 4 \\ 1 & , \text{ otherwise.} \end{cases}$

The Hexagonal approximation of this curve H(A) is given by, (here $r = r_1 = r_2 = .5$)

By using Lemma 4.1 - Lemma 4.3 we get $a_1 = 1$, $a_2 = \frac{5}{4}$, $a_3 = 2$, $a_4 = 3$, $a_5 = \frac{15}{4}$, $a_6 = 4$, $b_1 = 1$, $b_2 = \frac{5}{4}$, $b_3 = 2$, $b_4 = 3$, $b_5 = \frac{15}{4}$, $b_6 = 4$, $c_1 = 1$, $c_2 = \frac{5}{4}$, $c_3 = 2$, $c_4 = 3$, $c_5 = \frac{15}{4}$ $c_6 = 4$, where the membership function is defined as,

$$T_{H(A)} = \begin{cases} .5\left(\frac{x-1}{\frac{5}{4}-1}\right) & , \text{ if } 1 \le x \le \frac{5}{4} \\ .5 + .5\left(\frac{x-\frac{5}{4}}{2-\frac{5}{4}}\right) & , \text{ if } \frac{5}{4} \le x \le 2 \\ 1 & , \text{ if } 2 \le x \le 3 \\ .5 + .5\left(\frac{x-\frac{15}{4}}{3-\frac{15}{4}}\right) & , \text{ if } 3 \le x \le \frac{15}{4} \\ .5\left(\frac{x-4}{\frac{15}{4}-4}\right) & , \text{ if } \frac{15}{4} \le x \le 4 \\ 0 & , \text{ otherwise.} \end{cases}$$

The comparison of membership function with LHNNS, Trapezoidal (TI) and triangular (Tr) neutrosophic numbers is given in FIGURE 2a. Indeterminacy function can be described as,

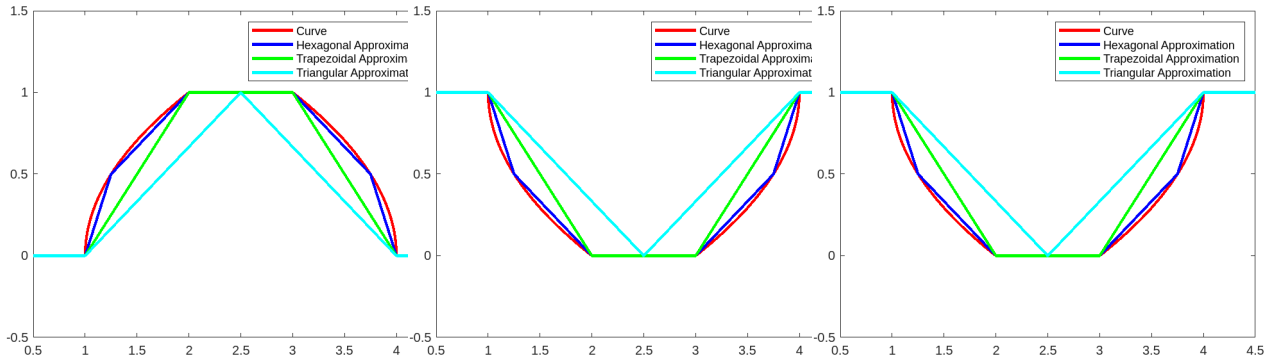
$$I_{H(A)} = \begin{cases} 1-.5\left(\frac{x-1}{\frac{5}{4}-1}\right) & , \text{ if } 1 \le x \le \frac{5}{4} \\ 1-(.5 + .5\left(\frac{x-\frac{5}{4}}{2-\frac{5}{4}}\right)) & , \text{ if } \frac{5}{4} \le x \le 2 \\ 0 & , \text{ if } 2 \le x \le 3 \\ 1-(.5 + .5\left(\frac{x-\frac{15}{4}}{3-\frac{15}{4}}\right)) & , \text{ if } 3 \le x \le \frac{15}{4} \\ 1-(.5\left(\frac{x-4}{\frac{15}{4}-4}\right)) & , \text{ if } \frac{15}{4} \le x \le 4 \\ 1 & , \text{ otherwise.} \end{cases}$$

The comparison of Indeterminacy function with LHNNS, Trapezoidal and triangular neutrosophic numbers is given in FIGURE 2b. Non-membership function can be described as,

$$F_{H(A)} = \begin{cases} 1-.5\left(\frac{x-1}{\frac{5}{4}-1}\right) & , \text{ if } 1 \le x \le \frac{5}{4} \\ 1-(.5 + .5\left(\frac{x-\frac{5}{4}}{2-\frac{5}{4}}\right)) & , \text{ if } \frac{5}{4} \le x \le 2 \\ 0 & , \text{ if } 2 \le x \le 3 \\ 1-(.5 + .5\left(\frac{x-\frac{15}{4}}{3-\frac{15}{4}}\right)) & , \text{ if } 3 \le x \le \frac{15}{4} \\ 1-(.5\left(\frac{x-4}{\frac{15}{4}-4}\right)) & , \text{ if } \frac{15}{4} \le x \le 4 \\ 1 & , \text{ otherwise.} \end{cases}$$

The comparison of non-membership function with LHNNS, Trapezoidal ($a_2 = a_3, a_4 = a_5, r = s = 1, b_2 = b_3, b_4 = b_5, r_1 = s_1 = 1, c_2 = c_3, c_4 = c_5, r_2 = s_2 = 1$) and triangular ($a_2 = a_3 = a_4 = a_5, r = s = 1, b_2 = b_3 = b_4 = b_5, r_1 = s_1 = 1, c_2 = c_3 = c_4 = c_5, r_2 = s_2 = 1$)

neutrosophic numbers is given in the FIGURE 2c.



(a) Membership function (b) Indeterminacy function (c) Non-membership function

Figure 2. Approximation of Neutrosophic Numbers

$$d(A, H(A)) = \sqrt{\frac{1}{3}(M + N + O)} \text{ where,}$$

$M = \int_0^r (A_{T+} - (a_6 + \left(\frac{\alpha}{r}\right)(a_5 - a_6))^2 d\alpha + \int_r^1 (A_{T+} - (a_5 + \left(\frac{\alpha - r}{1-r}\right)(a_4 - a_5)))^2 d\alpha + \int_0^r (A_{T-} - (a_1 + \left(\frac{\alpha}{r}\right)(a_2 - a_1)))^2 d\alpha + \int_r^1 (A_{T-} - (a_2 + \left(\frac{\alpha - r}{1-r}\right)(a_3 - a_2)))^2 d\alpha,$

$N = \int_0^{1-r_1} (A_{I+} - (b_5 + \left(\frac{1-\beta-r_1}{1-s_1}\right)(b_4 - b_5)))^2 d\beta + \int_{1-r_1}^1 (A_{I+} - (b_6 + \left(\frac{1-\beta}{r_1}\right)(b_5 - b_6)))^2 d\beta + \int_0^{1-r_1} (A_{I-} - (b_2 + \left(\frac{1-\beta-r_1}{1-r_1}\right)(b_3 - b_2)))^2 d\beta + \int_{1-r_1}^1 (A_{I-} - (b_1 + \left(\frac{1-\beta}{r_1}\right)(b_2 - b_1)))^2 d\beta$ and

$O = \int_0^{1-r_2} (A_{F+} - (c_5 + \left(\frac{1-\gamma-r_2}{1-r_2}\right)(c_4 - c_5)))^2 d\gamma + \int_{1-r_2}^1 (A_{F+} - (c_6 + \left(\frac{1-\gamma}{r_2}\right)(c_5 - c_6)))^2 d\gamma + \int_0^{1-r_2} (A_{F-} - (c_2 + \left(\frac{1-\gamma-r_2}{1-r_2}\right)(c_3 - c_2)))^2 d\gamma + \int_{1-r_2}^1 (A_{F-} - (c_1 + \left(\frac{1-\gamma}{r_2}\right)(c_2 - c_1)))^2 d\gamma$

d(A, H(A))= 0.0645; d(A, Tl(A))= 0.3651; d(A, Tr(A))= 0.6831.

Example 4.6. Suppose A be a neutrosophic number defined as given below.

Where the membership function is defined as, $T_A = \begin{cases} (x-1)^2 & , \text{ if } 1 \le x \le 2 \\ (4-x)^2 & , 2 \le x \le 3 \\ 0 & , \text{ otherwise.} \end{cases}$

The indeterminacy function is be described as, $I_A = \begin{cases} 1-(x-1)^2 & , \text{ if } 1 \le x \le 2 \\ 1-(4-x)^2 & , 2 \le x \le 3 \\ 1 & , \text{ otherwise.} \end{cases}$

The non-membership function is described as, $F_A = \begin{cases} 1-(x-1)^2 & , \text{ if } 1 \le x \le 2 \\ 1-(4-x)^2 & , 2 \le x \le 3 \\ 1 & , \text{ otherwise.} \end{cases}$

The Hexagonal approximation of this curve H(A) is given by, (here $r = r_1 = r_2 = .5$)

By using Lemma 4.1 - Lemma 4.3 we get

$a_1 = 1$, $a_2 = 1 + \frac{1}{\sqrt{2}}$, $a_3 = 2$, $a_4 = 2$, $a_5 = 3 - \frac{1}{\sqrt{2}}$, $a_6 = 3$,$b_1 = 1$, $b_2 = 1 + \frac{1}{\sqrt{2}}$, $b_3 = 2$, $b_4 = 2$,

$b_5 = 3 - \frac{1}{\sqrt{2}}$, $b_6 = 3$, $c_1 = 1$, $c_2 = 1 + \frac{1}{\sqrt{2}}$, $c_3 = 2$, $c_4 = 2$, $c_5 = 3 - \frac{1}{\sqrt{2}}$, $c_6 = 3$, where the membership function is defined as,

$$T_{H(A)} = \begin{cases} \frac{1}{\sqrt{2}}(x-1) & , \text{ if } 1 \le x \le 1 + \frac{1}{\sqrt{2}} \\ 1 + \dfrac{x-2}{2-\sqrt{2}} & , \text{ if } 1 + \frac{1}{\sqrt{2}} \le x \le 2 \\ 1 + \dfrac{2-x}{2-\sqrt{2}} & , \text{ if } 2 \le x \le 3 - \frac{1}{\sqrt{2}} \\ \dfrac{3-x}{\sqrt{2}} & , \text{ if } 3 - \frac{1}{\sqrt{2}} \le x \le 3 \\ 0 & , \text{ otherwise.} \end{cases}$$

The comparison of membership function with LHNNS, Trapezoidal and triangular neutrosophic numbers is given in the FIGURE 3a. The indeterminacy function can be described as,

$$I_{H(A)} = \begin{cases} 1 - \frac{1}{\sqrt{2}}(x-1) & , \text{ if } 1 \le x \le 1 + \frac{1}{\sqrt{2}} \\ \dfrac{2-x}{2-\sqrt{2}} & , \text{ if } 1 + \frac{1}{\sqrt{2}} \le x \le 2 \\ \dfrac{x-2}{2-\sqrt{2}} & , \text{ if } 2 \le x \le 3 - \frac{1}{\sqrt{2}} \\ 1 - \dfrac{3-x}{\sqrt{2}} & , \text{ if } 3 - \frac{1}{\sqrt{2}} \le x \le 3 \\ 1 & , \text{ otherwise.} \end{cases}$$

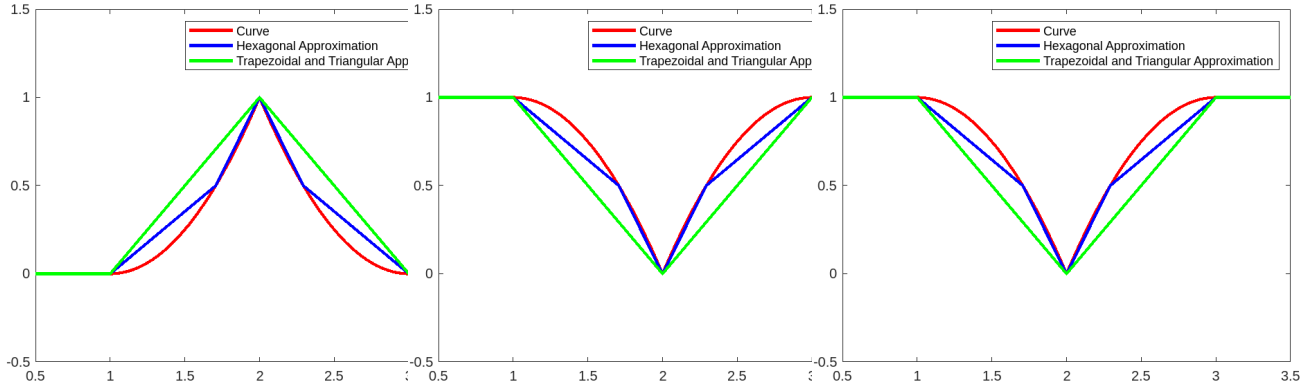
The comparison of the Indeterminacy function with LHNNS, Trapezoidal and triangular neutrosophic numbers is given in FIGURE 3b. Non-membership function can be described as,

$$F_{H(A)} = \begin{cases} 1 - \frac{1}{\sqrt{2}}(x-1) & , \text{ if } 1 \le x \le 1 + \frac{1}{\sqrt{2}} \\ \dfrac{2-x}{2-\sqrt{2}} & , \text{ if } 1 + \frac{1}{\sqrt{2}} \le x \le 2 \\ \dfrac{x-2}{2-\sqrt{2}} & , \text{ if } 2 \le x \le 3 - \frac{1}{\sqrt{2}} \\ 1 - \dfrac{3-x}{\sqrt{2}} & , \text{ if } 3 - \frac{1}{\sqrt{2}} \le x \le 3 \\ 1 & , \text{ otherwise.} \end{cases}$$

The comparison of non-membership function with LHNNS, Trapezoidal and triangular neutrosophic numbers is given in FIGURE 3c. d(A, H(A))= 0.1294; d(A, Tl(A))= 0.2582; d(A, Tr(A))= 0.2582.

From the above two examples, we can conclude that the distance in the hexagonal case is minimal.

Theorem 4.7. *Let* $\{T(a_1, a_2, a_3, a_4, a_5, a_6; r), I(b_1, b_2, b_3, b_4, b_5, b_6; r_1), F(c_1, c_2, c_3, c_4, c_5, c_6; r_2)\}$ *be a Linear Hexagonal Neutrosophic Number with Symmetry and T(A) denotes the approximation with LHNNS then,* $V_T(T(A)) = ((0.3333(r-1)^2 - 0.5000r(r-1))(6A_1 - 2B_1 - 12C_1 + 4D_1 - 6A_1r + 2B_1r + 6C_1r - 4E_1r))/(2*r-2) - 0.0833r(6A - 6B + 6Ar - 2Br - 6Cr + 4rD) - 0.0833r(6A_1 - 6B_1 + 6A_1r - 2B_1r - 6C_1r + 4E_1r) - (6A - 2B - 6C + 4E)(0.3333(r-1)^2 - 0.5000r(r-1) + 0.5000r^2 - 0.5000) - (0.3333(r-1)^2 - 0.5000r(r -$



(a) Membership function (b) Indeterminacy function (c) Non-membership function

Figure 3. Functions for Neutrosophic Numbers

$1)+0.5000r^2-0.5000)(6A_1-2B_1-6C_1+4E_1)+((0.3333(r-1)^2-0.5000r(r-1))(6A-2B-12C+4E-6Ar+2Br+6Cr-4rE))/(2r-2)$

Proof. We know,

$$a_1=-\frac{6A-6B+6Ar-2Br-6Cr+4Dr}{2r}, a_2=6A-2B-6C+4D,$$

$$a_3=\frac{6A-2B-12C+4D-6Ar+2Br+Cr-4Dr}{2(r-1)},$$

$$a_4=\frac{6A_1-2B_1-12C_1+4D_1-6A_1r+2B_1r+6C_1r-4D_1r}{2(r-1)},$$

$$a_5=6A_1-2B_1-6C_1+4D_1 \text{ and } a_6=-\frac{6A_1-6B_1+6Ar-2B_1r-6C_1r+4D_1r}{2r},$$

from definition 2.3 we get

$V_T(T(A)) = \left[\frac{1}{2}-\frac{1}{3}\right]r^2a_1 + \left[\frac{1-r^2}{2}-\frac{(1-r)^2}{3}-r\frac{1-r}{2}\right]a_2 + \left[\frac{(1-r)^2}{3}+r\frac{1-r}{2}\right]a_3 + \left[\frac{(1-r)^2}{3}+r\frac{1-r}{2}\right]a_4 + \left[\frac{1-r^2}{2}-\frac{(1-r)^2}{3}-r\frac{1-r}{2}\right]a_5 + \left[\frac{1}{2}-\frac{1}{3}\right]r^2a_6$

Substituting the values of a_1-a_6 and simplifying we get, $V_T(T(A)) = ((0.3333(r-1)^2 - 0.5000r(r-1))(6A_1-2B_1-12C_1+4D_1-6A_1r+2B_1r+6C_1r-4E_1r))/(2*r-2)-0.0833r(6A-6B+6Ar-2Br-6Cr+4rD)-0.0833r(6A_1-6B_1+6A_1r-2B_1r-6C_1r+4E_1r)-(6A-2B-6C+4E)(0.3333(r-1)^2-0.5000r(r-1)+0.5000r^2-0.5000)-(0.3333(r-1)^2-0.5000r(r-1)+0.5000r^2-0.5000)(6A_1-2B_1-6C_1+4E_1)+((0.3333(r-1)^2-0.5000r(r-1))(6A-2B-12C+4E-6Ar+2Br+6Cr-4rE))/(2r-2)$ □

Theorem 4.8. *Let* $\{T(a_1,a_2,a_3,a_4,a_5,a_6;r), I(b_1,b_2,b_3,b_4,b_5,b_6;r_1), F(c_1,c_2,c_3,c_4,c_5,c_6;r_2)\}$ *be a Linear Hexagonal Neutrosophic Number with Symmetry and T(A) denotes the approximation with LHNNS then,*

$V_I(T(A)) = ((0.3333(r-1)^2 - 0.5000r(r-1))(2AI-6BI+3CI-DI-2AIr_1+3BIr_1-3CIr_1+DIr_1))/(r_1-1)-(0.3333(r-1)^2-0.5000r(r-1)+0.5000r^2-0.5000)(4AI_1-6BI_1+$

$6CI_1 - 2DI_1) - (0.3333(r-1)^2 - 0.5000r(r-1) + 0.5000r^2 - 0.5000)(4AI - 6BI + 6CI - 2DI) + ((0.3333(r-1)^2 - 0.5000r(r-1))(2AI_1 - 6BI_1 + 3CI_1 - DI_1 - 2AI_1r_1 + 3BI_1r_1 - 3CI_1r_1 + DI_1r_1))/(r_1 - 1) - (0.1667r^2(3CI - 3DI + 2AIr_1 - 3BIr_1 + 3CIr_1 - DIr_1))/r_1 - (0.1667r^2(3CI_1 - 3DI_1 + AI_1r_1 - 3BI_1r_1 + 3CI_1r_1 - DI_1r_1))/r_1$

Proof. The proof is similar to theorem 4.7 □

Theorem 4.9. *Let* $\{T(a_1, a_2, a_3, a_4, a_5, a_6; r),$
$I(b_1, b_2, b_3, b_4, b_5, b_6; r_1), F(c_1, c_2, c_3, c_4, c_5, c_6; r_2)\}$ *be a Linear Hexagonal Neutrosophic Number with Symmetry and T(A) denotes the approximation with LHNNS then,*
$V_F(T(A)) = ((0.3333(r-1)^2 - 0.5000r(r-1))(2AF - 6BF + 3CF - DF - 2AFr_1 + 3BFr_1 - 3CFr_1 + DFr_1))/(r_1 - 1) - (0.3333(r-1)^2 - 0.5000r(r-1) + 0.5000r^2 - 0.5000)(4AF_1 - 6BF_1 + 6CF_1 - 2DF_1) - (0.3333(r-1)^2 - 0.5000r(r-1) + 0.5000r^2 - 0.5000)(4AF - 6BF + 6CF - 2DF) + ((0.3333(r-1)^2 - 0.5000r(r-1))(2AF_1 - 6BF_1 + 3CF_1 - DF_1 - 2AF_1r_1 + 3BF_1r_1 - 3CF_1r_1 + DF_1r_1))/(r_1 - 1) - (0.1667r^2(3CF - 3DF + 2AFr_1 - 3BFr_1 + 3CFr_1 - DFr_1))/r_1 - (0.1667r^2(3CF_1 - 3DF_1 + AF_1r_1 - 3BF_1r_1 + 3CF_1r_1 - DF_1r_1))/r_1$

Proof. The proof is similar to theorem 4.7 □

5. Quantitative Analysis of Decision-Making Research

This section discusses a decision-making problem to show the effectiveness of approximation in LHNNS. Here decision-makers can use any linear or non-linear neutrosophic number as parameters. The Neutrosophic numbers, which are employed to express linguistic variables, are approximated to LHNNS. Then we will find an LHNNS for each characteristic using weighted arithmetic addition and use the values of LHNNS to accomplish single-valued neutrosophic numbers. By using the score function, deneutrosophication is done. Finally, we will compare the score corresponding to each available alternative and choose the best one. This approach can compare any decision-making issue using any neutrosophic number. This hexagonal approximation will get a better outcome if two separate data sets need to be analyzed. The following is an algorithm for determining the ideal result.

(1) Approximate each linguistic variable to LHNNS.
(2) Calculate the LHNNS corresponding to each attribute using scalar multiplication and addition operations.
(3) Find the values corresponding to each attribute which converts them into a single-valued neutrosophic number.
(4) Find the score function.
(5) Ranking is given according to score, and the best outcome is selected.

5.1. *Numerical Example*

Consider a problem for investing in a stock market that will give excellent yield. The stock markets were shortlisted into five $x_i (1 \leq i \leq 5)$ categories by considering the feasibility. Now, the investor has to choose the best stock market by evaluating the criteria of c_1 - Earnings Growth, c_2 - Revenue Growth, c_3 - Profit Margins, c_4 - Dividend Yield, and c_5 - Valuation Metrics.

Step:1 Let table 1 denote the decision-making matrix with linguistic variables given below. SU-Subdued; RE-Reasonable; EX-Exceptional; SUB-Substantial; SC-Scant.

Step:2 Corresponding to each criterion, the linguistic variables are given in terms of non-linear trapezoidal numbers in table 2.

Step:3 The weights $w_1 = 0.4$, $w_2 = 0.2$, $w_3 = 0.15$, $w_4 = 0.17$ and $w_5 = 0.08$ are assigned for each criteria c_1 to c_5 respectively.

Step:4 The approximation is applied to each linguistic variable and is given in Table 3.

Step:5 Table 4 gives hexagonal values corresponding to each x_i using addition and scalar multiplication.

Step:6 Table 5 shows the single-valued neutrosophic number obtained using Values (Definition 2.3).

Step:7 The score function is evaluated and shown in Table 6. Also, we get $x_2 > x_3 > x_4 > x_5 > x_1$. Therefore x_2 is the best possible outcome.

Table 1. Decision making Matrix

	c_1	c_2	c_3	c_4	c_5
x_1	SU	RE	SU	EX	RE
x_2	EX	EX	EX	RE	RE
x_3	SUB	SU	RE	EX	SC
x_4	SUB	SC	EX	SU	RE
x_5	SU	EX	SUB	RE	SC

Table 2. Linguistic Variable (LV) and the corresponding Nonlinear Trapezoidal Number (NLTN)

LV	NLTN
Scant	[T(0,0,0,0;0.5,0.5),I(3,5,6,8;0.5,0.5), F(3,5,6,8;0.5,0.5)]
Subdued	[T(0,2,3,5;0.5,0.5), I(2,4,5,7;0.5,0.5),F(2,4,5,7;0.5,0.5)]
Reasonable	[T(1,3,4,6;0.5,0.5), I(1,3,4,6;0.5,0.5), F(1,3,4,6;0.5,0.5)]
Substantial	[T(2,4,5,7;0.5,0.5), I(0,2,3,5;0.5,0.5), F(0,2,3,5;0.5,0.5)]
Exceptional	[T(3,5,6,8;0.5,0.5), I(0,0,0,0;0.5,0.5), F(0,0,0,0;0.5,0.5)]

Table 3. Approximation of NLTN to LHNNS

LV	LHNNS
Scant	[T(0,0,0,0,0,0;0.5,0.5), I(3,3.5,5,6,7.5,8;0.5,0.5),F(3,3.5,5,6,7.5,8;0.5,0.5)]
Subdued	[T(0,0.5,2,3,4.5,5;0.5,0.5),I(2,2.5,4,5,6.5,7;0.5,0.5), F(2,2.5,4,5,6.5,7;0.5,0.5)]
Reasonable	[T(1,1.5,3,4,5.5,6;0.5,0.5), I(1,1.5,3,4,5.5,6;0.5,0.5), F(1,1.5,3,4,5.5,6;0.5,0.5)]
Substantial	[T(2,2.5,4,5,6.5,7;0.5,0.5),I(0,0.5,2,3,4.5,5;0.5,0.5), F(0,0.5,2,3,4.5,5;0.5,0.5)]
Exceptional	[T(3,3.5,5,6,7.5,8;0.5,0.5), I(0,0,0,0,0,0;0.5,0.5), F(0,0,0,0,0,0;0.5,0.5)]

Table 4. Hexagonal Neutrosophic number corresponding to each x_i

Attribute	LHNNS
x_1	[T(0.79,1.29,2.79,3.79,5.29,5.79), I(1.38,1.79,3.04,3.87,5.11,5.53), F(1.38,1.79,3.04,3.87,5.11,5.53)]
x_2	[T(2.50,3.00,4.50,5.50,7.00,7.50), I(0.25,0.37,0.75,1.00,1.37,1.50), F(0.25,0.37,0.75,1.00,1.37,1.50)]
x_3	[T(1.46,1.92,3.30,4.22,5.60,6.06), I(0.79,1.20,2.45,3.28,4.52,4.94), F(0.79,1.20,2.45,3.28,4.52,4.94)]
x_4	[T(1.33,1.73,2.93,3.73,4.93,5.33), I(1.02,1.44,2.72,3.57,4.84,5.27), F(1.02,1.44,2.72,3.57,4.84,5.27)]
x_5	[T(1.07,1.53,2.91,3.83,5.21,5.67), I(1.21,1.61,2.81,3.61,4.81,5.21), F(1.21,1.61,2.81,3.61,4.81,5.21)]

Table 5. Value of LHNNS

Attribute	Neutrosophic Number
x_1	(2.7417,0.8797,0.8797)
x_2	(4.167, 0.7292, 0.7292)
x_3	(3.1333, 2.3875, 2.3875)
x_4	(2.775, 2.6208, 2.6208)
x_5	(2.8083,2.675,2.675)

Table 6. Score of table 5

Attribute	Score
x_1	-3.0167
x_2	2.7083
x_3	-1.6417
x_4	-2.4667
x_5	-2.542

6. **Conclusion**

In this study, we explored the approximation of generalized neutrosophic numbers with asymmetry and provided a comparison with other well-known approaches for approximating

neutrosophic numbers. The results emphasize the significance of taking uneven membership degrees into account within the neutrosophic context since it more precisely captures the complexities of actual uncertainty. The complexity and subjectivity associated with depicting uncertainty using neutrosophic numbers have come to light due to research of numerous theories relating to value assignment and approximation points.

We have illustrated the benefits and limitations of several approximation strategies using numerical examples. When nonlinearity in membership degrees is typical, the suggested method for approximating generalized neutrosophic numbers with asymmetry shows accuracy and broad application.

The neutrosophic theory offers a versatile framework for modelling uncertainty in various domains. Future research may focus on hybrid strategies that combine neutrosophic numbers with alternative uncertainty representations and broaden the application of these models to more challenging decision-making scenarios.

References

1. Zadeh, L. A. (1965). Fuzzy sets. Information and control, 8(3), 338-353.
2. Liu, F., Liu, Z. L., & Wu, Y. H. (2018). A group decision making model based on triangular fuzzy additive reciprocal matrices with additive approximation-consistency. Applied Soft Computing, 65, 349-359.
3. Ghadle, K. P., & Pathade, P. A. (2017). Solving transportation problem with generalized hexagonal and generalized octagonal fuzzy numbers by ranking method. Global journal of pure and applied mathematics, 13(9), 6367-6376.
4. Nagalakshmi, T., Sudharani, R., & Ambika, G. (2022, June). A Comparative Approach for Solving Fuzzy Transportation Problem with Hexagonal Fuzzy Numbers and Neutrosophic Triangular Fuzzy Numbers. In Proceedings of 2nd International Conference on Mathematical Modeling and Computational Science: ICMMCS 2021 (pp. 315-324). Singapore: Springer Nature Singapore.
5. Velu, L. G. N., & Ramalingam, B. (2023). Total Ordering on Generalized 'n'Gonal Linear Fuzzy Numbers. International Journal of Computational Intelligence Systems, 16(1), 23.
6. Grzegorzewski, P. (2002). Nearest interval approximation of a fuzzy number. Fuzzy Sets and systems, 130(3), 321-330.
7. Grzegorzewski, P. (2003). Approximation of a fuzzy number preserving entropy-like nonspecifity. Badania Operacyjne i Decyzje, (4), 49-59.
8. Chanas, S. (2001). On the interval approximation of a fuzzy number. Fuzzy Sets and Systems, 122(2), 353-356.
9. Nayagam, V. L. G., & Murugan, J. (2021). Hexagonal fuzzy approximation of fuzzy numbers and its applications in MCDM. Complex & Intelligent Systems, 7(3), 1459-1487.
10. Coroianu, L. (2021). Trapezoidal approximations of fuzzy numbers using quadratic programs. Fuzzy Sets and Systems, 417, 71-92.
11. Huang, H., Wu, C., Xie, J., & Zhang, D. (2017). Approximation of fuzzy numbers using the convolution method. Fuzzy Sets and Systems, 310, 14-46.
12. Heilpern, S. (1992). The expected value of a fuzzy number. Fuzzy sets and Systems, 47(1), 81-86.
13. Ban, A. (2008). Approximation of fuzzy numbers by trapezoidal fuzzy numbers preserving the expected interval. Fuzzy Sets and Systems, 159(11), 1327-1344.

14. Grzegorzewski, P., & Mrówka, E. (2005). Trapezoidal approximations of fuzzy numbers. Fuzzy sets and systems, 153(1), 115-135.
15. Yeh, C. T. (2008). On improving trapezoidal and triangular approximations of fuzzy numbers. International Journal of Approximate Reasoning, 48(1), 297-313.
16. Ban, A. I., & Coroianu, L. (2015). Existence, uniqueness, calculus and properties of triangular approximations of fuzzy numbers under a general condition. International Journal of Approximate Reasoning, 62, 1-26.
17. Yeh, C. T. (2017). Existence of interval, triangular, and trapezoidal approximations of fuzzy numbers under a general condition. Fuzzy Sets and Systems, 310, 1-13.
18. Abbasbandy, S., & Asady, B. (2004). The nearest trapezoidal fuzzy number to a fuzzy quantity. Applied mathematics and computation, 156(2), 381-386.
19. Ban, A. I. (2009). Triangular and parametric approximations of fuzzy numbers—inadvertences and corrections. Fuzzy Sets and Systems, 160(21), 3048-3058.
20. Yeh, C. T., & Chu, H. M. (2014). Approximations by LR-type fuzzy numbers. Fuzzy Sets and Systems, 257, 23-40.
21. Ban, A. I., Coroianu, L., & Khastan, A. (2016). Conditioned weighted L–R approximations of fuzzy numbers. Fuzzy Sets and Systems, 283, 56-82.
22. Jin, Q., & Li, L. (2021). L-fuzzy rough approximation operators based on co-implication and their (single) axiomatic characterizations. Axioms, 10(3), 134.
23. Atanassov, K. T., & Atanassov, K. T. (1999). Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20(1), 87–96.
24. Lakshmana Gomathi Nayagam, V., & Murugan, J. (2021). Triangular approximation of intuitionistic fuzzy numbers on multi-criteria decision making problem. Soft Computing, 25(15), 9887-9914.
25. Wu, W. Z., Gu, S. M., Li, T. J., & Xu, Y. H. (2014). Intuitionistic fuzzy rough approximation operators determined by intuitionistic fuzzy triangular norms. In Rough Sets and Knowledge Technology: 9th International Conference, RSKT 2014, Shanghai, China, October 24-26, 2014, Proceedings 9 (pp. 653-662). Springer International Publishing.
26. Smarandache, F. (1999). A unifying field in logics. neutrosophy: Neutrosophic probability, set and logic.
27. Smarandache, F. (2010). Neutrosophic set–a generalization of the intuitionistic fuzzy set. Journal of Defense Resources Management (JoDRM), 1(1), 107-116.
28. Şahin, M., Kargın, A., & Smarandache, F. (2018). Generalized single valued triangular neutrosophic numbers and aggregation operators for application to multi-attribute group decision making. Infinite Study.
29. Ye, J. (2015). Trapezoidal neutrosophic set and its application to multiple attribute decision-making. Neural computing and Applications, 26, 1157-1166.
30. Abdel-Basset, M., Mohamed, M., Hussien, A. N., & Sangaiah, A. K. (2018). A novel group decision-making model based on triangular neutrosophic numbers. Soft Computing, 22, 6629-6643.
31. Vidhya, R., Hepzibah, I., & Gani, N. (2017) Neutrosophic multi objective linear programming problems. Global Journal of Pure and Applied Mathematics Volume 13.
32. Deveci, M., Erdogan, N., Cali, U., Stekli, J., & Zhong, S. (2021). Type-2 neutrosophic number based multi-attributive border approximation area comparison (MABAC) approach for offshore wind farm site selection in USA. Engineering Applications of Artificial Intelligence, 103, 104311.
33. Nagarajan, D., Kanchana, A., Jacob, K., Kausar, N., Edalatpanah, S. A., & Shah, M. A. (2023). A novel approach based on neutrosophic Bonferroni mean operator of trapezoidal and triangular neutrosophic interval environments in multi-attribute group decision making. Scientific reports, 13(1), 10455.
34. Sumathi, I. R., & Antony Crispin Sweety, C. (2019). New approach on differential equation via trapezoidal neutrosophic number. Complex & Intelligent Systems, 5, 417-424.

35. Sumathi, I. R., & Priya, V. M. (2018). A new perspective on neutrosophic differential equation, International Journal of Engineering and Technology.
36. Kurian, A, Sumathi.I.R, Edalatpanah, S.A. A Novel Technique on MCDM using Non-linear Hexagonal Neutrosophic Numbers. (Communicated).
37. Sumathi.I.R, Kurian, A. & Parvathi (2023). An Enhanced Generalized Neutrosophic Number and its role in Multi-Criteria Decision-Making Challenges. Neutrosophic Sets and Systems, 61(1), 18.
38. Khan, S., Abdullah, S., Ashraf, S., Chinram, R., & Baupradist, S. (2020). Decision support technique based on neutrosophic Yager aggregation operators: Application in solar power plant locations—Case study of Bahawalpur, Pakistan. Mathematical Problems in Engineering, 2020, 1-21.
39. Saqlain, M., Riaz, M., Kiran, N., Kumam, P., & Yang, M. S., "Water Quality Evaluation Using Generalized Correlation Coefficient for M-Polar Neutrosophic Hypersoft Sets," Neutrosophic Sets and Syst., vol. 55, pp. 58-89, 2023.
40. Saqlain, M., Garg, H., Kumam, P., Kumam, W., "Uncertainty and decision-making with multi-polar interval-valued neutrosophic hypersoft set: A distance, similarity measure, and machine learning approach," Alex. Eng. J., vol. 84, pp. 323-332, 2023.
41. Murugan, Jagadeeswari & Velu, Lakshmana Gomathi Nayagam & Nayagam, Gomathi. (2019). Trapezoidal Approximation of Neutrosophic Numbers on Transportation problems. Journal of Advanced Research in Dynamical and Control Systems. 11. 377-394.

Received: June 27, 2024. Accepted: Oct 11, 2024

NSS
Neutrosophic Sets and Systems, Vol. 75, 2025
University of New Mexico

Enhanced T2NN-LogTODIM-TOPSIS Framework for Assessing Vocational College Students' Employment Quality Using Type-2 Neutrosophic Numbers

Ziyan Wang and Xifeng He*
Hebei Vocational University of Industry and Technology, Shijiazhuang, 050091, Hebei, China (Ziyan Wang, E-email: ziyan_8882004@163.com)
*Correspondence: E-mail: hexifengxg@163.com; hexifeng2024@ldy.edu.rs

Abstract: With the ongoing development of vocational education in our country, the employment quality of vocational college graduates has emerged as a crucial metric for assessing the overall strength and effectiveness of these institutions. Establishing a robust employment quality evaluation system for vocational colleges not only complements and enhances their employment-related efforts, but also represents a critical step toward improving the broader educational and teaching practices within these institutions. Currently, the construction of such evaluation systems tends to focus primarily on professional education and teaching. While this approach has undoubtedly contributed to improving the quality of vocational education, it often overlooks the dimensions of development, innovation, and entrepreneurship education, thereby limiting its potential to produce optimal educational outcomes. In light of this, vocational colleges must proactively design and implement an employment quality evaluation system that incorporates the principles of entrepreneurship and innovation, fostering the healthy and sustainable growth of vocational education. The employment quality assessment of vocational college students, within the framework of innovation and entrepreneurship (often referred to as "double innovation"), constitutes a multiple-attribute decision-making (MADM) problem. To address such complex issues, methods like Logarithmic TODIM (LogTODIM) and TOPSIS have been proposed for MADM scenarios. In this context, Type-2 neutrosophic numbers (T2NNs) are utilized to represent and manage the inherent uncertainty and vagueness in the employment quality evaluation process for vocational college students under the double innovation framework. This study introduces a novel method—the Type-2 neutrosophic number Logarithmic TODIM-TOPSIS (T2NN-LogTODIM-TOPSIS) approaches a solution for multi-attribute group decision-making (MAGDM) under T2NN conditions. Finally, a numerical case study is conducted to illustrate the application and effectiveness of the T2NN-LogTODIM-TOPSIS approach in evaluating the employment quality of vocational college students within the context of innovation and entrepreneurship.

Keywords: MADM problem; T2NNs; LogTODIM model; TOPSIS model; employment quality evaluation

1. Introduction

The so-called employment quality belongs to a relatively complex concept. Employment enables people to enhance their quality of life while meeting material needs, thereby better satisfying their own value and social value[1-3]. It can be said that employment is an important process of meeting individual material needs, and it is also a key form of

ensuring that individuals receive social recognition, thereby providing them with spiritual satisfaction[4, 5]. The quality of employment refers to the process in which individuals, through the integration of labor and means of production, engage in work and ensure the effective distribution of means of production, while ultimately satisfying their own value and the mutual unity of social value[6, 7]. The key purpose of establishing a comprehensive evaluation system for the employment quality of vocational college students is to quantify the various indicators mentioned in the evaluation system, transform them into detectable indicators, and better meet the current employment requirements of vocational college students. This can provide more favorable basis for the improvement of the employment quality of vocational college students and the construction of economy and culture. However, in the current environment, there is no specialized system for evaluating the employment quality of vocational college students, and main employment quality evaluation system only focuses on college students, including the definition of subjective and objective indicators for college student employment. Generally, it includes relevant quality indicators for college students' job positions and their employment satisfaction, but no corresponding algorithm is provided; In addition, the algorithm provided is the classification form of the employment quality evaluation system. Some scholars will focus on indicators such as employment rate, structure and distribution, and salary benefits, while others will establish first and second level evaluation standards from a fuzzy perspective for quality evaluation. For the actual employment rate of graduates in current vocational colleges, it is a more intuitive method for analyzing the quality of social employment and a fundamental indicator for evaluating the employment quality of vocational college students[8, 9]. The problem of employment difficulties for vocational college students has been preliminarily solved, which is a key achievement in promoting vocational education and the level of vocational education. However, the employment rate of students is difficult to intuitively and comprehensively demonstrate the employment problems of vocational college students[10-12]. We cannot explore the employment quality of vocational college students solely from the perspective of employment rate. In the current context of entrepreneurship and entrepreneurship, vocational colleges can evaluate employment rates from two perspectives: first, the evaluation of innovation ability, which means that after vocational college students are employed, the campus should integrate innovation evaluation forms based on previous evaluation models, such as incorporating project indicators for graduates to participate in innovation into the original evaluation system, to evaluate whether the overall innovation ability of graduates has improved. Secondly, the evaluation of entrepreneurship indicators, that is, in vocational colleges, the evaluation of employment quality for students should incorporate indicators of graduate entrepreneurship projects[13, 14]. After analyzing the indicators of entrepreneurship projects and the proportion of employment and innovation entrepreneurship, it can better evaluate the specific entrepreneurial ability of students. Through this form, going to vocational colleges can better measure students' ability to innovate and start businesses, thereby achieving a reasonable evaluation of

employment rates[15, 16]. The current industry structure is relatively complex and diverse, which leads to diverse employment structures. Due to the employment structure, it can not only display the specific career trends and regional differences of vocational college graduates, but also more accurately reflect the actual nature of their employment, as well as their innovation ability and entrepreneurial choices[17, 18]. By analyzing the employment structure of vocational college students, the specific employment quality can be maximized, and the competitiveness of graduates in employment can be effectively judged based on their career choices. The current overall economic development in our country is showing signs of imbalance. Exploring the employment structure based on regional economic levels can effectively demonstrate the employment orientation of vocational college students and highlight the competition and employment pressure between regions. If the employment rate in economically developed regions is relatively high, then the employment prospects for graduates are better. From this, it can be understood that the employment structure can continuously demonstrate the need for graduates to improve their own qualities, as well as the contribution of various professional talents to local development. The evaluation of the employment quality of vocational college students in the current context of entrepreneurship and entrepreneurship is an analysis of the employment situation of vocational college students[19, 20]. Through the construction of a clear evaluation system, it can better understand the employment situation of vocational college students, and finally provide favorable improvement strategies for vocational education. Only by formulating comprehensive educational strategies for students can we provide guarantees for the stable development of vocational college students in the future.

The evaluation of employment quality for vocational college students within the context of innovation and entrepreneurship ("double innovation") can be framed as a MADM problem. To address this, techniques such as LogTODIM [21] and TOPSIS [22, 23] have been proposed as effective MADM methods. However, the existing approaches have yet to fully integrate the handling of fuzzy and uncertain information, which is essential for accurately evaluating employment quality in this dynamic context. To address this gap, T2NNs [24] have been introduced as a means to represent and manage the inherent fuzziness of the data involved in the employment quality evaluation process for vocational college students under the double innovation framework. Despite the advancements in the LogTODIM and TOPSIS techniques, few or no studies have yet explored their application in combination with an entropy-based weighting model and T2NNs. Therefore, this study proposes a novel T2NN-LogTODIM-TOPSIS model, which integrates T2NNs to enhance the MADM process. A numerical example illustrating the employment quality evaluation of vocational college students in the double innovation context, along with a comparative analysis, is provided to validate the effectiveness and applicability of this approach.

1.2 Importance of Neutrosophic Theory

The neutrosophic theory is vital for addressing uncertainties, improving decision-making, and fostering innovation across various domains. Its flexibility and comprehensive approach make it an essential tool for researchers and practitioners.

- Neutrosophic theory effectively handles uncertainty and imprecision in data, allowing for more accurate modeling of real-world situations.
- It provides a robust framework for decision-making processes that involve multiple criteria, accommodating various perspectives and preferences.
- Applicable across various fields, including economics, engineering, medicine, and social sciences, enhancing analysis in diverse contexts.
- Allows for a more nuanced interpretation of data by incorporating degrees of truth, indeterminacy, and falsehood, leading to better conclusions.
- Promotes the development of new methodologies and approaches to complex problems, fostering innovative solutions in research and practice.
- Clarifies ambiguous concepts and improves communication among stakeholders, facilitating collaboration in multidisciplinary teams.
- Compatible with AI and machine learning, enhancing their ability to process and analyze complex, uncertain data.

The structure of this study is organized as follows:

Section 2 introduces the formulation of T2NNs, providing the foundational framework for handling uncertainty and imprecision in decision-making processes.

Section 3 presents the development of the T2NN-Logarithmic TODIM-TOPSIS model, specifically designed for MADM under the T2NN environment.

Section 4 offers a detailed numerical case study that evaluates the employment quality of vocational college students in the context of innovation and entrepreneurship ("double innovation"). This section also includes various comparative analyses to validate and demonstrate the effectiveness of the proposed model.

Finally, Section 5 concludes the study, summarizing key findings and offering insights into the future application of the T2NN-LogTODIM-TOPSIS approach.

2. Preliminaries

Wang et al. [25] built the SVNSs

Definition 1 [25]. The SVNSs is raised:

$$HA=\left\{\left(\theta,HT_{HA}\left(\theta\right),HI_{HA}\left(\theta\right),HF_{HA}\left(\theta\right)\right)\middle|\theta\in\Theta\right\} \tag{1}$$

where the $HT_{HA}\left(\theta\right),HI_{HA}\left(\theta\right),HF_{HA}\left(\theta\right)$ is truth-membership (TM), indeterminacy-membership (IM) and falsity-membership (FM), $HT_{HA}\left(\theta\right),HI_{HA}\left(\theta\right),HF_{HA}\left(\theta\right)\in\left[0,1\right]$, $0\le HT_{HA}\left(\theta\right)+HI_{HA}\left(\theta\right)+HF_{HA}\left(\theta\right)\le 3$. The SVNN is implemented as $HA=\left(HT_A,HI_A,HF_A\right)$, where $HT_A,HI_A,HF_A\in\left[0,1\right]$, and $0\le HT_A+HI_A+HF_A\le 3$.

Abdel-Basset et al. [24] raised the T2NNs.

Definition 1[24]. **T**he T2NN is raised:

$$HH=\left\{\left(\theta,HA\left(\theta\right),HB\left(\theta\right),HC\left(\theta\right)\right)\middle|\theta\in\Theta\right\} \tag{2}$$

where $HA\left(\theta\right),HB\left(\theta\right),HC\left(\theta\right)\in\left[0,1\right]$ be TM, IM and FM with triangular fuzzy numbers (TFNs).

$$HA\left(\theta\right)=\left(HA^{L}\left(\theta\right),HA^{M}\left(\theta\right),HA^{U}\left(\theta\right)\right),0\le HA^{L}\left(\theta\right)\le HA^{M}\left(\theta\right)\le HA^{U}\left(\theta\right)\le1 \tag{3}$$

$$HB\left(\theta\right)=\left(HB^{L}\left(\theta\right),HB^{M}\left(\theta\right),HB^{U}\left(\theta\right)\right),0\le HB^{L}\left(\theta\right)\le HB^{M}\left(\theta\right)\le HB^{U}\left(\theta\right)\le1 \tag{4}$$

$$HC\left(\theta\right)=\left(HC^{L}\left(\theta\right),HC^{M}\left(\theta\right),HC^{U}\left(\theta\right)\right),0\le HC^{L}\left(\theta\right)\le HC^{M}\left(\theta\right)\le HC^{U}\left(\theta\right)\le1 \tag{5}$$

We let $HH=\begin{Bmatrix}\left(HA^{L},HA^{M},HA^{U}\right),\\\left(HB^{L},HB^{M},HB^{U}\right),\left(HC^{L},HC^{M},HC^{U}\right)\end{Bmatrix}$ be a T2NN, $0\le HA^{U}+HB^{U}+HC^{U}\le3$.

Definition 2[24]. Let $HH_{1}=\begin{Bmatrix}\left(HA_{1}^{L},HA_{1}^{M},HA_{1}^{U}\right),\\\left(HB_{1}^{L},HB_{1}^{M},HB_{1}^{U}\right),\left(HC_{1}^{L},HC_{1}^{M},HC_{1}^{U}\right)\end{Bmatrix}$,

$HH_{2}=\begin{Bmatrix}\left(HA_{2}^{L},HA_{2}^{M},HA_{2}^{U}\right),\\\left(HB_{2}^{L},HB_{2}^{M},HB_{2}^{U}\right),\left(HC_{2}^{L},HC_{2}^{M},HC_{2}^{U}\right)\end{Bmatrix}$ and $HH=\begin{Bmatrix}\left(HA^{L},HA^{M},HA^{U}\right),\\\left(HB^{L},HB^{M},HB^{U}\right),\left(HC^{L},HC^{M},HC^{U}\right)\end{Bmatrix}$ be TFNNs, the operation laws are raised:

$$(1)\ HH_{1}\oplus HH_{2}=\begin{Bmatrix}\begin{pmatrix}HA_{1}^{L}+HA_{2}^{L}\text{-}HA_{1}^{L}HA_{2}^{L},HA_{1}^{M}+HA_{2}^{M}\text{-}HA_{1}^{M}HA_{2}^{M},\\HA_{1}^{U}+HA_{2}^{U}\text{-}HA_{1}^{U}HA_{2}^{U}\end{pmatrix},\\\left(HB_{1}^{L}HB_{2}^{L},HB_{1}^{M}HB_{2}^{M},HB_{1}^{U}HB_{2}^{U}\right),\\\left(HC_{1}^{L}HC_{2}^{L},HC_{1}^{M}HC_{2}^{M},HC_{1}^{U}HC_{2}^{U}\right)\end{Bmatrix};$$

$$(2)HH_{1}\otimes HH_{2}=\begin{Bmatrix}\left(HA_{1}^{L}HA_{2}^{L},HA_{1}^{M}HA_{2}^{M},HA_{1}^{U}HA_{2}^{U}\right),\\\begin{pmatrix}HB_{1}^{L}+HB_{2}^{L}\text{-}HB_{1}^{L}HB_{2}^{L},HB_{1}^{M}+HB_{2}^{M}\text{-}HB_{1}^{M}HB_{2}^{M},\\HB_{1}^{U}+HB_{2}^{U}\text{-}HB_{1}^{U}HB_{2}^{U}\end{pmatrix},\\\begin{pmatrix}HC_{1}^{L}+HC_{2}^{L}\text{-}HC_{1}^{L}HC_{2}^{L},HC_{1}^{M}+HC_{2}^{M}\text{-}HC_{1}^{M}HC_{2}^{M},\\HC_{1}^{U}+HC_{2}^{U}\text{-}HC_{1}^{U}HC_{2}^{U}\end{pmatrix}\end{Bmatrix};$$

$$(3)\ \lambda HH=\left\{\begin{array}{l}\left(1-\left(1-HA^{L}\right)^{\lambda},1-\left(1-HA^{M}\right)^{\lambda},1-\left(1-HA^{U}\right)^{\lambda}\right),\\ \left(\left(HB^{L}\right)^{\lambda},\left(HB^{M}\right)^{\lambda},\left(HB^{U}\right)^{\lambda}\right),\\ \left(\left(HC^{L}\right)^{\lambda},\left(HC^{M}\right)^{\lambda},\left(HC^{U}\right)^{\lambda}\right)\end{array}\right\},\lambda>0;$$

$$(4)HH^{\lambda}=\left\{\begin{array}{l}\left(\left(HA^{L}\right)^{\lambda},\left(HA^{M}\right)^{\lambda},\left(HA^{U}\right)^{\lambda}\right),\\ \left(1-\left(1-HB^{L}\right)^{\lambda},1-\left(1-HB^{M}\right)^{\lambda},1-\left(1-HB^{U}\right)^{\lambda}\right),\\ \left(1-\left(1-HC^{L}\right)^{\lambda},1-\left(1-HC^{M}\right)^{\lambda},1-\left(1-HC^{U}\right)^{\lambda}\right)\end{array}\right\},\lambda>0.$$

The operational laws raised following properties.

$$(1)\ HH_1\oplus HH_2=HH_2\oplus HH_1, HH_1\otimes HH_2=HH_2\otimes HH_1,\left(\left(HH_1\right)^{\lambda_1}\right)^{\lambda_2}=\left(HH_1\right)^{\lambda_1\lambda_2};\tag{6}$$

$$(2)\ \lambda\left(HH_1\oplus HH_2\right)=\lambda HH_1\oplus\lambda HH_2,\left(HH_1\otimes HH_2\right)^{\lambda}=\left(HH_1\right)^{\lambda}\otimes\left(HH_2\right)^{\lambda};\tag{7}$$

$$(3)\ \lambda_1 HH_1\oplus\lambda_2 HH_1=\left(\lambda_1+\lambda_2\right)HH_1,\left(HH_1\right)^{\lambda_1}\otimes\left(HH_1\right)^{\lambda_2}=\left(HH_1\right)^{\left(\lambda_1+\lambda_2\right)}.\tag{8}$$

Definition 3[24]. Let $HH=\left\{\begin{array}{l}\left(HA^{L},HA^{M},HA^{U}\right),\\ \left(HB^{L},HB^{M},HB^{U}\right),\left(HC^{L},HC^{M},HC^{U}\right)\end{array}\right\}$ be TFNN, the score function and accuracy function are raised:

$$HSF\left(HH\right)=\frac{1}{12}\left[\begin{array}{l}8+\left(HA^{L}+2HA^{M}+HA^{U}\right)\\ -\left(HB^{L}+HVB^{M}+HB^{U}\right)\\ -\left(HC^{L}+2HC^{M}+HC^{U}\right)\end{array}\right],HSF\left(HH\right)\in\left[0,1\right]\tag{9}$$

$$HAF\left(HH\right)=\frac{1}{4}\left[\begin{array}{l}\left(HA^{L}+2HA^{M}+HA^{U}\right)\\ -\left(HB^{L}+2HB^{M}+HB^{U}\right)\end{array}\right],HAF\left(HH\right)\in\left[-1,1\right]\tag{10}$$

For the TFNNs HH_1 and HH_2, in light with Definition 3, the following results are raised:

(1) *if* $HSF\left(HH_1\right)\prec HSF\left(HH_2\right),\ HH_1\prec HH_2;$

(2) *if* $HSF\left(HH_1\right)=HSF\left(HH_2\right),HAF\left(HH_1\right)\prec HAF\left(HH_2\right),\ HH_1\prec HH_2;$

(3) *if* $HSF\left(HH_1\right)=HSF\left(HH_2\right),HAF\left(HH_1\right)=HAF\left(HH_2\right),\ HH_1=HH_2.$

Definition 4[24]. Let $HH_1=\left\{\begin{matrix}\left(HA_1^L,HA_1^M,HA_1^U\right),\\ \left(HB_1^L,HB_1^M,HB_1^U\right),\left(HC_1^L,HC_1^M,HC_1^U\right)\end{matrix}\right\}$, $HH_2=\left\{\begin{matrix}\left(HA_2^L,HA_2^M,HA_2^U\right),\\ \left(HB_2^L,HB_2^M,HB_2^U\right),\left(HC_2^L,HC_2^M,HC_2^U\right)\end{matrix}\right\}$ be T2NNs, the T2NNs Hamming distance (T2NNHD) is raised:

$$\begin{aligned}&T2NNHD\left(HH_1,HH_2\right)\\&=\frac{1}{9}\left(\begin{matrix}\left|HA_1^L-HA_2^L\right|+\left|HA_1^M-HA_2^M\right|+\left|HA_1^U-HA_2^U\right|\\+\left|HB_1^L-HB_2^L\right|+\left|HB_1^M-HB_2^M\right|+\left|HB_1^U-HB_2^U\right|\\+\left|HC_1^L-HC_2^L\right|+\left|HC_1^M-HC_2^M\right|+\left|HC_1^U-HC_2^U\right|\end{matrix}\right)\end{aligned}\tag{11}$$

3. T2NN-LogTODIM-TOPSIS approach for MADM with entropy

The T2NN-LogTODIM-TOPSIS is raised for MAGDM. Let $HA=\left\{HA_1,HA_2,\cdots,HA_m\right\}$ be alternative and $HG=\left\{HG_1,HG_2,\cdots,HG_n\right\}$ be attributes with weight hw, $hw_j\in[0,1]$, $\sum_{j=1}^{n}hw_j=1$.Then, T2NN-LogTODIM-TOPSIS is raised for MADM.

Step 1. Cultivate the T2NN-matrix $HM=\left[HM_{ij}\right]_{m\times n}$:

$$HM=\left[HM_{ij}\right]_{m\times n}=\begin{matrix}\\HA_1\\HA_2\\\vdots\\HA_m\end{matrix}\begin{matrix}\begin{matrix}HG_1&HG_2&\cdots&HG_n\end{matrix}\\\begin{bmatrix}HM_{11}&HM_{12}&\cdots&HM_{1n}\\HM_{21}&HM_{22}&\cdots&HM_{2n}\\\vdots&\vdots&\vdots&\vdots\\HM_{m1}&HM_{m2}&\cdots&HM_{mn}\end{bmatrix}\end{matrix}\tag{12-a}$$

$$HM_{ij}=\left\{\begin{matrix}\left(\left(HA_{ij}^L\right),\left(HA_{ij}^M\right),\left(HA_{ij}^U\right)\right),\\ \left(\left(HB_{ij}^L\right),\left(HB_{ij}^M\right),\left(HB_{ij}^U\right)\right),\\ \left(\left(HC_{ij}^L\right),\left(HC_{ij}^M\right),\left(HC_{ij}^U\right)\right)\end{matrix}\right\}\tag{12-b}$$

Step 2. Normalize $HM=\left[HM_{ij}\right]_{m\times n}$ into $NHM=\left[NHM_{ij}\right]_{m\times n}$.

Aiming at benefit attributes:

$$NHM_{ij}=\left\{\begin{matrix}\left(\left(NHA_{ij}^L\right),\left(NHA_{ij}^M\right),\left(NHA_{ij}^U\right)\right),\\ \left(\left(NHB_{ij}^L\right),\left(NHB_{ij}^M\right),\left(NHB_{ij}^U\right)\right),\\ \left(\left(NHC_{ij}^L\right),\left(NHC_{ij}^M\right),\left(NHC_{ij}^U\right)\right)\end{matrix}\right\}=\left\{\begin{matrix}\left(\left(HA_{ij}^L\right),\left(HA_{ij}^M\right),\left(HA_{ij}^U\right)\right),\\ \left(\left(HB_{ij}^L\right),\left(HB_{ij}^M\right),\left(HB_{ij}^U\right)\right),\\ \left(\left(HC_{ij}^L\right),\left(HC_{ij}^M\right),\left(HC_{ij}^U\right)\right)\end{matrix}\right\}\tag{13}$$

Aiming at cost attributes:

$$NHM_{ij}=\begin{Bmatrix}\left(\left(NHA_{ij}^{L}\right),\left(NHA_{ij}^{M}\right),\left(NHA_{ij}^{U}\right)\right),\\ \left(\left(NHB_{ij}^{L}\right),\left(NHB_{ij}^{M}\right),\left(NHB_{ij}^{U}\right)\right),\\ \left(\left(NHC_{ij}^{L}\right),\left(NHC_{ij}^{M}\right),\left(NHC_{ij}^{U}\right)\right)\end{Bmatrix}=\begin{Bmatrix}\left(\left(HC_{ij}^{L}\right),\left(HC_{ij}^{M}\right),\left(HC_{ij}^{U}\right)\right),\\ \left(\left(HB_{ij}^{L}\right),\left(HB_{ij}^{M}\right),\left(HB_{ij}^{U}\right)\right),\\ \left(\left(HA_{ij}^{L}\right),\left(HA_{ij}^{M}\right),\left(HA_{ij}^{U}\right)\right)\end{Bmatrix} \quad (14)$$

Step 3. Cultivate the weight numbers with entropy.

Entropy [26] is raised for weight numbers. The T2NN-matrix is raised: $T2NNNHM_{ij}$

$$=\frac{\left(HSF\begin{Bmatrix}\left(\left(NHA_{ij}^{L}\right),\left(NHA_{ij}^{M}\right),\left(NHA_{ij}^{U}\right)\right),\\ \left(\left(NHB_{ij}^{L}\right),\left(NHB_{ij}^{M}\right),\left(NHB_{ij}^{U}\right)\right),\\ \left(\left(NHC_{ij}^{L}\right),\left(NHC_{ij}^{M}\right),\left(NHC_{ij}^{U}\right)\right)\end{Bmatrix}+HAF\begin{Bmatrix}\left(\left(NHA_{ij}^{L}\right),\left(NHA_{ij}^{M}\right),\left(NHA_{ij}^{U}\right)\right),\\ \left(\left(NHB_{ij}^{L}\right),\left(NHB_{ij}^{M}\right),\left(NHB_{ij}^{U}\right)\right),\\ \left(\left(NHC_{ij}^{L}\right),\left(NHC_{ij}^{M}\right),\left(NHC_{ij}^{U}\right)\right)\end{Bmatrix}+1\right)}{\sum_{i=1}^{m}\left(HSF\begin{Bmatrix}\left(\left(NHA_{ij}^{L}\right),\left(NHA_{ij}^{M}\right),\left(NHA_{ij}^{U}\right)\right),\\ \left(\left(NHB_{ij}^{L}\right),\left(NHB_{ij}^{M}\right),\left(NHB_{ij}^{U}\right)\right),\\ \left(\left(NHC_{ij}^{L}\right),\left(NHC_{ij}^{M}\right),\left(NHC_{ij}^{U}\right)\right)\end{Bmatrix}+HAF\begin{Bmatrix}\left(\left(NHA_{ij}^{L}\right),\left(NHA_{ij}^{M}\right),\left(NHA_{ij}^{U}\right)\right),\\ \left(\left(NHB_{ij}^{L}\right),\left(NHB_{ij}^{M}\right),\left(NHB_{ij}^{U}\right)\right),\\ \left(\left(NHC_{ij}^{L}\right),\left(NHC_{ij}^{M}\right),\left(NHC_{ij}^{U}\right)\right)\end{Bmatrix}+1\right)}, \quad (15)$$

The T2NN Shannon decision entropy (T2NNSDE) is raised:

$$T2NNSDE_{j}=-\frac{1}{\ln m}\sum_{i=1}^{m}T2NNM_{ij}\ln T2NNM_{ij} \quad (16)$$

and $T2NNM_{ij}\ln T2NNM_{ij}=0$ if $T2NNM_{ij}=0$.

Then, the weight numbers $hw=\left(hw_1,hw_2,\cdots,hw_n\right)$ is raised:

$$hw_{j}=\frac{1-T2NNSDE_{j}}{\sum_{j=1}^{n}\left(1-T2NNSDE_{j}\right)} \quad (17)$$

Step 4. Raise relative weight:

$$rhw_{j}=hw_{j}\Big/\max_{j}hw_{j}, \quad (18)$$

Step 5. Raise the T2NN dominance decision degree (T2NNDDD).

(1) The T2NNDDD of HA_i over HA_t for HG_j is raised:

$$T2NNDDD_{j}\left(HA_{i},HA_{t}\right)=\begin{cases}\dfrac{rhw_{j}\times\log\left(1+10\rho T2NNHD\left(NHM_{ij},NHM_{tj}\right)\right)}{\sum_{j=1}^{n}rhw_{j}} & if\ HSF\left(NHM_{ij}\right)>HSF\left(NHM_{tj}\right)\\ 0 & if\ HSF\left(NHM_{ij}\right)=HSF\left(NHM_{tj}\right)\\ -\dfrac{rhw_{j}\times\lambda\log\left(1+10\rho T2NNHD\left(NHM_{ij},NHM_{tj}\right)\right)}{\sum_{j=1}^{n}rhw_{j}} & if\ HSF\left(NHM_{ij}\right)<HSF\left(NHM_{tj}\right)\end{cases} \quad (19)$$

where $\lambda \in [1,5]$ and $\rho \in N^+$ is raised from Ref.[27].

(2) The $T2NNDDD_j(HA_i)(j=1,2,\cdots,n)$ under HG_j is raised:

$$T2NNDDD_j(HA_i) = \left[T2NNDDD_j(HA_i, HA_t)\right]_{m\times m}$$

$$= \begin{array}{c} \\ HA_1 \\ HA_2 \\ \vdots \\ HA_m \end{array} \begin{array}{c} \begin{array}{cccc} HA_1 & HA_2 & \cdots & HA_m \end{array} \\ \begin{bmatrix} 0 & T2NNDDD_j(HA_1, HA_2) & \cdots & T2NNDDD_j(HA_1, HA_m) \\ T2NNDDD_j(HA_2, HA_1) & 0 & \cdots & T2NNDDD_j(HA_2, HA_m) \\ \vdots & \vdots & \cdots & \vdots \\ T2NNDDD_j(HA_m, HA_1) & T2NNDDD_j(HA_m, HA_2) & \cdots & 0 \end{bmatrix} \end{array}$$

(3) Raise the overall T2NNDDD of HA_i over others under HG_j:

$$T2NNDDD_j(HA_i) = \sum_{t=1}^{m} T2NNDDD_j(HA_i, HA_t) \tag{20}$$

(4) The T2NNDDD matrix is raised:

$$T2NNDDD = \left(T2NNDDD_{ij}\right)_{m\times n}$$

$$= \begin{bmatrix} & HG_1 & HG_2 & \cdots & HG_n \\ HA_1 & \sum_{t=1}^{m} T2NNDDD_1(HA_1, HA_t) & \sum_{t=1}^{m} T2NNDDD_2(HA_1, HA_t) & \cdots & \sum_{t=1}^{m} T2NNDDD_n(HA_1, HA_t) \\ HA_2 & \sum_{t=1}^{m} T2NNDDD_1(HA_2, HA_t) & \sum_{t=1}^{m} T2NNDDD_2(HA_2, HA_t) & \cdots & \sum_{t=1}^{m} T2NNDDD_n(HA_2, HA_t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ HA_m & \sum_{t=1}^{m} T2NNDDD_1(HA_m, HA_t) & \sum_{t=1}^{m} T2NNDDD_2(HA_m, HA_t) & \cdots & \sum_{t=1}^{m} T2NNDDD_n(HA_m, HA_t) \end{bmatrix}$$

Step 6. Raise the T2NNPIDS (T2NN positive ideal decision solution) and T2NNNIDS (T2NN negative ideal decision solution):

$$T2NNPIDS = \left(T2NNPIDS_1, T2NNPIDS_1, \cdots, T2NNPIDS_n\right) \tag{21}$$

$$T2NNNIDS = \left(T2NNNIDS_1, T2NNNIDS_1, \cdots, T2NNNIDS_n\right) \tag{22}$$

$$T2NNPIDS_j = \max_{j=1}^{n} T2NNDDD_{ij}, \tag{23}$$

$$T2NNNIDS_j = \min_{j=1}^{n} T2NNDDD_{ij} \tag{24}$$

Step 7. Raise the T2NNEDV (T2NN Euclidean distance values) for T2NNPIDS and T2NNNIDS.

$$T2NNEDV(HA_i, T2NNPIDS) = \sqrt{\sum_{j=1}^{n}\left(T2NNDDD_{ij} - T2NNPIDS_j\right)^2} \tag{25}$$

$$T2NNEDV(HA_i, T2NNNIDS) = \sqrt{\sum_{j=1}^{n}\left(T2NNDDD_{ij} - T2NNNIDS_j\right)^2} \tag{26}$$

Step 8. Raise the T2NNCCV (T2NN closeness coefficient values) from T2NNPIDS.

$$T2NNCCV\left(HA_i, T2NNPIDS\right) = \frac{T2NNEDV\left(HA_i, T2NNNIDS\right)}{\left(T2NNEDV\left(HA_i, T2NNPIDS\right) + T2NNEDV\left(HA_i, T2NNPIDS\right)\right)}$$

$$= \frac{\sqrt{\sum_{j=1}^{n}\left(T2NNDDD_{ij} - T2NNNIDS_j\right)^2}}{\left(\sqrt{\sum_{j=1}^{n}\left(T2NNDDD_{ij} - T2NNNIDS_j\right)^2} + \sqrt{\sum_{j=1}^{n}\left(T2NNDDD_{ij} - T2NNPIDS_j\right)^2}\right)} \quad (27)$$

Step 9. Choose the best scheme with maximum T2NNCCV.

4. Analytical Comparison with Numerical Examples

In this section, we will compare different methods using numerical examples. This approach will help us clearly see the advantages and disadvantages of each method, making it easier to understand their practical applications.

4.1. Numerical example

The concept of employment quality is inherently complex and can be examined through two main dimensions: employment and quality [28]. Employment refers to an individual's participation in social and economic activities, where labor is performed under certain conditions of material production in exchange for remuneration, resulting in a specific level of economic income. It is the product of the interaction between labor and the means of production. Employment not only satisfies material needs but also enhances quality of life, enabling individuals to realize their self-worth and contribute to society [29]. Beyond the material benefits, employment offers individuals social recognition and spiritual fulfillment. Quality, on the other hand, refers to the degree to which a set of inherent characteristics meets certain requirements [30, 31]. The quality of employment thus pertains to how individuals engage in work by combining labor with the means of production, ensuring the effective and rational distribution of resources, and ultimately achieving a balance between personal value and societal contribution. In summary, the standard of employment quality is a multifaceted concept. It encompasses not only tangible factors such as employment rates, job trend analysis, professional alignment, and job satisfaction, but also considers longer-term aspects like opportunities for advancement, career development prospects, and levels of social security. In essence, employment quality is a broad concept with extensive implications, capable of forming a comprehensive evaluation system [32, 33]. Such a system addresses both the immediate employment needs of vocational college students and their potential for future development. This dual focus is essential for fostering the growth of vocational students and advancing the sustainable development of the

national economy. While many developed Western countries have not explicitly proposed an evaluation system specifically for the employment quality of vocational college students, they have introduced the concept of a "decent work" evaluation framework. This provides a valuable reference point for China's efforts to establish an employment quality evaluation system for vocational students. The "decent work" evaluation framework includes 11 indicators related to social welfare, and in 1999, the European Union introduced work quality indicators. The European Foundation further developed a set of indicators for "Work and Employment Quality," which are organized into four main dimensions: occupational and employment security, health and welfare, technical quality, and work-life balance. These frameworks offer useful insights for developing a comprehensive employment quality evaluation system for vocational college graduates. The primary goal of constructing an employment quality evaluation system for vocational college students is to quantify various indicators, transforming them into measurable metrics that can meet students' employment needs. Such a system would provide a reference for understanding employment quality and contribute to the broader economic and cultural development of vocational students. However, at present, there is no specific, well-defined evaluation system for the employment quality of vocational college students in China [34, 35]. Existing frameworks, which are largely aimed at general college students, focus on subjective and objective indicators such as job quality, employment satisfaction, the proportion of entrepreneurial students, and rates of postgraduate study and overseas education. However, these frameworks do not include specific algorithms to guide evaluation. Both domestic and international employment quality evaluation systems tend to cater to the needs of general workers and are insufficient for addressing the unique employment needs of vocational college students, especially those who have received education in innovation and entrepreneurship [36, 37]. Graduates from vocational colleges who have undergone such training typically exhibit greater creativity, more dynamic thinking, and a heightened desire for self-fulfillment, along with a stronger sense of social responsibility. This highlights a clear shortcoming in the current evaluation systems, which fail to take a developmental perspective in evaluating employment quality [38, 39]. To address these gaps, it is crucial to adopt a dynamic approach that continuously improves the employment quality evaluation system for vocational college students. By refining the system and defining specific indicators, we can ensure that the evaluation process is aligned with the evolving needs of vocational students, ultimately leading to more effective and meaningful results.

The employment quality evaluation of vocational college students is MADM. Five vocational colleges $HA_i\left(i=1,2,3,4,5\right)$ are assessed with different attributes: ①HG_1 is employment rate of graduates from vocational colleges; ②HG_2 is evaluation of salary status; ③HG_3 is management cost of employment quality; ④HG_4 is evaluation of social

recognition level. The T2NN-LogTODIM-TOPSIS is raised for employment quality evaluation of vocational college students.

Step 1. Cultivate the T2NN-matrix $HM = \left[HM_{ij} \right]_{5\times 4}$ (Table 1).

Table 1. T2NN information

	HG1	**HG2**
HA1	$\left\{\begin{matrix}(0.48,0.65,0.71),\\(0.56,0.68,0.74),\\(0.29,0.36,0.65)\end{matrix}\right\}$	$\left\{\begin{matrix}(0.48,0.52,0.57),\\(0.42,0.49,0.58),\\(0.29,0.41,0.43)\end{matrix}\right\}$
HA2	$\left\{\begin{matrix}(0.38,0.49,0.56),\\(0.40,0.67,0.75),\\(0.45,0.48,0.62)\end{matrix}\right\}$	$\left\{\begin{matrix}(0.49,0.54,0.69),\\(0.42,0.52,0.63),\\(0.50,0.65,0.78)\end{matrix}\right\}$
HA3	$\left\{\begin{matrix}(0.24,0.37,0.48),\\(0.28,0.51,0.62),\\(0.56,0.61,0.65)\end{matrix}\right\}$	$\left\{\begin{matrix}(0.21,0.35,0.49),\\(0.48,0.56,0.67),\\(0.69,0.75,0.78)\end{matrix}\right\}$
HA4	$\left\{\begin{matrix}(0.32,0.56,0.67),\\(0.49,0.54,0.59),\\(0.38,0.45,0.51)\end{matrix}\right\}$	$\left\{\begin{matrix}(0.46,0.49,0.54),\\(0.28,0.37,0.48),\\(0.45,0.48,0.51)\end{matrix}\right\}$
HA5	$\left\{\begin{matrix}(0.67,0.74,0.78),\\(0.72,0.79,0.87),\\(0.68,0.69,0.76)\end{matrix}\right\}$	$\left\{\begin{matrix}(0.62,0.65,0.74),\\(0.35,0.43,0.54),\\(0.19,0.25,0.36)\end{matrix}\right\}$

	HG3	**HG4**
HA1	$\left\{\begin{matrix}(0.48,0.65,0.69),\\(0.31,0.43,0.65),\\(0.24,0.37,0.54)\end{matrix}\right\}$	$\left\{\begin{matrix}(0.59,0.76,0.85),\\(0.38,0.49,0.61),\\(0.34,0.38,0.52)\end{matrix}\right\}$
HA2	$\left\{\begin{matrix}(0.45,0.49,0.57),\\(0.26,0.34,0.45),\\(0.41,0.43,0.47)\end{matrix}\right\}$	$\left\{\begin{matrix}(0.67,0.78,0.86),\\(0.54,0.56,0.67),\\(0.58,0.65,0.74)\end{matrix}\right\}$

HA3	$\begin{Bmatrix}(0.39,0.58,0.86),\\(0.46,0.62,0.73),\\(0.42,0.48,0.56)\end{Bmatrix}$	$\begin{Bmatrix}(0.43,0.56,0.64),\\(0.26,0.45,0.57),\\(0.46,0.52,0.56)\end{Bmatrix}$
HA4	$\begin{Bmatrix}(0.29,0.39,0.68),\\(0.26,0.37,0.64),\\(0.23,0.34,0.45)\end{Bmatrix}$	$\begin{Bmatrix}(0.47,0.56,0.67),\\(0.57,0.68,0.76),\\(0.46,0.54,0.62)\end{Bmatrix}$
HA5	$\begin{Bmatrix}(0.49,0.56,0.75),\\(0.53,0.54,0.65),\\(0.32,0.36,0.53)\end{Bmatrix}$	$\begin{Bmatrix}(0.38,0.56,0.64),\\(0.49,0.65,0.78),\\(0.53,0.57,0.72)\end{Bmatrix}$

Step 2. Normalize the $HM=\left[HM_{ij}\right]_{5\times4}$ into $NHM=\left[NHM_{ij}\right]_{5\times4}$ (See Table 2).

Table 2. The normalized T2NN

	HG1	**HG2**
HA1	$\begin{Bmatrix}(0.48,0.65,0.71),\\(0.56,0.68,0.74),\\(0.29,0.36,0.65)\end{Bmatrix}$	$\begin{Bmatrix}(0.48,0.52,0.57),\\(0.42,0.49,0.58),\\(0.29,0.41,0.43)\end{Bmatrix}$
HA2	$\begin{Bmatrix}(0.38,0.49,0.56),\\(0.40,0.67,0.75),\\(0.45,0.48,0.62)\end{Bmatrix}$	$\begin{Bmatrix}(0.49,0.54,0.69),\\(0.42,0.52,0.63),\\(0.50,0.65,0.78)\end{Bmatrix}$
HA3	$\begin{Bmatrix}(0.24,0.37,0.48),\\(0.28,0.51,0.62),\\(0.56,0.61,0.65)\end{Bmatrix}$	$\begin{Bmatrix}(0.21,0.35,0.49),\\(0.48,0.56,0.67),\\(0.69,0.75,0.78)\end{Bmatrix}$
HA4	$\begin{Bmatrix}(0.32,0.56,0.67),\\(0.49,0.54,0.59),\\(0.38,0.45,0.51)\end{Bmatrix}$	$\begin{Bmatrix}(0.46,0.49,0.54),\\(0.28,0.37,0.48),\\(0.45,0.48,0.51)\end{Bmatrix}$
HA5	$\begin{Bmatrix}(0.67,0.74,0.78),\\(0.72,0.79,0.87),\\(0.68,0.69,0.76)\end{Bmatrix}$	$\begin{Bmatrix}(0.62,0.65,0.74),\\(0.35,0.43,0.54),\\(0.19,0.25,0.36)\end{Bmatrix}$

	HG3	**HG4**

HA1	$\left\{\begin{matrix}(0.24,0.37,0.54),\\(0.31,0.43,0.65),\\(0.48,0.65,0.69)\end{matrix}\right\}$	$\left\{\begin{matrix}(0.59,0.76,0.85),\\(0.38,0.49,0.61),\\(0.34,0.38,0.52)\end{matrix}\right\}$
HA2	$\left\{\begin{matrix}(0.41,0.43,0.47),\\(0.26,0.34,0.45),\\(0.45,0.49,0.57)\end{matrix}\right\}$	$\left\{\begin{matrix}(0.67,0.78,0.86),\\(0.54,0.56,0.67),\\(0.58,0.65,0.74)\end{matrix}\right\}$
HA3	$\left\{\begin{matrix}(0.42,0.48,0.56),\\(0.46,0.62,0.73),\\(0.39,0.58,0.86)\end{matrix}\right\}$	$\left\{\begin{matrix}(0.43,0.56,0.64),\\(0.26,0.45,0.57),\\(0.46,0.52,0.56)\end{matrix}\right\}$
HA4	$\left\{\begin{matrix}(0.23,0.34,0.45),\\(0.26,0.37,0.64),\\(0.29,0.39,0.68)\end{matrix}\right\}$	$\left\{\begin{matrix}(0.47,0.56,0.67),\\(0.57,0.68,0.76),\\(0.46,0.54,0.62)\end{matrix}\right\}$
HA5	$\left\{\begin{matrix}(0.32,0.36,0.53),\\(0.53,0.54,0.65),\\(0.49,0.56,0.75)\end{matrix}\right\}$	$\left\{\begin{matrix}(0.38,0.56,0.64),\\(0.49,0.65,0.78),\\(0.53,0.57,0.72)\end{matrix}\right\}$

Step 3. Raise the weight:

$$hw_1 = 0.2505, hw_2 = 0.3326$$
$$hw_3 = 0.2206, hw_4 = 0.1963.$$

Step 4. Raise the relative weight numbers: $rhw = (0.7532, 1.0000, 0.6633, 0.5902)$

Step 5. Raise the $T2NNDDD = \left(T2NNDDD_{ij}\right)_{5\times4}$ (Table 3):

Table 3. The $T2NNDDD = \left(T2NNDDD_{ij}\right)_{5\times4}$

	HG1	**HG2**	**HG3**	**HG4**
HA1	-0.5885	-1.6068	-0.0164	-0.9392
HA2	0.4094	1.3050	-0.5645	-0.2002
HA3	0.5366	0.3569	-1.8893	0.3606
HA4	1.1354	1.0133	-0.4681	-0.4072
HA5	-1.3186	-1.3611	0.9138	-0.7574

Step 6. Raise the T2NNPIDS & T2NNNIDS (Table 4).

Table 4. T2NNPIDS & T2NNNIDS

	HG1	**HG2**	**HG3**	**HG4**
T2NNPIDS	1.1354	1.3050	0.9138	0.3606
T2NNNIDS	-1.3186	-1.6068	-1.8893	-0.9392

Step 7. Raise the $T2NNEDV\left(HA_i, T2NNPIDS\right)$ and $T2NNEDV\left(HA_i, T2NNNIDS\right)$ (Table 5).

Table 5. $T2NNEDV\left(HA_i, T2NNPIDS\right)$ and $T2NNEDV\left(HA_i, T2NNNIDS\right)$

	$T2NNEDV\left(HA_i, T2NNPIDS\right)$	$T2NNEDV\left(HA_i, T2NNNIDS\right)$
HA1	3.7423	2.0102
HA2	1.7398	3.7102
HA3	3.0190	2.9980
HA4	1.6075	3.8974
HA5	3.7922	2.8196

Step 8. Raise the $T2NNCCV\left(HA_i, T2NNPIDS\right)$ (See Table 6).

Table 6. $T2NNCCV\left(HA_i, T2NNPIDS\right)$

	$T2NNCCV\left(HA_i, T2NNPIDS\right)$	**Order**
HA1	0.3494	5
HA2	0.6808	2
HA3	0.4983	3
HA4	0.7080	1
HA5	0.4265	4

Step 9. In light with $T2NNCCV\left(HA_i, T2NNPIDS\right)$, the order is: $HA_4 > HA_2 > HA_3 > HA_5 > HA_1$ and the optimal vocational college is HA_4.

4.2. Comparative analysis

The T2NN-LogTODIM-TOPSIS strategy is compared with theT2NNWA strategy [24], T2NNWG strategy [24], T2NN-CE strategy [40], T2NN-TOPSIS strategy [24], T2NN-EDAS strategy [41], T2NN-MABAC strategy [42], T2NN-TODIM strategy [43] and T2NN-TODIM-VIKOR strategy [44]. The comparative results are raised in Table 7.

Table 7. Order for different strategies

Strategies	Order
T2NNWA strategy [24]	$HA_4 > HA_2 > HA_3 > HA_5 > HA_1$
T2NNWG strategy [24]	$HA_4 > HA_2 > HA_5 > HA_3 > HA_1$
T2NN-CE strategy [40]	$HA_4 > HA_2 > HA_3 > HA_5 > HA_1$
T2NN-TOPSIS strategy [24]	$HA_4 > HA_2 > HA_3 > HA_5 > HA_1$
T2NN-EDAS strategy [41]	$HA_4 > HA_2 > HA_5 > HA_3 > HA_1$
T2NN-MABAC strategy [42]	$HA_4 > HA_2 > HA_3 > HA_5 > HA_1$
T2NN-TODIM strategy [43]	$HA_4 > HA_2 > HA_3 > HA_5 > HA_1$
T2NN-TODIM-VIKOR strategy [44]	$HA_4 > HA_2 > HA_3 > HA_5 > HA_1$
T2NN-LogTODIM-TOPSIS strategy	$HA_4 > HA_2 > HA_3 > HA_5 > HA_1$

Based on the detailed analysis above, it can be concluded that while the rankings produced by the different approaches vary slightly, they consistently identify the same vocational college as the best and the worst performer. This consistency confirms the effectiveness of the T2NN-LogTODIM-TOPSIS approach for evaluating the employment quality of vocational college students within the context of innovation and entrepreneurship ("double innovation").

5. Conclusion

The forms and structures of various industries are inherently complex and multifaceted, leading to a similarly diversified employment landscape. The employment structure not only comprehensively reflects the career trajectories and regional disparities among vocational college graduates, but also captures the characteristics of their employers, as well as their capacities for innovation and entrepreneurial endeavors. Analyzing this employment structure provides a comprehensive view of employment quality, while the competitiveness of vocational graduates can be inferred from the professional paths they pursue. At present, the economic development across different regions in China remains uneven. By examining employment structures through the lens of regional economic levels, we can gain deeper insights into the employment preferences of vocational college students and the varying levels of competition and job market pressures between different areas. Currently, there is a

noticeable trend where vocational graduates gravitate toward regions with higher economic development for employment opportunities. In these economically advanced areas, higher employment rates correlate with more favorable job prospects and career growth for graduates. This suggests that the employment structure not only serves as an indicator of the improvement in graduate quality but also highlights the contributions of various professional talents to regional economic growth. When evaluating the employment quality of vocational college students in the context of innovation and entrepreneurship ("double innovation"), it becomes a MADM problem. To address this, the LogTODIM and TOPSIS techniques have been proposed for such MADM scenarios. To better handle the fuzzy and uncertain information that arises during the employment quality evaluation, T2NNs have been introduced. In this study, a T2NN-LogTODIM-TOPSIS model is presented as a solution for MADM, integrating the capabilities of T2NNs to manage uncertainty. A numerical example is provided to illustrate the employment quality evaluation of vocational college students under the double innovation framework, and a comparative analysis is conducted to verify the effectiveness of the T2NN-LogTODIM-TOPSIS approach.

References

[1] R.K. Johnson, H. Smiciklaswright, A.C. Crouter, F.K. Willits, Maternal employment and the quality of young childrens diets - empirical-evidence based on the 1987-1988 nationwide food-consumption survey, Pediatrics, 90 (1992) 245-249.

[2] C. Blake, M.B. Codd, A. Cassidy, Y.M. O'Meara, Physical function, employment and quality of life in end-stage renal disease, Journal of Nephrology, 13 (2000) 142-149.

[3] S. Reisine, J. Fifield, S. Walsh, D. Dauser, A. Micha, The effects of employment on quality of life among female fibromyalgia (fm) patients (pts) and controls (cs), Arthritis and Rheumatism, 48 (2003) S426-S426.

[4] B.Y. Cartwright, B.S.K. Kim, Selected factors associated with quality employment outcomes, Journal of Rehabilitation, 72 (2006) 41-47.

[5] S. Reisine, J. Fifield, S. Walsh, D. Dauser, M. Abeles, Employment and quality of life among women with recent onset rheumatoid arthritis (ra) and controls, Arthritis and Rheumatism, 54 (2006) S773-S773.

[6] P.M. Meiners, J.M. Meijer, J. Slater, F.K.L. Spijkervet, C.G.M. Kallenberg, A. Vissink, H. Bootsma, Quality of life, employment and disability in patients with sjogren's syndrome, Arthritis and Rheumatism, 58 (2008) S789-S789.

[7] J. Yazdany, E. Yelin, Health-related quality of life and employment among persons with systemic lupus erythematosus, Rheumatic Disease Clinics of North America, 36 (2010) 15-

32.

[8] S.B. Andrea, J. Eisenberg-Guyot, T. Peckham, V.M. Oddo, A. Hajat, Intersectional trends in employment quality in older adults in the united states, Ssm-Population Health, 15 (2021) 3.

[9] L. Roncolato, C. Koh, Underground employment: Analyzing the job quality of new york city subway dancers, Feminist Economics, 27 (2021) 77-101.

[10] L.H. Song, University employment quality evaluation system based on multicriteria decision and data analysis, Scientific Programming, 2021 (2021) 7.

[11] S. Baron, I. Cuervo, G. Winkel, D. Flores, A. Gonzalez, H. Harari, Employment quality and mental and self-reported health inequities among latinx housecleaners: The safe and just cleaners study, International Journal of Environmental Research and Public Health, 19 (2022) 17.

[12] M. Bisello, V. Maccarrone, E. Fernández-Macías, Occupational mobility, employment transitions and job quality in europe: The impact of the great recession, Economic and Industrial Democracy, 43 (2022) 585-611.

[13] M. Hasrati, Putting the employment status of faculty into the quality of education: A canadian case study, Innovations in Education and Teaching International, 59 (2022) 658-668.

[14] W. Kim, K.T. Han, S. Kim, Health-related quality of life among cancer patients and survivors and its relationship with current employment status, Supportive Care in Cancer, 30 (2022) 4547-4555.

[15] Y.Y. Yu, X.Z. Meng, Optimal decision making by farmer households: Nonfarm employment quality and agricultural production investment flexibility, Custos E Agronegocio on Line, 19 (2022) 2-22.

[16] S. Zaman, Q.U. Zaman, L.L. Zhang, Z.L. Wang, N. Jehan, Interaction between agricultural production, female employment, renewable energy, and environmental quality: Policy directions in context of developing economies, Renewable Energy, 186 (2022) 288-298.

[17] L. Zhang, Research on the influence of rural revitalization and high-quality employment on college students' employment anxiety, Psychiatria Danubina, 34 (2022) S828-S829.

[18] Y. Cao, N.D. DiPiro, J.S. Krause, Longitudinal changes in employment, health, participation, and quality-of-life and the relationships with long-term survival after spinal cord injury, Spinal Cord, 61 (2023) 430-435.

[19] X.H. Gu, X.K. Wang, S. Liang, Employment quality evaluation model based on hybrid intelligent algorithm, Cmc-Computers Materials & Continua, 74 (2023) 131-139.

[20] D.J. Kinitz, F.V. Shahidi, L.E. Ross, Job quality and precarious employment among lesbian, gay, and bisexual workers: A national study, Ssm-Population Health, 24 (2023) 9.

[21] A.B. Leoneti, L.F. Autran Monteiro Gomes, A novel version of the todim method based on the exponential model of prospect theory: The exptodim method, European Journal of Operational Research, 295 (2021) 1042-1055.

[22] Y.-J. Lai, T.-Y. Liu, C.-L. Hwang, Topsis for modm, European Journal of Operational Research, 76 (1994) 486-500.

[23] K.P. Yoon, C.-L. Hwang, Multiple attribute decision making: An introduction, Sage publications, 1995.

[24] M. Abdel-Basset, M. Saleh, A. Gamal, F. Smarandache, An approach of topsis technique for developing supplier selection with group decision making under type-2 neutrosophic number, Applied Soft Computing, 77 (2019) 438-452.

[25] H. Wang, F. Smarandache, Y.Q. Zhang, R. Sunderraman, Single valued neutrosophic sets, Multispace Multistruct, (2010) 410-413.

[26] C.E. Shannon, A mathematical theory of communication, Bell System Technical Journal, 27 (1948) 379-423.

[27] A.B. Leoneti, L. Gomes, A novel version of the todim method based on the exponential model of prospect theory: The exptodim method, European Journal of Operational Research, 295 (2021) 1042-1055.

[28] M. Jolly, J.A. Block, Employment, work disability and health related quality of life systemic lupus erythematosus, Arthritis and Rheumatism, 58 (2008) S806-S806.

[29] A. Raies, Sustainable employment in developing and emerging countries: Testing augmented okun's law in light of institutional quality, Sustainability, 15 (2023) 13.

[30] L.J. Zhao, S. Du, An improved todim-topsis method for quality evaluation of college students employment and entrepreneurship education with probabilistic hesitant fuzzy sets, Journal of Intelligent & Fuzzy Systems, 45 (2023) 7547-7562.

[31] E. Vigna, A. Meek, S. Beyer, Supported employment, quality of jobs and employment typicalness: The experience of the engage to change project, Journal of Applied Research in Intellectual Disabilities, 37 (2024) 12.

[32] J.C. Dixon, D.B. McCollum, A.S. Fullerton, Who is a part-time worker around the world and why does it matter? Examining the quality of employment measures and workers' perceived job quality, Sociological Spectrum, 38 (2018) 1-23.

[33] D. Min, E. Cho, Patterns in quality of life according to employment among the older adults: The korean longitudinal study of aging (2008-2014), Bmc Public Health, 18 (2018) 10.

[34] L. Cichocki, A. Arciszewska, P. Bladzinski, M. Hat, A. Kalisz, A. Cechnicki, Differences in subjective quality of life of people with a schizophrenia diagnosis between participants in occupational therapy workshops and those working in a sheltered employment establishment, Psychiatria Polska, 53 (2019) 81-92.

[35] J.M.P. Poritz, L. Vos, E. Ngan, L. Leon-Novelo, M. Sherer, Gender differences in employment and economic quality of life following traumatic brain injury, Rehabilitation Psychology, 64 (2019) 65-71.

[36] K.K. Xue, D.D. Xu, S.Q. Liu, Social network influences on non-agricultural employment quality for part-time peasants: A case study of sichuan province, china, Sustainability, 11 (2019) 22.

[37] T. Ye, H.M. Dai, L. Lin, J.F. Lu, Employment of κ-carrageenan and high pressure processing for quality improvement of reduced nacl surimi gels, Journal of Food Processing and Preservation, 43 (2019) 9.

[38] M. Meade, K.S. Reed, P. Rumrill, R. Aust, J.S. Krause, Perceptions of quality of employment

outcomes after multiple sclerosis: A qualitative study, Journal of Rehabilitation, 82 (2016) 31-40.

[39] L. Roncolato, J. Willoughby, Job quality complexities: Self-employment within the low-income communities surrounding cape town, south africa, Review of Radical Political Economics, 49 (2017) 30-53.

[40] X. Li, Modified cross-entropy method for multiple-attribute decision making with type-2 neutrosophic number and applications to risk assessment of internet supply chain finance, International Journal of Knowledge-Based and Intelligent Engineering Systems, 27 (2023) 207-218.

[41] U. Cali, M. Deveci, S.S. Saha, U. Halden, F. Smarandache, Prioritizing energy blockchain use cases using type-2 neutrosophic number-based edas, Ieee Access, 10 (2022) 34260-34276.

[42] V. Simic, I. Gokasar, M. Deveci, A. Karakurt, An integrated critic and mabac based type-2 neutrosophic model for public transportation pricing system selection, Socio-Economic Planning Sciences, 80 (2022) 22.

[43] Z.Y. Wang, Q. Cai, G.W. Wei, Modified todim method based on cumulative prospect theory with type-2 neutrosophic number for green supplier selection, Engineering Applications of Artificial Intelligence, 126 (2023) 18.

[44] Z.Y. Wang, Q. Cai, G.W. Wei, Enhanced todim based on vikor method for multi-attribute decision making with type-2 neutrosophic number and applications to green supplier selection, Soft Computing, (2023) 15.

Received: June 22, 2024. Accepted: Oct 13, 202

NSS
Neutrosophic Sets and Systems, Vol. 75, 2025
University of New Mexico

Enhancing DVNN-WCSM Technique for Double-Valued Neutrosophic Multiple-Attribute Decision-Making in Digital Economy: A Case Study on Enhancing the Quality of Development of Henan's Cultural and Tourism Industry

Yourong Guo*

School of Cultural Heritage and Communication, Luoyang Polytechnic, Luoyang 471000, Henan, China
Corresponding author, e-mail: 202352001@lypt.edu.cn
https://orcid.org/0009-0000-7798-4851

Abstract: Henan Province is the core area of the Yellow River Basin and an important area for inheritance of Yellow River culture. Promoting the development of the cultural and tourism industry in Henan Province essentially involves integrating the cultural resources of the Yellow River in Henan Province, creating cultural tourism products and projects with the cultural characteristics of the Yellow River region, thereby enriching the content of Henan's tourism industry and promoting its economic development. In the future development of the cultural and tourism industry, Henan Province not only needs to solve current problems of insufficient industrial competitiveness, lack of product innovation, and incomplete brand shaping, but also needs to pay attention to the economic and social benefits of the cultural and tourism industry. Only in this way could we truly achieve the high-quality development goals of the cultural and tourism industry. The high-quality development quality evaluation of Henan's cultural and tourism industry driven by digital economy is multiple-attribute decision-making (MADM). The cosine similarity measure (CSM) has been put forward MADM. The double-valued neutrosophic sets (DVNSs) are put forward characterizing fuzzy information during the high-quality development quality evaluation of Henan's cultural and tourism industry driven by digital economy. In this study, the double-valued neutrosophic number CSM (DVNN-CSM) technique and double-valued neutrosophic number weighted CSM (DVNN-WCSM) technique are put forward the MADM along with DVNSs. Finally, a numerical study for high-quality development quality evaluation of Henan's cultural and tourism industry driven by digital economy is put forward validating the DVNNN-WCSM.

Keywords: Multiple-attribute decision-making (MADM); double-valued neutrosophic sets (DVNSs); cosine similarity measure (CSM); high-quality development quality evaluation

1. Introduction

The 20th National Congress of the Communist Party of China's report underscores the initiative to establish a digital China by accelerating the digital economy's growth and fostering its integration with the real economy. In response, the Ministry of Culture and Tourism in 2020 released a directive to enhance the digital cultural industry's quality development. It aims to merge cultural and tourism sectors within the digital economy framework and to expand superior digital cultural and tourism initiatives. Zhengzhou, recognized for its unique cultural and tourism assets labeled as "the soul of the Yellow River, the root of China, the center of heaven and earth, the capital of kung fu, and the source of civilization," stands out in this context. Consequently, the Henan Provincial Government unveiled the 14th Five Year Plan for Cultural Tourism Integration Development of Henan Province. This plan is set to deepen the synergy and

innovation between the "Internet plus tourism" and the digital cultural industry, aiding Henan's ambition to become a global cultural tourism nexus and a national beacon for cultural innovation. Furthermore, the Henan Province Cultural and Tourism Industry Development Work Conference emphasized a commitment to innovation-driven growth, leveraging digital technology to cultivate new cultural and tourism formats. Seizing this momentum, various departments within Henan Province have issued strategic documents advocating for a robust cultural and tourism city. These include positioning the Yellow River as a historic and cultural landmark and setting long-term objectives to establish a formidable cultural and tourism city [1-3]. Therefore, by advancing the high-quality development of the cultural and tourism industry in Henan Province through the digital economy, crafting a compelling narrative about the "Mountains and Rivers of the Motherland," and establishing the brand "Walking in Henan Province, Understanding the Earliest China," Henan can effectively integrate into the "Three Cities, Three Hundred Miles, Three Thousand Years" cultural belt of Zhengzhou and Bianluo. This integration will support Zhengzhou's efforts to become a nationally recognized historical and cultural landmark city and a central city at the national level. Located in the Yellow River Basin, Henan Province's cultural and tourism sector is deeply rooted in the culture of the Yellow River Basin [4]. To fully leverage the advantages of Henan's cultural and tourism natural resources, it is necessary to focus on ecological environment of Yellow River Basin, strengthen environmental governance, and create a green cultural and tourism industry. Henan Province aims to promote the green concept and encourage industrial entities to consciously comply with environmental protection requirements, develop cultural and tourism routes, and develop innovative cultural and tourism products based on not damaging the ecological environment and not affecting ecological balance[5-7]. Only in this way can we achieve good economic benefits while enhancing the social benefits of Henan's cultural and tourism industry, and achieve healthy and sustainable industry development. In the actual development process, cultural and tourism routes can be developed with themes such as "protecting the environment", "Yellow River governance", and "wetland protection"[8-10]. While promoting Yellow River culture, the process of environmental changes in Yellow River Basin can be explained, and tourists can be supported to contribute to ecological environment governance of Yellow River Basin through methods such as "crowdfunding", "small donations", and "collective tree planting". Relevant departments can also collaborate with new media platforms such as WeChat and Xiaohongshu to design activities such as "Henan Cultural and Tourism Steps Exchange for Tree Planting Points" and "Scenic Area Check in Records Exchange for Geopark Maintenance Points", in order to stimulate the enthusiasm of tourists and guide them to make their own contributions to the improvement of the ecological environment in Yellow River Basin. Compared to other inland provinces, Henan Province has certain advantages in the development of cultural and tourism industries[11-13]. This is because Henan is located on a transportation hub: cities such as Zhengzhou and Luoyang have been important hubs for the circulation of goods and economic exchanges since ancient times, and Kaifeng has a long history of cultural, economic, and social development. In addition, many cities in Henan are distributed along the Yellow River Basin and have rich cultural resources in the Yellow River Basin[14-16]. These are the unique conditions for Henan's cultural and tourism industry to create distinctive brands. Therefore, it is necessary to strengthen brand building, based on the Yellow River culture, to create the "Greater Yellow River Provincial Cultural and Tourism" brand, and based on this, to strengthen cooperation and exchange

among cities in Henan Province, and to create tourism routes between cities. If we can take advantage of the flow of people in Luoyang and attract tourists with its popularity, we can transport tourists from Luoyang to unknown urban attractions in the surrounding area, which can play a driving role[17, 18]. Afterwards, relevant departments of various cities can build sub-brands based on the "Greater Yellow River Provincial Cultural and Tourism" brand. For instance, in Kaifeng City, leveraging the historical significance of the "Bianjing Ancient City" in conjunction with the Sui and Tang dynasties can create a powerful brand effect. To enable high-quality development of Henan's cultural and tourism industry from an integration perspective, it is essential for the province to blend modern and historical cultures. This includes merging traditional culture with high-tech, and integrating the cultural and creative industries with tourism, catering, hotel industries, and other sectors.

From this perspective of fusion, a strategic step could be to establish a "Yellow River Cultural Landmark" to foster the cohesion of urban cultural and tourism resources within Henan Province. Echoing the insights shared by Secretary Xu Liyi at the "Cultural Tourism Enterprise Symposium," there is a critical need to capitalize on the strengths of historical cities, align with industrial trends, and effectively develop the cultural tourism industry [19-21]. In the process of developing the cultural and tourism industry, it is necessary to fully leverage Zhengzhou's geographical advantages, Kaifeng's historical advantages, and Luoyang's climate advantages, highlight urban characteristics, and build a Yellow River cultural landmark that integrates multiple urban features[22-24]. Secondly, we need to strengthen industrial integration and enrich the content of cultural and tourism integration system. It is recommended to rely on the infrastructure of the cultural and tourism industry to further carry out the construction of projects such as Henan Film and Television City, Republican style streets, and movie towns, achieving the integration of multiple industries such as film and television industry, construction industry, retail industry, and cultural and tourism industry[25-27].

The high-quality development quality evaluation of Henan’s cultural and tourism industry driven by the digital economy is MADM [28, 29]. The CSM approach was put forward by the MADM. The DVNSs [30] are put forward managing fuzzy information during the high-quality development quality evaluation of Henan’s cultural and tourism industry driven by the digital economy. However, it’s known that the existing studies about the CSM approach along with DVNSs do not exist. Hence, it’s very necessary to take the CSM approach along with DVNSs into account. Thus, in this study, the double-valued neutrosophic number CSM (DVNN-CSM) technique and double-valued neutrosophic number weighted CSM (DVNN-WCSM) technique are put forward the MADM along with DVNSs. Finally, a numerical study for high-quality development quality evaluation of Henan’s cultural and tourism industry driven by the digital economy is put forward validating the DVNNN-WCSM. The major contributions of this study are administrated: (1) the CSM approach was enhanced along with DVNSs; (2) the entropy put forward the weight values along with DVNSs; (3) the DVNN-WCSM approach put forward the MAGDM along with DVNSs; (4) numerical example for high-quality development quality evaluation of Henan’s cultural and tourism industry driven by digital economy is administrated to show the DVNN-WCSM approach. (5) the DVNN-WCSM approach is validated through different comparative studies.

The study structure of this work is administrated. In Sect. 2, the DVNSs is administrated. In Sect. 3, the DVNN-CSM and DVNN-WCSM model is put forward along with DVNSs. In Sect. 4, the DVNN-

WCSM model is put forward the MADM along with DVNSs. Sect. 5 administrates numerical examples for high-quality development quality evaluation of Henan's cultural and tourism industry driven by the digital economy and different comparative techniques. Comparative analysis is administrated in Sect. 6. Remarks are administrated in Sect. 7.

2. Preliminaries

Kandasamy [30] administrated the DVNSs.

Definition 1 [30]. The DVNSs DA in Θ is administrated:

$$DA=\left\{\left(\theta,DT_A(\theta),DIT_A(\theta),DIF_A(\theta),DF_A(\theta)\right)\middle|\theta\in\Theta\right\}. \tag{1}$$

where $DT_A(\theta)$ is truth-membership, $DIT_A(\theta)$ is indeterminacy leaning towards $DT_A(\theta)$, $DIF_A(\theta)$ is indeterminacy leaning towards $DF_A(\theta)$, $DF_A(\theta)$ is falsity-membership, $DT_A(\theta),DIT_A(\theta),DIF_A(\theta),DF_A(\theta)\in[0,1]$, $0\le DT_A(\theta)+DIT_A(\theta)+DIF_A(\theta)+DF_A(\theta)\le 4$.

The DVNN is administrated as: $DA=\left(DT_A,DIT_A,DIF_A,DF_A\right)$, $DT_A,DIT_A,DIF_A,DF_A\in[0,1]$, $0\le DT_A+DIT_A+DIF_A+DF_A\le 4$.

Definition 2[30]. Let $DA=\left(DT_A,DIT_A,DIF_A,DF_A\right)$ be DVNN, the score value is administrated:

$$SV(DA)=\frac{\left(2+DT_A+DIT_A-DIF_A-DF_A\right)}{4},\ SV(DA)\in[0,1]. \tag{2}$$

Definition 3[30]. Let $DA=\left(DT_A,DIT_A,DIF_A,DF_A\right)$ be DVNN, the accuracy value is administrated:

$$AV(DA)=\frac{\left(DT_A+DIT_A+DIF_A+DF_A\right)}{4},\ AV(DA)\in[0,1]\ . \tag{3}$$

The order relation for DVNNs is administrated.

Definition 4[30]. Let $DA=\left(DT_A,DIT_A,DIF_A,DF_A\right)$ and $DB=\left(DT_B,DIT_B,DIF_B,DF_B\right)$,

$SV(DA)=\frac{\left(2+DT_A+DIT_A-DIF_A-DF_A\right)}{4}$, $SV(DB)=\frac{\left(2+DT_B+DIT_B-DIF_B-DF_B\right)}{4}$,

$AV(DA)=\frac{\left(DT_A+DIT_A+DIF_A+DF_A\right)}{4}$, $AV(DB)=\frac{\left(DT_B+DIT_B+DIF_B+DF_B\right)}{4}$, if

$SV\left(DA\right)<SV\left(DB\right)$, $DA<DB$; if $SV\left(DA\right)=SV\left(DB\right)$, (1)if $AV\left(DA\right)=AV\left(DB\right)$, $DA=DB$; (2) if $AV\left(DA\right)<AV\left(DB\right)$, $DA<DB$.

Definition 5[30]. Let $DA=\left(DT_A,DIT_A,DIF_A,DF_A\right)$ and $DB=\left(DT_B,DIT_B,DIF_B,DF_B\right)$ be two DVNNs, the operations are administrated:

(1) $DA\oplus DB=\left(DT_A+DT_B-DT_ADT_B,DIT_A+DIT_B-DIT_ADIT_B,DIF_ADIF_B,DF_ADF_B\right)$;

(2) $DA\otimes DB=\left(DT_ADT_B,DIT_ADIT_B,DIF_A+DIF_B-DIF_ADIF_B,DF_A+DF_B-DF_ADF_B\right)$;

(3) $\lambda DA=\left(1-\left(1-DT_A\right)^{\lambda},1-\left(1-DIT_A\right)^{\lambda},\left(DIF_A\right)^{\lambda},\left(DF_A\right)^{\lambda}\right),\lambda>0$;

(4) $\left(DA\right)^{\lambda}=\left(\left(DT_A\right)^{\lambda},\left(DIT_A\right)^{\lambda},1-\left(1-DIF_A\right)^{\lambda},1-\left(1-DF_A\right)^{\lambda}\right),\lambda>0.$

3. Some CSM approaches for DVNSs based on cosine function

In light with the idea of CSM [31-37], DVNN-CSM and DVNN-WCSM are administrated.

Definition 6. Let $DA=\left\{\left\langle\xi_j,\left(DT_A\left(\xi_j\right),DIT_A\left(\xi_j\right),DIF_A\left(\xi_j\right),DF_A\left(\xi_j\right)\right)\right\rangle\middle|\xi_j\in\xi\right\}$ and $DB=\left\{\left\langle\xi_j,\left(DT_B\left(\xi_j\right),DIT_B\left(\xi_j\right),DIF_B\left(\xi_j\right),DF_B\left(\xi_j\right)\right)\right\rangle\middle|\xi_j\in\xi\right\}$ be DVNSs in $\xi=\left\{\xi_1,\xi_2,\cdots,\xi_n\right\}$. Then, DVNN-CSM and DVNN-WCSM are administrated between DA and DB:

$$DVNN\text{-}CSM\left(DA,DB\right)=\frac{1}{n}\sum_{j=1}^{n}\cos\left[\frac{\pi}{4}\begin{pmatrix}\left|DT_A\left(\xi_j\right)-DT_B\left(\xi_j\right)\right|\vee\\ \left|DIT_A\left(\xi_j\right)-DIT_B\left(\xi_j\right)\right|\vee\\ \left|DIF_A\left(\xi_j\right)-DIF_B\left(\xi_j\right)\right|\vee\\ \left|DF_A\left(\xi_j\right)-DF_B\left(\xi_j\right)\right|\end{pmatrix}+\frac{\pi}{16}\begin{pmatrix}\left|DT_A\left(\xi_j\right)-DT_B\left(\xi_j\right)\right|+\\ \left|DIT_A\left(\xi_j\right)-DIT_B\left(\xi_j\right)\right|+\\ \left|DIF_A\left(\xi_j\right)-DIF_B\left(\xi_j\right)\right|+\\ \left|DF_A\left(\xi_j\right)-DF_B\left(\xi_j\right)\right|\end{pmatrix}\right] \tag{4}$$

Proposition 1. For DA and DB in $\xi=\left\{\xi_1,\xi_2,\cdots,\xi_n\right\}$, the $DVNN\text{-}CSM\left(DA,DB\right)$ meet the properties (1)-(4):

(1) $0\le DVNN\text{-}CSM\left(DA,DB\right)\le 1$;

(2) $DVNN\text{-}CSM\left(DA,DB\right)=1$, if and only if $DA=DB$;

(3) $DVNN\text{-}CSM\left(DA,DB\right)=DVNN\text{-}CSM\left(DB,DA\right)$;

(4) If DC is DVNS in $\xi=\left\{\xi_1,\xi_2,\cdots,\xi_n\right\}$, $DA\subseteq DB\subseteq DC$, then

$$DVNN\text{-}CSM\left(DA,DC\right)\le DVNN\text{-}CSM\left(DA,DB\right),$$

$$DVNN\text{-}CSM\left(DA,DC\right)\le DVNN\text{-}CSM\left(DB,DC\right).$$

Proof:

(1) Since cosine value is [0,1], $0\le DVNN\text{-}CSM\left(DA,DB\right)\le 1$.

(2) For DVNSs DA and DB in $\xi=\{\xi_1,\xi_2,\cdots,\xi_n\}$, if $DA=DB$, then

$$DT_A\left(\xi_j\right)=DT_B\left(\xi_j\right),DIT_A\left(\xi_j\right)=DIT_B\left(\xi_j\right),DIF_A\left(\xi_j\right)=DIF_B\left(\xi_j\right),DF_A\left(\xi_j\right)=DF_B\left(\xi_j\right).$$

Thus, $\left|DT_A\left(\xi_j\right)-DT_B\left(\xi_j\right)\right|=0$, $\left|DIT_A\left(\xi_j\right)-DIT_B\left(\xi_j\right)\right|=0$, $\left|DIF_A\left(\xi_j\right)-DIF_B\left(\xi_j\right)\right|=0$, $\left|DF_A\left(\xi_j\right)-DF_B\left(\xi_j\right)\right|=0$. Thus, $DVNN\text{-}CSM\left(DA,DB\right)=1$.

If $DVNN\text{-}CSM\left(DA,DB\right)=1$, then $\left|DT_A\left(\xi_j\right)-DT_B\left(\xi_j\right)\right|=0$, $\left|DIT_A\left(\xi_j\right)-DIT_B\left(\xi_j\right)\right|=0$, $\left|DIF_A\left(\xi_j\right)-DIF_B\left(\xi_j\right)\right|=0$, $\left|DF_A\left(xx_j\right)-DF_B\left(xx_j\right)\right|=0$. Since $\cos(0)=1$. Then, $DT_A\left(\xi_j\right)=DT_B\left(\xi_j\right),DIT_A\left(\xi_j\right)=DIT_B\left(\xi_j\right),\ DIF_A\left(\xi_j\right)=DIF_B\left(\xi_j\right),DF_A\left(\xi_j\right)=DF_B\left(\xi_j\right)$ Hence $DA=DB$.

(3) Proof is apparent.

(4) If $DA\subseteq DB\subseteq DC$, then $DT_A\left(\xi_j\right)\le DT_B\left(\xi_j\right)\le DT_C\left(\xi_j\right)$, $DIT_A\left(\xi_j\right)\le DIT_B\left(\xi_j\right)\le DIT_C\left(\xi_j\right)$, $DIF_A\left(\xi_j\right)\ge DIF_B\left(\xi_j\right)\le DIF_C\left(\xi_j\right)$, $DF_A\left(\xi_j\right)\ge DF_B\left(\xi_j\right)\le DF_C\left(\xi_j\right)$ for $j=1,2,\cdots,n$. Then, we have

$$\left|DT_A\left(\xi_j\right)-DT_B\left(\xi_j\right)\right|\le\left|DT_A\left(\xi_j\right)-DT_C\left(\xi_j\right)\right|,$$

$$\left|DT_B\left(\xi_j\right)-DT_C\left(\xi_j\right)\right|\le\left|DT_A\left(\xi_j\right)-DT_C\left(\xi_j\right)\right|,$$

$$\left|DIT_A\left(\xi_j\right)-DIT_B\left(\xi_j\right)\right|\le\left|DIT_A\left(\xi_j\right)-DIT_C\left(\xi_j\right)\right|,$$

$$\left|DIT_B\left(\xi_j\right)-DIT_C\left(\xi_j\right)\right|\le\left|DIT_A\left(\xi_j\right)-DIT_C\left(\xi_j\right)\right|$$

$$\left|DIF_A\left(\xi_j\right)-DIF_B\left(\xi\right)\right|\le\left|DIF_A\left(\xi_j\right)-DIF_C\left(\xi_j\right)\right|,$$

$$\left|DIF_B\left(\xi_j\right)-DIF_C\left(\xi_j\right)\right|\le\left|DIF_A\left(\xi_j\right)-DIF_C\left(\xi_j\right)\right|,$$

$$\left|DF_A\left(\xi_j\right)-DF_B\left(\xi_j\right)\right|\le\left|DF_A\left(\xi_j\right)-DF_C\left(\xi_j\right)\right|,$$

$$\left|DF_B\left(\xi_j\right)-DF_C\left(\xi_j\right)\right|\le\left|DF_A\left(\xi_j\right)-DF_C\left(\xi_j\right)\right|,$$

Hence, $DVNN\text{-}CSM\left(DA,DC\right)\le DVNN\text{-}CSM\left(DA,DB\right)$,

$DVNN\text{-}CSM\left(DA,DC\right)\le DVNN\text{-}CSM\left(DB,DC\right)$, for $k=1,2$, as cosine function is decreasing for $\left[0,\frac{\pi}{2}\right]$.

In more and more MADM settings, the weight values for $\xi_j\in\xi$ need be considered. Then, the DVNN-WCSM is administrated:

$$DVNN\text{-}WCSM\left(DA,DB\right)$$
$$=\sum_{j=1}^{n}dw_j\cos\left[\frac{\pi}{4}\begin{pmatrix}\left|DT_A\left(\xi_j\right)-DT_B\left(\xi_j\right)\right|\vee\\ \left|DIT_A\left(\xi_j\right)-DIT_B\left(\xi_j\right)\right|\vee\\ \left|DIF_A\left(\xi_j\right)-DIF_B\left(\xi_j\right)\right|\vee\\ \left|DF_A\left(\xi_j\right)-DF_B\left(\xi_j\right)\right|\end{pmatrix}+\frac{\pi}{16}\begin{pmatrix}\left|DT_A\left(\xi_j\right)-DT_B\left(\xi_j\right)\right|+\\ \left|DIT_A\left(\xi_j\right)-DIT_B\left(\xi_j\right)\right|+\\ \left|DIF_A\left(\xi_j\right)-DIF_B\left(\xi_j\right)\right|+\\ \left|DF_A\left(\xi_j\right)-DF_B\left(\xi_j\right)\right|\end{pmatrix}\right] \tag{5}$$

where $dw=\left(dw_1,dw_2,\cdots,dw_n\right)^T$ is weight values of ξ_j, $dw_j\in[0,1]$, $\sum_{j=1}^{n}dw_j=1$ and the symbol "$\vee$ "is maximum operation. If $dw=\left(1/n,1/n,\cdots,1/n\right)^T$, then $DVNN\text{-}WCSM\left(DA,DB\right)=DVNN\text{-}CSM\left(DA,DB\right)$.

The $DVNN\text{-}WCSM\left(DA,DB\right)$ meet the Proposition 2.

Proposition 2. For DVNSs DA and DB in $\xi=\left\{\xi_1,\xi_2,\cdots,\xi_n\right\}$, $DVNN\text{-}WCSM\left(DA,DB\right)$ meet properties (1-4):

(5) $0\le DVNN\text{-}WCSM\left(DA,DB\right)\le 1$;

(6) $DVNN\text{-}WCSM\left(DA,DB\right)=1$, then $DA=DB$;

(7) $DVNN\text{-}WCSM\left(DA,DB\right)=DVNN\text{-}WCSM\left(DB,DA\right)$;

(8) If DC is the DVNS in XX, $DA\subseteq DB\subseteq DC$, we have: $DVNN\text{-}WCSM\left(DA,DC\right)\le DVNN\text{-}WCSM\left(DA,DB\right)$ and $DVNN\text{-}WCSM\left(DA,DC\right)\le DVNN\text{-}WCSM\left(DB,DC\right)$.

4. Proposed DVNN-WCSM technique for MADM

The DVNN-WCSM technique is administrated for MADM. Let $DA=\left\{DA_1,DA_2,\cdots,DA_m\right\}$ be alternatives, $DG=\left\{DG_1,DG_2,\cdots,DG_n\right\}$ be attributes with weight dw, where $dw_j\in\left[0,1\right]$, $\sum_{j=1}^{n}dw_j=1$. Suppose that assessed values are administrated with DVNNs $DR=\left(DR_{ij}\right)_{m\times n}=\left(DT_{ij},DIT_{ij},DIF_{ij},DF_{ij}\right)_{m\times n}$. Then, DVNN-WCSM technique is administrated for MADM.

Step 1. Administrate the DVNN-matrix $DR=\left(DR_{ij}\right)_{m\times n}=\left(DT_{ij},DIT_{ij},DIF_{ij},DF_{ij}\right)_{m\times n}$.

$$DR=\left[DR_{ij}\right]_{m\times n}=\begin{bmatrix}DR_{11} & DR_{12} & \cdots & DR_{1n}\\ DR_{21} & DR_{22} & \cdots & DR_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ DR_{m1} & DR_{m2} & \cdots & DR_{mn}\end{bmatrix} \tag{6}$$

$$DR_{ij}=\left(DT_{ij},DIT_{ij},DIT_{ij},DF_{ij}\right) \tag{7}$$

Step 2. Normalize the $DR=\left(DR_{ij}\right)_{m\times n}=\left(DT_{ij},DIT_{ij},DIF_{ij},DF_{ij}\right)_{m\times n}$ into $NDR=\left(NDR_{ij}\right)_{m\times n}=\left(NDT_{ij},NDIT_{ij},NDIF_{ij},NDF_{ij}\right)_{m\times n}$.

$$\begin{aligned}NDR_{ij}&=\left(NDT_{ij},NDIT_{ij},NDIF_{ij},NDF_{ij}\right)\\&=\begin{cases}\left(DT_{ij},DIT_{ij},DIF_{ij},DF_{ij}\right), & DG_j\ is\ benefit\ attribute\\ \left(DF_{ij},DIF_{ij},DIT_{ij},DT_{ij}\right), & DG_j\ is\ \cos t\ attribue\end{cases}\end{aligned} \tag{8}$$

Step 3. Administrate the weight numbers through entropy.

The weight is really important for MADM [38-42]. Entropy [43] is put forward weight along with HFSs [44]. The normalized DVNN-matrix $DVNNM_{ij}$ is administrated:

$$DVNNM_{ij} = \frac{SV\left(NDT_{ij}, NDIT_{ij}, NDIF_{ij}, NDF_{ij}\right)+1}{\sum_{i=1}^{m}\left(SV\left(NDT_{ij}, NDIT_{ij}, NDIF_{ij}, NDF_{ij}\right)+1\right)}, \tag{9}$$

Then, DVNN Shannon information entropy (DVNNSIE) is administrated:

$$DVNNSIE_j = -\frac{1}{\ln m}\sum_{i=1}^{m} DVNNM_{ij} \ln DVNNM_{ij} \tag{10}$$

and $DVNNM_{ij} \ln DVNNM_{ij} = 0$ if $DVNNM_{ij} = 0$.

Then, the weight numbers are administrated:

$$dw_j = \frac{1 - DVNNSIE_j}{\sum_{j=1}^{n}\left(1 - DVNNSIE_j\right)} \tag{11}$$

Step 4. Administrate the DVNN positive ideal alternative (DVNNPIA):

$$DVNNPIA = \left(DVNNPIA_1, DVNNPIA_2, \cdots, DVNNPIA_n\right) \tag{12}$$

$$DVNNPIA_j = \left(NDT_j, NDIT_j, NDIF_j, NDF_j\right) \tag{13}$$

$$\begin{aligned} &SV\left(NDT_j, NDIT_j, NDIF_j, NDF_j\right) \\ &= \max_i SV\left(NDT_{ij}, NDIT_{ij}, NDIF_{ij}, NDF_{ij}\right) \end{aligned} \tag{14}$$

Step 5. Administrate the $DVNN\text{-}WCSM\left(DA_i, DVNNPIA\right)$:

$$\begin{aligned} &DVNN\text{-}WCSM\left(DA_i, DVNNPIA\right) \\ &= \sum_{j=1}^{n} ow_j \cos\left[\frac{\pi}{4}\begin{pmatrix} \left|NDT_j - NDT_{ij}\right| \vee \\ \left|NDIT_j - NDIT_{ij}\right| \vee \\ \left|NDIF_j - NDIF_{ij}\right| \vee \\ \left|NDF_j - NDF_{ij}\right| \end{pmatrix} + \frac{\pi}{16}\begin{pmatrix} \left|NDT_j - NDT_{ij}\right| + \\ \left|NDIT_j - NDIT_{ij}\right| + \\ \left|NDIF_j - NDIF_{ij}\right| + \\ \left|NDF_j - NDF_{ij}\right| \end{pmatrix}\right] \end{aligned} \tag{15}$$

Step 6. In line with $DVNN\text{-}WCSM\left(DA_i, DVNNPIA\right)$, the order is obtained. The largest $DVNN\text{-}WCSM\left(DA_i, DVNNPIA\right)$ is optimal one.

5. Application on double-valued Neutrosophic set

Guided by documents like the "2020 Key Points for Ecological Protection and High-Quality Development of the Yellow River Basin in Henan Province" and the "Henan Tourism Regulations," there is a pressing

need for relevant departments to enhance the development of cultural and tourism industry resources in Henan Province. The focus is on expanding the cultural and tourism sectors while also considering the ecological environment and regional economic growth, aiming to significantly boost the economic and social benefits of these industries. There are, however, existing challenges within Henan's cultural and tourism sector that demand attention. Addressing these challenges requires the formulation of high-quality development strategies tailored to the region's unique characteristics, thereby fostering continuous progress in this sector. The current state of Henan's cultural and tourism industry, based on the statistical bulletin, presents mixed developments across different cities: (1) Zhengzhou and Luoyang showcase supply advantages in the cultural and tourism sectors, with numerous enterprises in Zhengzhou and robust public service capabilities in Luoyang. Despite this, Luoyang has fewer cultural and tourism enterprises compared to Zhengzhou. (2) Both cities are noted for their strong industrial innovation capabilities. Luoyang, in particular, excels in developing smart scenic spots, while Zhengzhou leads in cultural and creative development. (3) The industrial structure in Zhengzhou and Luoyang is relatively balanced, contrasting with other cities that face geographic limitations and reduced market influence, leading to notable imbalances. To support and enhance the cultural and tourism industry in Henan, several strategic initiatives are necessary: (1) Government Support: Increased backing for leading cultural and tourism enterprises is crucial. This includes leveraging national industrial parks and cultural industry bases within Henan to consolidate resources, boost investment in leading companies, and initiate projects like the Yellow River National Cultural Park and the Yellow River Ecological Cultural Tourism Corridor. These efforts aim to create large-scale, branded cultural and tourism projects. (2) Utilization of Public Resources: It's vital to maximize the use of public cultural and tourism resources, integrating the infrastructure of libraries, museums, and cultural centers across Henan. This approach should foster collaborations across multiple venues, developing comprehensive cultural and tourism projects, renowned tourism routes, and distinctive cultural products, thereby enriching the industry ecosystem and enhancing Henan's overall competitiveness in these sectors. The ongoing evaluation of Henan's cultural and tourism industry's quality development, driven by the digital economy, is a MADM process. Seven Henan's cultural and tourism industries are assessed with different attributes: DG_1 is thinking on the digital development of the cultural and tourism industry. DG_2 is the input of production factors required for the digital development of the

cultural and tourism industry. DG_3 is the supply of digital development in the cultural and tourism industry. DG_4 is a digital management service for the cultural and tourism industry. The DVNN-WCSM approach is putting forward the high-quality development quality evaluation of Henan's cultural and tourism industry driven by the digital economy.

Step 1. Administrate the DVNN-matrix $DR = \left(DR_{ij}\right)_{7\times4}$ (Table 1).

Table 1. $DR = \left(DR_{ij}\right)_{7\times4}$

	DG_1	DG_2
DA_1	(0.44, 0.56,0.07, 0.48)	(0.44, 0.67,0.39, 0.48)
DA_2	(0.43, 0.67,0.44, 0.44)	(0.44, 0.56,0.44, 0.46)
DA_3	(0.49, 0.49,0.44, 0.49)	(0.43, 0.76,0.44, 0.44)
DA_4	(0.46, 0.45,0.44, 0.40)	(0.44, 0.44, 0.25,0.44)
DA_5	(0.43, 0.54,0.07, 0.49)	(0.42, 0.45,0.34, 0.49)
DA_6	(0.26, 0.43, 0.34,0.47)	(0.43, 0.52,0.58, 0.57)
DA_7	(0.22, 0.39, 0.31,0.43)	(0.39, 0.48,0.54, 0.53)
	DG_3	DG_4
DA_1	(0.44, 0.48,0.07, 0.47)	(0.44, 0.65,0.43, 0.48)
DA_2	(0.43, 0.47,0.28, 0.44)	(0.43,0.28,0.63, 0.48)
DA_3	(0.45, 0.65,0.09, 0.44)	(0.43, 0.42,0.05, 0.39)
DA_4	(0.44,0.45,0.45, 0.47)	(0.36, 0.24,0.35, 0.34)
DA_5	(0.44, 0.46,0.35, 0.48)	(0.48,0.47,0.44, 0.39)
DA_6	(0.37,0.31,0.43, 0.35)	(0.31, 0.25,0.37, 0.38)
DA_7	(0.24, 0.32,0.37, 0.41)	(0.36,0.35,0.32, 0.28)

Step 2. Normalize the $DR = \left(DR_{ij}\right)_{7\times4}$ into $NDR = \left(NDR_{ij}\right)_{7\times4}$ (See Table 2).

Table 2. $NDR = \left(NDR_{ij}\right)_{7\times4}$

	DG_1	DG_2
DA_1	(0.44, 0.56,0.07, 0.48)	(0.44, 0.67,0.39, 0.48)
DA_2	(0.43, 0.67,0.44, 0.44)	(0.44, 0.56,0.44, 0.46)
DA_3	(0.49, 0.49,0.44, 0.49)	(0.43, 0.76,0.44, 0.44)
DA_4	(0.46, 0.45,0.44, 0.40)	(0.44, 0.44, 0.25,0.44)
DA_5	(0.43, 0.54,0.07, 0.49)	(0.42, 0.45,0.34, 0.49)
DA_6	(0.26, 0.43, 0.34,0.47)	(0.43, 0.52,0.58, 0.57)
DA_7	(0.22, 0.39, 0.31,0.43)	(0.39, 0.48,0.54, 0.53)
	DG_3	DG_4
DA_1	(0.44, 0.48,0.07, 0.47)	(0.44, 0.65,0.43, 0.48)

DA_2	(0.43, 0.47,0.28, 0.44)	(0.43,0.28,0.63, 0.48)
DA_3	(0.45, 0.65,0.09, 0.44)	(0.43, 0.42,0.05, 0.39)
DA_4	(0.44,0.45,0.45, 0.47)	(0.36, 0.24,0.35, 0.34)
DA_5	(0.44, 0.46,0.35, 0.48)	(0.48,0.47,0.44, 0.39)
DA_6	(0.37,0.31,0.43, 0.35)	(0.31, 0.25,0.37, 0.38)
DA_7	(0.24, 0.32,0.37, 0.41)	(0.36,0.35,0.32, 0.28)

Step 3. Administrate the weight numbers (Table 3).

Table 3. The weight numbers

	DG_1	DG_2	DG_3	DG_4
Weight	0.2356	0.3177	0.2355	0.2112

Step 4. Administrate the DVNNPIA (Table 4):

Table 4. The DVNNPIA

	DG_1	**DG_2**
DVNNPIA	(0.49, 0.49,0.44, 0.49)	(0.44, 0.67,0.39, 0.48)

	DG_3	**DG_4**
DVNNPIA	(0.45, 0.65,0.09, 0.44)	(0.48,0.47,0.44, 0.39)

Step 5. Administrate the $DVNN\text{-}WCSM\left(DA_i, DVNNPIA\right)$ (Table 5):

Table 5. The $DVNN\text{-}WCSM\left(DA_i, DVNNPIA\right)$

Similarity measures	$DVNN\text{-}WCSM\left(DA_i, DVNNPIA\right)$	Order
$\left(DA_1, DVNNPIA\right)$	0.5252	4
$\left(DA_2, DVNNPIA\right)$	0.8238	1
$\left(DA_3, DVNNPIA\right)$	0.4449	5
$\left(DA_4, DVNNPIA\right)$	0.7267	2
$\left(DA_5, DVNNPIA\right)$	0.6563	3

$\left(DA_6, DVNNPIA\right)$	0.4216	6
$\left(DA_7, DVNNPIA\right)$	0.4097	7

Step 6. In line with $DVNN\text{-}WCSM\left(DA_i, DVNNPIA\right)$, the order is $DA_2 > DA_4 > DA_5 > DA_1 > DA_3 > DA_6 > DA_7$, and the optimal Henan's cultural and tourism industry is DA_2.

6. Comparative analysis

Then, $DVNSWCSM\left(DA_i, DVNNPIA\right)$ is compared with weighted Dice similarity measures $WD_{\mathrm{DVNS}_1}\left(DA_i, DVNNPIA\right)$, $WD_{\mathrm{DVNS}_2}\left(DA_i, DVNNPIA\right)$ and weighted generalized Dice similarity measures $WGD_{\mathrm{DVNS}_1}\left(DA_i, DVNNPIA\right)$, $WGD_{\mathrm{DVNS}_2}\left(DA_i, DVNNPIA\right)$ [45] (See Table 6).

Table 6. Order for different techniques

Techniques	order
$WD_{\mathrm{DVNS}_1}\left(DA_i, DVNNPIA\right)$	$DA_2 > DA_4 > DA_5 > DA_1 > DA_3 > DA_6 > DA_7$
$WD_{\mathrm{DVNS}_2}\left(DA_i, DVNNPIA\right)$	$DA_2 > DA_4 > DA_5 > DA_1 > DA_3 > DA_6 > DA_7$
$WGD_{\mathrm{DVNS}_1}\left(DA_i, DVNNPIA\right)$	$DA_2 > DA_4 > DA_5 > DA_1 > DA_3 > DA_6 > DA_7$
$WGD_{\mathrm{DVNS}_2}\left(DA_i, DVNNPIA\right)$	$DA_2 > DA_4 > DA_5 > DA_1 > DA_3 > DA_6 > DA_7$
$DVNSWCSM\left(DA_i, DVNNPIA\right)$	$DA_2 > DA_4 > DA_5 > DA_1 > DA_3 > DA_6 > DA_7$

From the comparative analysis, it could be administrated that these techniques have the same optimal Henan's cultural and tourism industry and worst Henan's cultural and tourism industry. This verified the DVNN-WCSM approach is reasonable and effective.

7. Conclusion

Industrial integration is the foundation of the development of cultural and tourism industry. Strengthening the integration of culture and tourism industry, developing cultural tourism integration products and service projects, is the main means of comprehensively developing the cultural tourism

industry. Cultural and tourism brands are an important element in the development of the cultural and tourism industry in Henan Province. A good brand effect can attract more tourists, while also introducing foreign funds and technology to drive the high-quality development of cultural and tourism industry. From the actual situation, although Henan Province has rich cultural resources in the Yellow River Basin and distinct cultural characteristics, there is a lack of integration and utilization of these characteristic resources, and no brand of "Yellow River Basin cultural tourism" has been established. There is insufficient promotion of the ecological environment, historical culture, cultural development, and other aspects of the region, and no brand effect has been formed. The high-quality development quality evaluation of Henan's cultural and tourism industry driven by the digital economy is MADM. In this study, entropy is put forward the weight values along with DVNSs, and the DVNN-CSM technique and DVNN-WCSM technique are put forward along with DVNSs. Then, the DVNN-WCSM approach is put forward the MAGDM along with DVNSs. Finally, a numerical study for high-quality development quality evaluation of Henan's cultural and tourism industry driven by the digital economy is put forward validating the DVNNN-WCSM. The major contributions of this study are administrated: (1) the CSM approach was enhanced along with DVNSs; (2) the entropy put forward the weight values along with DVNSs; (3) the DVNN-WCSM approach put forward the MAGDM along with DVNSs; (4) numerical example for high-quality development quality evaluation of Henan's cultural and tourism industry driven by digital economy is administrated to show the DVNN-WCSM approach. (5) the DVNN-WCSM approach is validated through different comparative studies.

Acknowledgments

This work was supported by the Soft Science Project of the Henan Provincial Department of Science and Technology, Project Number 242400410527.

References

[1] Z. Cao, H. Xu, B.S.X. Teo, Sentiment of chinese tourists towards malaysia cultural heritage based on online travel reviews, Sustainability, 15 (2023) 17.

[2] Y.L. Kong, Real-time processing system and internet of things application in the cultural tourism industry development, Soft Computing, 27 (2023) 10347-10357.

[3] J.Q. Li, T.J. Lee, N. Chen, K.S. Park, Pro-environmental behaviour of the residents in sensitive tourism destinations, Journal of Vacation Marketing, 29 (2023) 291-308.

[4] G.L. Liu, Research on the role of red tourism on the inheritance and development of red culture from the perspective of psychological education, International Journal of Mental Health Nursing, 32 (2023) 30-30.

[5] X. Liu, Tourism development, environmental regulations, and natural resource management: Evidence from

g20 countries, Resources Policy, 86 (2023) 11.

[6] W. Luo, Z.H. Huang, S.Y. Cheng, Z.Q. Gan, Oriental management strategies for urban resilience: Based on the wuli-shili-renli methodology and coupled coordination degree model, Heliyon, 9 (2023) 12.

[7] Z.Y. Mao, K. Ma, Y.S. Li, C.M. Kuang, Y.J. Tang, S.W. Zhu, X.F. Zhang, P. Zhang, Q.P. Qin, Environmental evaluation and regulation method of ancient buildings based on wireless rechargeable sensor network, Ieee Sensors Journal, 23 (2023) 20865-20873.

[8] X.Y. Meng, Application of vr recognition based on image object detection algorithm in urban street landscape art design, Soft Computing, (2023) 9.

[9] P. Munoz, E. Donaque, A. Larranaga, J. Martinez, A. Mejias, Tourism-related placeness feature extraction from social media data using machine learning models, International Journal of Interactive Multimedia and Artificial Intelligence, 8 (2023) 217.

[10] P.F. Wang, X.D. Cao, Z.Y. Gao, X.Y. Su, X. Wei, The influence of leisure patterns on the subjective well-being of the floating population-a social integration perspective, Leisure Studies, (2023) 16.

[11] Y. Wang, Y.L. Gao, R. Faraj, Cross-cultural aspects of tourism and hospitality: A services marketing and management perspective, Journal of Tourism and Cultural Change, 21 (2023) 623-625.

[12] Y.X. Wang, W.W. Hu, K.S. Park, Q. Yuan, N. Chen, Examining residents? Support for night tourism: An application of the social exchange theory and emotional solidarity, Journal of Destination Marketing & Management, 28 (2023) 12.

[13] B.B. Xie, W. Wei, Y.Y. Li, C.Y. Liu, S.L. Ju, Research on spatial distribution characteristics and correlation degree of the historical and cultural towns (villages) in china, Sustainability, 15 (2023) 17.

[14] L. Yang, Y.S. Yang, S.J. Zhang, H.M. Li, H.H. Cao, Y.F. Cui, F.W. Liu, M.M. Ma, Asynchronous transformation of cropping patterns from 5800-2200 cal bp on the southern loess plateau, china, Land, 12 (2023) 18.

[15] Y.Z. Yang, S.J. Liu, X.T. Song, The co-creation of museum experience value from the perspective of visitor motivation, Sage Open, 13 (2023) 19.

[16] Q. Zhang, J. Zhang, S. Lu, Y. Liu, L. Liu, Y.Y. Wang, M.Y. Cao, Multi-resolution feature extraction and fusion for traditional village landscape analysis in remote sensing imagery, Traitement Du Signal, 40 (2023) 1259-1266.

[17] Y.X. Zhang, M.Y. Yang, Z.Y. Li, W.Z. Li, C.C. Lu, Z.G. Li, H.D. Li, F.F. Zhai, Study on the spatial distribution characteristics and influencing factors in the reuse of national industrial heritage sites in china, Sustainability, 15 (2023) 23.

[18] H. Zhu, Z.C. Chen, Population genetics and pedigree geography of trionychia japonica in the four mountains of henan province and the taihang mountains, Open Geosciences, 15 (2023) 13.

[19] Y. Li, L.Y. Jiao, M. Zhou, How does cultural heritage destination digitalization influence tourists' attitudes? The role of constructive authenticity and technology-destination fit, Journal of Travel & Tourism Marketing, 41 (2024) 221-234.

[20] W. Luo, Q.D. Ran, S.Y. Cheng, What kind of support can promote academic entrepreneurial intention: An analysis of the qualitative comparative analysis configuration effect of multiple support, Managerial and Decision Economics, 45 (2024) 1414-1427.

[21] L.X. Ouyang, S. Zhang, S.C. Zhu, Z. Liu, J. Li, Digital technology in tourism dance performance: Exploring the influence of tourists' flow experience and meaningful experience on revisit intention, Ieee Access, 12 (2024) 46347-46361.

[22] F. Wang, Analysis of the role of psychological occupational therapy in the process of cultural tourism with

tourists, Journal of Intelligent & Fuzzy Systems, 46 (2024) 2775-2788.

[23] J.J. Wang, Z. Chen, T.Z. Chen, W. Wang, B.L. Liu, Empirical analysis of factors influencing industrial eco-efficiency in the yellow river basin from a social embeddedness perspective, Heliyon, 10 (2024) 15.

[24] Z.W. Wu, X.Y. Shang, Q. Hou, J.G. Xu, Z.L. Kang, H.J. Ma, Using ultrasonic-assisted sodium bicarbonate treatment to improve the gel and rheological properties of reduced-salt pork myofibrillar protein, Meat Science, 212 (2024) 7.

[25] Q. Xu, Incorporating cnn-lstm and svm with wavelet transform methods for tourist passenger flow prediction, Soft Computing, 28 (2024) 2719-2736.

[26] X.Y. Zhang, B.F. Peng, L.L. Zhou, C.Y. Lu, Y.L. Wang, R. Liu, H. Xiang, Tourism development potential and obstacle factors of cultural heritage: Evidence from traditional music in xiangxi, Journal of Geographical Sciences, 34 (2024) 309-328.

[27] D.J. Zhong, Q.Y. He, Y.Y. Li, Y.B. Wang, J. Chen, Cultural values and homestead retention: Insights from rural tourism residents, Current Psychology, (2024) 21.

[28] S. Bhuvaneshwari, C.A.C. Sweety, A. Singh, S. Broumi, M. Talea, P.K. Raut, A novel and an efficient codas technique to solve real-life magdm problems in fermatean neutrosophic environment, Neutrosophic Sets and Systems, 72 (2024) 41-59.

[29] M. Abouhawwash, M. Jameel, Evaluation factors of solar power plants to reduce cost under neutrosophic multi-criteria decision making model, Sustainable Machine Intelligence Journal, 2 (2023) 1-11.

[30] I. Kandasamy, Double-valued neutrosophic sets, their minimum spanning trees, and clustering algorithm, Journal of Intelligent Systems, 27 (2018) 163-182.

[31] J. Ye, Cosine similarity measures for intuitionistic fuzzy sets and their applications, Mathematical and Computer Modelling, 53 (2011) 91-97.

[32] K.C. Hung, K.P. Lin, Ieee, A new intuitionistic fuzzy cosine similarity measures and its application, in: 2012 ieee international conference on industrial engineering and engineering management, 2012, pp. 2194-2198.

[33] J. Ye, Interval-valued intuitionistic fuzzy cosine similarity measures for multiple attribute decision-making, International Journal of General Systems, 42 (2013) 883-891.

[34] J. Ye, Improved cosine similarity measures of simplified neutrosophic sets for medical diagnoses, Artificial Intelligence in Medicine, 63 (2015) 171-179.

[35] S. Jimenez, F.A. Gonzalez, A. Gelbukh, Mathematical properties of soft cardinality: Enhancing jaccard, dice and cosine similarity measures with element-wise distance, Information Sciences, 367 (2016) 373-389.

[36] W. Qiu, G.C. Zhang, L. Zhu, Cosine similarity measures for dual hesitant fuzzy sets, in: J. Zhu, G. Yao (Eds.) Proceedings of the 2016 4th international conference on machinery, materials and computing technology, 2016, pp. 511-514.

[37] J. Ye, Similarity measures of intuitionistic fuzzy sets based on cosine function for the decision making of mechanical design schemes, Journal of Intelligent & Fuzzy Systems, 30 (2016) 151-158.

[38] Tehreem, A. Hussain, A. Alsanad, M.A.A. Mosleh, Spherical cubic fuzzy extended topsis method and its application in multicriteria decision-making, Mathematical Problems in Engineering, 2021 (2021) 14.

[39] R.P. Tan, W.D. Zhang, S.Q. Chen, Decision-making method based on grey relation analysis and trapezoidal fuzzy neutrosophic numbers under double incomplete information and its application in typhoon disaster assessment, Ieee Access, 8 (2020) 3606-3628.

[40] J.H. Kim, B.S. Ahn, The hierarchical vikor method with incomplete information: Supplier selection problem, Sustainability, 12 (2020) 15.

[41] M.S.A. Khan, F. Khan, J. Lemley, S. Abdullah, F. Hussain, Extended topsis method based on pythagorean cubic fuzzy multi-criteria decision making with incomplete weight information, Journal of Intelligent & Fuzzy Systems, 38 (2020) 2285-2296.

[42] P.D. Liu, W.Q. Liu, Multiple-attribute group decision-making method of linguistic q-rung orthopair fuzzy power muirhead mean operators based on entropy weight, International Journal of Intelligent Systems, 34 (2019) 1755-1794.

[43] C.E. Shannon, A mathematical theory of communication, Bell System Technical Journal, 27 (1948) 379-423.

[44] R. Wang, X. Rong, Extended group decision making method for quality evaluation of mental health education of college students with hesitant triangular fuzzy information, Journal of Intelligent & Fuzzy Systems, (2023) https://doi.org/10.3233/JIFS-231719.

[45] Q. Khan, P. Liu, T. Mahmood, Some generalized dice measures for double-valued neutrosophic sets and their applications, Mathematics, 6 (2018) 121.

Received: June 21, 2024. Accepted: Oct 12, 2024

NSS
Neutrosophic Sets and Systems, Vol. 75, 2025
University of New Mexico

Neutrosophic-Based Enhanced Framework for Multi-Attribute Decision-Making Using Single-Valued Neutrosophic Sets in Evaluating Quality of University General Education Courses

Yimo Zhao*
School of Engineering, Northeast Agricultural University, Harbin, 150030, Heilongjiang, China
*Corresponding author, E-mail: zhaoyimo@neau.edu.cn
https://orcid.org/0009-0001-3377-3013

Abstract: The quality evaluation of university general education courses is a process that comprehensively assesses aspects such as teaching effectiveness, content design, teacher qualifications, and student feedback. Its purpose is to ensure that the courses effectively enhance students' overall competencies, such as critical thinking, communication skills, and social responsibility. Through a scientific evaluation system, course design can be optimized, teaching methods improved, and overall educational quality enhanced, fostering students' all-round development. The quality evaluation of university general education courses is useful MADM. The single-valued neutrosophic sets (SVNSs) is useful approach to administrate fuzzy information during the quality evaluation of university general education courses. Currently, the Aczel-Alsina operations and power geometric (PG) approach was administrated to put forward the multiple-attribute decision-making (MADM). In this study, the single-valued neutrosophic number Aczel-Alsina power geometric (SVNNAAPG) approach is administrated in light with Aczel-Alsina operations and PG approach and SVNSs. The SVNNAAPG approach is administrated for MADM. Finally, numerical example for quality evaluation of university general education courses is administrated to conduct the SVNNAAPG approach. The major contributions of this work are tackled: (1) Aczel-Alsina operations and PG approach are enhanced under SVNSs; (2) SVNNAAPG approach is administrated in light with Aczel-Alsina operations and PG approach; (3) SVNNAAPG approach is tackled for MADM with SVNSs; (4) Finally, numerical example for quality evaluation of university general education courses is administrated to conduct the SVNNAAPG approach.

Keywords: multiple-attribute decision-making; single-valued neutrosophic sets (SVNSs); power geometric (PG); quality evaluation of university general education courses

1. Introduction

1.1. Background and Motivation

The evaluation of the quality of general education courses in universities is a systematic, comprehensive, and scientific process aimed at improving course quality and promoting the all-round development of students' overall abilities. The evaluation focuses on various aspects, including course design, teaching implementation, teaching effectiveness, and student feedback. First, the evaluation of course design looks at whether the course objectives are clear, the content is rich, the structure is reasonable, and whether it meets the actual needs and developmental requirements of students. Second, the evaluation of teaching implementation examines the teaching methods, attitudes, use of resources, and classroom management. Teachers should adopt diverse teaching methods, stimulate students' interest and initiative in learning, and emphasize the integration of theory and practice. The evaluation of teaching effectiveness is measured through students' learning outcomes, knowledge acquisition, skill enhancement, and personal development. This can be assessed through exams, assignments, project presentations, and practical operations. Lastly, student feedback is collected through surveys, interviews, and discussions to gather students' opinions and suggestions on the courses, understand their satisfaction levels, and identify areas for improvement. In summary, the evaluation of general education course quality in universities should be student-centered, considering multiple factors comprehensively. It should adopt scientific and reasonable evaluation methods, continuously improve course quality, and promote students' holistic development. Zhao [1] explored the construction of university general education curriculum systems, particularly focusing on the training of pre-service teachers in basic education. Zhao discussed several issues present in the design of general education curricula, such as randomness in course selection, unclear objectives, and little relevance to teacher training. The study emphasized the need to clarify the role of general education in teacher training, improve course management, and build a high-quality teaching workforce to better equip future teachers with comprehensive skills. Wang, Tang and Zhang [2] examined the reform of practical teaching systems for electronic information courses under the framework of general education. Using the case of Henan University of Technology, the authors emphasized the importance of constructing a comprehensive practical teaching system. They argued that reforms in teaching methods and evaluation systems, as well as the integration of curriculum groups, were crucial for enhancing students' engagement in practical activities and improving overall teaching effectiveness. Ma [3] investigated the design of general education courses in universities. Drawing inspiration from the "Harvard General Education Red Book" and the undergraduate course reforms in the U.S., Ma analyzed the implications of these developments for Chinese universities. The study aimed to promote a more comprehensive, specific, and efficient approach to general education curriculum design in China, focusing on the cultivation of students' overall competencies. Yang [4] explored the integration of MOOCs (Massive Open Online Courses) into university general education. Using the case of the course "Ten Lectures on Chinese and Foreign Cultural Spirit" from Yangtze University, the study highlighted how MOOCs could align with the goals of general education. Yang argued that the interactive and expansive nature of MOOCs made them suitable for fostering research-based and autonomous learning, aligning well with the broader goals of general education. Zu and Feng [5] focused on the construction of general education curriculum systems in the context of emerging engineering disciplines ("New Engineering"). They pointed out that there was insufficient integration between the general education curriculum and the needs of new engineering disciplines. The study called for strategies to bridge this gap, promoting the development of general education to meet the evolving demands of engineering education. Yan and Zhou [6] conducted a study on the influence of general education on students' health-related lifestyles. They examined the "Five-Classroom Teaching" fitness course at Shandong Modern College to assess how general education reforms

could positively impact students' exercise habits and overall health. The study observed that students were increasingly disengaged from physical activities due to digital immersion and proposed that well-structured general education courses could help instill healthier lifestyles. Hao [7] discussed the general education curriculum design for property management majors in universities. Hao emphasized the importance of cultivating interdisciplinary talents for the modern service industry, where comprehensive competencies are highly valued. The study highlighted the need for integrating ideological and political education into the curriculum, as well as focusing on practical skills that would prepare students for real-world challenges in property management. Gao [8] explored the goals and implementation pathways of general education within the ideological and political courses in universities. Gao argued that these courses play a crucial role in shaping future builders and successors of socialist causes, and that general education should focus not only on political literacy but also on humanistic and scientific competencies. The study proposed that ability education should take precedence over knowledge-based education to cultivate well-rounded individuals. Zeng [9] examined the optimization and innovation of college employment guidance courses under the framework of general education. Zeng analyzed the challenges faced by current employment guidance curricula and proposed strategies for improving and innovating these courses to better prepare students for career development. The study emphasized that optimizing these courses was essential for achieving higher employment rates for graduates. Gao et al. [10] analyzed the construction logic, current situation, and practical strategies for general education courses in local universities. The study identified several challenges in the current design and implementation of general education courses, such as the lack of a scientific curriculum system and inadequate teaching models. The authors proposed strategies for building a more robust general education system, including optimizing course evaluation mechanisms and strengthening implementation guarantees.

1.2. Objectives of the Study

MADM involves evaluating and selecting alternatives based on multiple criteria or attributes[11-14]. Decision-makers must consider the importance and weight of each attribute to determine the optimal solution [15-17]. This approach is often used in complex situations like resource allocation, project evaluation, and strategic planning, requiring effective handling of uncertainty and fuzziness to ensure rational and fair decision outcomes [18-24]. The quality evaluation of university general education courses is useful MADM. The SVNSs [25] is useful approach to administrate fuzzy information during the quality evaluation of university general education courses. Currently, the Aczel-Alsina operations [26, 27] and power geometric (PG) approach [28, 29] was administrated to put forward the MADM. Furthermore, many approaches administrated the Aczel-Alsina operations [30, 31] and PG approach [28, 29] to put up with the MADM, respectively. Until now, no or few techniques have been tackled on Aczel-Alsina operations and PG approach under SVNSs. In this study, the combination of Single-Valued Neutrosophic MADM with Aczel-Alsina operations and the power geometric method offers significant advantages. Firstly, SVNSs effectively handle uncertainty by providing a three-dimensional approach to fuzzy information, which is crucial for evaluating complex and subjective teaching quality. Secondly, Aczel-Alsina operations flexibly combine information, enhancing decision-making accuracy for diverse educational evaluation scenarios. The power geometric method ensures that all factors are reasonably considered and integrated through weighted averaging, resulting in comprehensive and fair outcomes. Additionally, this method is highly adaptable, allowing adjustments based on changing market and educational needs, thus providing a dynamic evaluation tool. These advantages make this approach highly valuable for assessing business English teaching quality.

Therefore, in this study, the SVNNAAPG approach is administrated in light with Aczel-Alsina operations and PG approach and SVNSs. The SVNNAAPG approach is administrated for MADM. Finally, numerical example for quality evaluation of university general education courses is administrated to conduct the SVNNAAPG approach. The major contributions of this work are tackled: (1) Aczel-Alsina operations and PG approach are enhanced under SVNSs; (2) SVNNAAPG approach is administrated in light with Aczel-Alsina operations and PG approach; (3) SVNNAAPG approach is tackled for MADM with SVNSs; (4) Finally, numerical example for quality evaluation of university general education courses is administrated to conduct the SVNNAAPG approach.

1.3. Structure of the Paper

The research structure of this work is put forward. In Sect. 2, the SVNSs is put forward. In Sect. 3, SVNNAAPG approach is administrated in light with Aczel-Alsina operations and PG approach. In Sect. 4, SVNNAAPG approach is tackled for MADM with SVNSs. Sect. 5 puts forward the numerical example for quality evaluation of university general education courses through comparative analysis. Some remarks are put forward in Sect. 6.

2. Preliminaries

Wang et al. [25] resolved the SVNSs

Definition 1 [25]. The SVNSs is resolved:

$$RA=\left\{\left(\theta,RT_A\left(\theta\right),RI_A\left(\theta\right),RF_A\left(\theta\right)\right)\middle|\theta\in X\right\} \tag{1}$$

where $RT_A\left(\theta\right),RI_A\left(\theta\right),RF_A\left(\theta\right)$ is truth-membership (TM), indeterminacy-membership (IM) and falsity-membership (FM), $RT_A\left(\theta\right),RI_A\left(\theta\right),RF_A\left(\theta\right)\in\left[0,1\right]$, $0\le RT_A\left(\theta\right)+RI_A\left(\theta\right)+RF_A\left(\theta\right)\le 3$.

The SVNN is structured as $RA=\left(RT_A,RI_A,RF_A\right)$, where $RT_A\in\left[0,1\right],RI_A\in\left[0,1\right],RF_A\in\left[0,1\right]$, and $0\le RT_A+RI_A+RF_A\le 3$.

Definition 2 [32]. Let $RA=\left(RT_A,RI_A,RF_A\right)$ and $RB=\left(RT_B,RI_B,RF_B\right)$, the score value is resolved:

$$RSV\left(RA\right)=\frac{\left(2+RT_A-RI_A-RF_A\right)}{3},\quad OSV\left(OA\right)\in\left[0,1\right]. \tag{2}$$

$$RSV\left(RB\right)=\frac{\left(2+RT_B-RI_B-RF_B\right)}{3},\qquad RSV\left(RA\right)\in\left[0,1\right]. \tag{3}$$

Definition 3[32]. Let $RA=\left(RT_A,RI_A,RF_A\right)$ and $RB=\left(RT_B,RI_B,RF_B\right)$, the accuracy value is resolved:

$$RAV\left(RA\right)=RT_A-RF_A,\ RAV\left(RA\right)\in\left[-1,1\right]\ . \tag{4}$$

$$RAV\left(RA\right)=RT_A-RF_A,\ RAV\left(RA\right)\in\left[-1,1\right]\ . \tag{5}$$

Peng et al.[32] resolved the order issues.

Definition 4[32]. Let $RA=\left(RT_A,RI_A,RF_A\right)$, $RB=\left(RT_B,RI_B,RF_B\right)$, let $RSV\left(RA\right)=\frac{\left(2+RT_A-RI_A-RF_A\right)}{3}$ and $RSV\left(RB\right)=\frac{\left(2+RT_B-RI_B-RF_B\right)}{3}$, and let

$RAV(RA) = RT_A - RF_A$ and $RAV(RA) = RT_A - RF_A$, if $RSV(RA) < RSV(RB)$, $RA < RB$; if $RSV(RA) = RSV(RB)$, (1)if $RAV(RA) = RAV(RB)$, $RA = RB$; (2) if $RAV(RA) > RAV(RB)$, $RA < RB$.

Definition 5[25, 33]. Let $RA = (RT_A, RI_A, RF_A)$ and $RB = (RT_B, RI_B, RF_B)$, $r\theta \geq 1, r\lambda > 0$, the Aczel-Alsina operations among with SVNNs are administrated:

(1) $$RA \oplus RB = \begin{pmatrix} 1 - e^{-\left(\left(-\ln(1-RT_A)\right)^{r\theta} + \left(-\ln(1-RT_B)\right)^{r\theta}\right)^{1/r\theta}}, \\ e^{-\left(\left(-\ln(RI_A)\right)^{r\theta} + \left(-\ln(RI_B)\right)^{r\theta}\right)^{1/r\theta}}, e^{-\left(\left(-\ln(RF_A)\right)^{r\theta} + \left(-\ln(RF_B)\right)^{r\theta}\right)^{1/r\theta}} \end{pmatrix};$$

(2) $$RA \otimes RB = \begin{pmatrix} e^{-\left(\left(-\ln(RT_A)\right)^{r\theta} + \left(-\ln(RT_B)\right)^{r\theta}\right)^{1/r\theta}}, \\ 1 - e^{-\left(\left(-\ln(1-RI_A)\right)^{r\theta} + \left(-\ln(1-RI_B)\right)^{r\theta}\right)^{1/r\theta}}, 1 - e^{-\left(\left(-\ln(1-RF_A)\right)^{r\theta} + \left(-\ln(1-RF_B)\right)^{r\theta}\right)^{1/r\theta}} \end{pmatrix}$$

(3) $$r\lambda RA = \begin{pmatrix} 1 - e^{-\left(r\lambda\left(-\ln(1-RT_A)\right)^{r\theta}\right)^{1/r\theta}}, e^{-\left(r\lambda\left(-\ln(RI_A)\right)^{r\theta}\right)^{1/r\theta}}, e^{-\left(r\lambda\left(-\ln(RF_A)\right)^{r\theta}\right)^{1/r\theta}} \end{pmatrix}$$

(4) $$(RA)^{r\lambda} = \begin{pmatrix} e^{-\left(r\lambda\left(-\ln(RT_A)\right)^{r\theta}\right)^{1/r\theta}}, 1 - e^{-\left(r\lambda\left(-\ln(1-RI_A)\right)^{r\theta}\right)^{1/r\theta}}, 1 - e^{-\left(r\lambda\left(-\ln(1-RF_A)\right)^{r\theta}\right)^{1/r\theta}} \end{pmatrix}.$$

3. SNNAAPG operator

Yong et al. [34] and Ashraf et al. [33] administrated the SVNNAAWG operator.

Definition 6. Let $RA_i = (RT_i, RI_i, RF_i)(i = 1, 2, \ldots, n)$ be the SVNNs with weight $rw_i = (rw_1, rw_2, \ldots, rw_n)^T$, $\sum_{i=1}^{n} rw_i = 1$, $r\theta \geq 1$. If

$$\begin{aligned} \text{SVNNAAWG}_{rw}(RA_1, RA_2, \cdots, RA_n) &= \overset{n}{\underset{i=1}{\otimes}} (RA_i)^{rw_i} \\ &= \begin{pmatrix} e^{-\left(\sum_{i=1}^{n} rw_i\left(-\ln(1-RT_j)\right)^{r\theta}\right)^{1/r\theta}}, \\ 1 - e^{-\left(\sum_{i=1}^{n} rw_i\left(-\ln(RI_j)\right)^{r\theta}\right)^{1/r\theta}}, 1 - e^{-\left(\sum_{i=1}^{n} rw_i\left(-\ln(RF_j)\right)^{r\theta}\right)^{1/r\theta}} \end{pmatrix} \end{aligned} \tag{6}$$

The SVNNAAWG has three properties.

Property 1. (idempotency). If $RA_i = RA = (RT, RI, RF)$,

$$\text{SVNNAAWG}_{rw}\left(RA_1, RA_2, \cdots, RA_n\right) = RA \tag{7}$$

Property2. (Monotonicity). Let $RA_i = (RT_{A_i}, RI_{A_i}, RF_{A_i})$, $RB_i = (RT_{B_i}, RI_{B_i}, RF_{B_i})$. If $RT_{A_i} \le RT_{B_i}, RI_{A_i} \ge RI_{B_i}, RF_{A_i} \ge RF_{B_i}$ holds for all i, then

$$\begin{aligned} &\text{SVNNAAWG}_{rw}\left(RA_1, RA_2, \cdots, RA_n\right) \\ &\le \text{SVNNAAWG}_{rw}\left(RB_1, RB_2, \cdots, RB_n\right) \end{aligned} \tag{8}$$

Property 3 (Boundedness). Let $RA_i = (RT_{A_i}, RI_{A_i}, RF_{A_i})$. If $RA^+ = (\max_i(RT_i), \min_i(RI_i), \min_i(RF_i))$, $RA^- = (\min_i(RT_i), \max_i(RI_i), \max_i(RF_i))$, then

$$RA^- \le \text{SVNNAAWG}_{rw}\left(RA_1, RA_2, \cdots, RA_n\right) \le RA^+ \tag{9}$$

Then, the SVNNAAPG operator is resolved on SVNNAAWG technique and PG technique [28, 29].

Definition 7. Let $RA_i = (RT_i, RI_i, RF_i)$ $(i = 1, 2, 3, \ldots, n)$ be SVNNs, $r\theta \ge 1$. If

$$\text{SVNNAAPG}\left(RA_1, RA_2, \cdots, RA_n\right) = \overset{n}{\underset{i=1}{\otimes}}\left(RA_i\right)^{\left(\pi + RT(RA_i)\right) \Big/ \sum_{i=1}^{n}\left(\pi + RT(RA_i)\right)} \tag{10}$$

where $\pi \in R, \pi \ge 0$, $RT\left(RA_a\right) = \sum_{\substack{j=1 \\ a \ne j}}^{m} Sup\left(RA_a, RA_j\right)$, $Sup\left(RA_a, RA_j\right)$ is support for RA_a from RA_j , with serval conditions: (1) $Sup\left(RA_a, RA_b\right) \in [0,1]$; (2) $Sup\left(RA_b, RA_a\right) = Sup\left(RA_a, RA_b\right)$; (3) $Sup\left(RA_a, RA_b\right) \ge Sup\left(RA_s, RA_t\right)$, if $d\left(RA_a, RA_b\right) \le d\left(RA_s, RA_t\right)$, where d is distance measure.

The Theorem 2 is resolved.

Theorem 2. Let $RA_i = (RT_i, RI_i, RF_i)(i = 1, 2, \ldots, n)$ be the SVNNs, $r\theta \ge 1$. If

$$\begin{aligned} &\text{SVNNAAPG}\left(RA_1, RA_2, \cdots, RA_n\right) \\ &= \overset{n}{\underset{i=1}{\otimes}}\left(RA_i\right)^{\left(\pi + RT(RA_i)\right) \Big/ \sum_{i=1}^{n}\left(\pi + RT(RA_i)\right)} \\ &= \left(\begin{array}{c} e^{-\left(\sum_{i=1}^{n} \frac{\left(\pi + RT(RA_i)\right)}{\sum_{i=1}^{n}\left(\pi + RT(RA_i)\right)} \left(-\ln\left(1 - RT_j\right)\right)^{r\theta} \right)^{1/r\theta}}, \\ 1 - e^{-\left(\sum_{i=1}^{n} \frac{\left(\pi + RT(RA_i)\right)}{\sum_{i=1}^{n}\left(\pi + RT(RA_i)\right)} \left(-\ln\left(RI_j\right)\right)^{r\theta} \right)^{1/r\theta}}, 1 - e^{-\left(\sum_{i=1}^{n} \frac{\left(\pi + RT(RA_i)\right)}{\sum_{i=1}^{n}\left(\pi + RT(RA_i)\right)} \left(-\ln\left(RF_j\right)\right)^{r\theta} \right)^{1/r\theta}} \end{array} \right) \end{aligned} \tag{11}$$

where $\pi \in R, \pi \ge 0$, $RT\left(RA_a\right) = \sum_{\substack{j=1 \\ a \ne j}}^{m} Sup\left(RA_a, RA_j\right)$, $Sup\left(RA_a, RA_j\right)$ is decision support for RA_a from RA_j , with given decision conditions: (1) $Sup\left(RA_a, RA_b\right) \in [0,1]$; (2)

$Sup\left(RA_b, RA_a\right) = Sup\left(RA_a, RA_b\right)$; (3) $Sup\left(RA_a, RA_b\right) \geq Sup\left(RA_s, RA_t\right)$, if $d\left(RA_a, RA_b\right) \leq d\left(RA_s, RA_t\right)$, where d is the distance information.

Proof:

(a) Let $i = 2$, we have:

$$\mathrm{SVNNAAPG}\left(RA_1, RA_2\right)$$

$$= \left(\left(RA_1\right)^{\left(\pi + RT\left(RA_1\right)\right) \Big/ \sum_{i=1}^{2}\left(\pi + RT\left(RA_1\right)\right)} \otimes \left(RA_2\right)^{\left(\pi + RT\left(RA_2\right)\right) \Big/ \sum_{i=1}^{2}\left(\pi + RT\left(RA_2\right)\right)} \right)$$

$$= \left(\begin{array}{c} e^{-\left(\frac{\left(\pi + RT\left(RA_1\right)\right)}{\sum_{i=1}^{2}\left(\pi + RT\left(RA_1\right)\right)} \left(-\ln\left(1 - RT_1\right)\right)^{r\theta} \right)^{1/r\theta}}, 1 - e^{-\left(\frac{\left(\pi + RT\left(RA_1\right)\right)}{\sum_{i=1}^{2}\left(\pi + RT\left(RA_1\right)\right)} \left(-\ln\left(RI_1\right)\right)^{r\theta} \right)^{1/r\theta}}, \\ 1 - e^{-\left(\frac{\left(\pi + RT\left(RA_1\right)\right)}{\sum_{i=1}^{2}\left(\pi + RT\left(RA_1\right)\right)} \left(-\ln\left(RF_1\right)\right)^{r\theta} \right)^{1/r\theta}} \end{array} \right) \otimes$$

$$\left(\begin{array}{c} e^{-\left(\frac{\left(\pi + RT\left(RA_2\right)\right)}{\sum_{i=1}^{2}\left(\pi + RT\left(RA_2\right)\right)} \left(-\ln\left(1 - RT_2\right)\right)^{r\theta} \right)^{1/r\theta}}, 1 - e^{-\left(\frac{\left(\pi + RT\left(RA_2\right)\right)}{\sum_{i=1}^{2}\left(\pi + RT\left(RA_2\right)\right)} \left(-\ln\left(RI_2\right)\right)^{r\theta} \right)^{1/r\theta}}, \\ 1 - e^{-\left(\frac{\left(\pi + RT\left(RA_2\right)\right)}{\sum_{i=1}^{2}\left(\pi + RT\left(RA_2\right)\right)} \left(-\ln\left(RF_2\right)\right)^{r\theta} \right)^{1/r\theta}} \end{array} \right)$$

$$= \left(\begin{array}{c} e^{-\left(\sum_{i=1}^{2} \frac{\left(\pi + RT\left(RA_i\right)\right)}{\sum_{i=1}^{n}\left(\pi + RT\left(RA_i\right)\right)} \left(-\ln\left(1 - RT_j\right)\right)^{r\theta} \right)^{1/r\theta}}, \\ 1 - e^{-\left(\sum_{i=1}^{2} \frac{\left(\pi + RT\left(RA_i\right)\right)}{\sum_{i=1}^{n}\left(\pi + RT\left(RA_i\right)\right)} \left(-\ln\left(OI_j\right)\right)^{r\theta} \right)^{1/r\theta}}, 1 - e^{-\left(\sum_{i=1}^{2} \frac{\left(\pi + RT\left(RA_i\right)\right)}{\sum_{i=1}^{n}\left(\pi + RT\left(RA_i\right)\right)} \left(-\ln\left(OF_j\right)\right)^{r\theta} \right)^{1/r\theta}} \end{array} \right)$$

If Eq. (11) hold for $i = k$, we have:

$$\mathrm{SVNNAAPG}\left(RA_1, RA_2, \cdots, RA_k\right) = \overset{k}{\underset{i=1}{\otimes}}\left(RA_i\right)^{\left(\pi+RT\left(RA_i\right)\right)\Big/\sum_{i=1}^{n}\left(\pi+RT\left(RA_i\right)\right)}$$

$$= \left(e^{-\left(\sum_{i=1}^{k}\frac{\left(\pi+RT\left(RA_i\right)\right)}{\sum_{i=1}^{k}\left(\pi+RT\left(RA_i\right)\right)}\left(-\ln\left(1-RT_j\right)\right)^{r\theta}\right)^{1/r\theta}}, 1-e^{-\left(\sum_{i=1}^{k}\frac{\left(\pi+RT\left(RA_i\right)\right)}{\sum_{i=1}^{k}\left(\pi+RT\left(RA_i\right)\right)}\left(-\ln\left(RI_j\right)\right)^{r\theta}\right)^{1/r\theta}}, 1-e^{-\left(\sum_{i=1}^{k}\frac{\left(\pi+RT\left(RA_i\right)\right)}{\sum_{i=1}^{k}\left(\pi+RT\left(RA_i\right)\right)}\left(-\ln\left(RF_j\right)\right)^{r\theta}\right)^{1/r\theta}} \right)$$

(b) Let $i = k+1$. From Definition 5 and Eq. (11), we obtain

$$\begin{aligned}
&\mathrm{SVNNAAPG}\left(OA_1, OA_2, \cdots, OA_k,\ OA_{k+1}\right)\\
&=\overset{k}{\underset{i=1}{\otimes}}\left(RA_i\right)^{\left(\pi+RT\left(RA_i\right)\right)\Big/\sum_{i=1}^{k+1}\left(\pi+RT\left(RA_i\right)\right)}\otimes\left(RA_{k+1}\right)^{\left(\pi+RT\left(RA_{k+1}\right)\right)\Big/\sum_{i=1}^{k+1}\left(\pi+RT\left(RA_i\right)\right)}\\
&=\left(\begin{array}{c}e^{-\left(\sum_{i=1}^{k}\frac{\left(\pi+RT\left(RA_i\right)\right)}{\sum_{i=1}^{k+1}\left(\pi+RT\left(RA_i\right)\right)}\left(-\ln\left(1-RT_j\right)\right)^{r\theta}\right)^{1/r\theta}},\\ 1-e^{-\left(\sum_{i=1}^{k}\frac{\left(\pi+RT\left(RA_i\right)\right)}{\sum_{i=1}^{k+1}\left(\pi+RT\left(RA_i\right)\right)}\left(-\ln\left(RI_j\right)\right)^{r\theta}\right)^{1/r\theta}},1-e^{-\left(\sum_{i=1}^{k}\frac{\left(\pi+RT\left(RA_i\right)\right)}{\sum_{i=1}^{k+1}\left(\pi+RT\left(RA_i\right)\right)}\left(-\ln\left(RF_j\right)\right)^{r\theta}\right)^{1/r\theta}}\end{array}\right)\otimes\\
&\left(\begin{array}{c}e^{-\left(\frac{\left(\pi+RT\left(RA_{k+1}\right)\right)}{\sum_{i=1}^{k+1}\left(\pi+RT\left(RA_i\right)\right)}\left(-\ln\left(1-RT_{k+1}\right)\right)^{r\theta}\right)^{1/r\theta}},\\ 1-e^{-\left(\frac{\left(\pi+RT\left(RA_{k+1}\right)\right)}{\sum_{i=1}^{k+1}\left(\pi+RT\left(RA_i\right)\right)}\left(-\ln\left(OI_{k+1}\right)\right)^{r\theta}\right)^{1/r\theta}},1-e^{-\left(\frac{\left(\pi+RT\left(RA_{k+1}\right)\right)}{\sum_{i=1}^{k+1}\left(\pi+RT\left(RA_i\right)\right)}\left(-\ln\left(OF_{k+1}\right)\right)^{r\theta}\right)^{1/r\theta}}\end{array}\right)\\
&=\left(\begin{array}{c}e^{-\left(\sum_{i=1}^{k+1}\frac{\left(1+RT\left(RA_i\right)\right)}{\sum_{i=1}^{k+1}\left(1+RT\left(RA_i\right)\right)}\left(-\ln\left(1-RT_j\right)\right)^{r\theta}\right)^{1/r\theta}},\\ 1-e^{-\left(\sum_{i=1}^{k+1}\frac{\left(1+RT\left(RA_i\right)\right)}{\sum_{i=1}^{k+1}\left(1+RT\left(RA_i\right)\right)}\left(-\ln\left(RI_j\right)\right)^{r\theta}\right)^{1/r\theta}},1-e^{-\left(\sum_{i=1}^{k+1}\frac{\left(1+RT\left(RA_i\right)\right)}{\sum_{i=1}^{k+1}\left(1+RT\left(RA_i\right)\right)}\left(-\ln\left(RF_j\right)\right)^{r\theta}\right)^{1/r\theta}}\end{array}\right).
\end{aligned}$$

In light with (a), (b), and (c), it could be known that Eq. (11) holds any *i*.

The SVNNAAPG has serval properties.

Property 4 (idempotency). If $RA_i = RA = (RT, RI, RF)$,

$$\mathrm{SVNNAAPG}\left(RA_1, RA_2, \cdots, RA_n\right) = RA \tag{11}$$

Property 5 (Monotonicity). Let $RA_i = (RT_{A_i}, RI_{A_i}, RF_{A_i})$, $RB_i = (RT_{B_i}, RI_{B_i}, RF_{B_i})$. If $RT_{A_i} \le RT_{B_i}, RI_{A_i} \ge RI_{B_i}, RF_{A_i} \ge RF_{B_i}$ holds for all i, then

$$\begin{aligned}
&\mathrm{SVNNAAPG}\left(RA_1, RA_2, \cdots, RA_n\right)\\
&\le \mathrm{SVNNAAPG}(RB_1, RB_2, \cdots, RB_n)
\end{aligned} \tag{12}$$

Property 6 (Boundedness). Let $RA_i = (RT_{A_i}, RI_{A_i}, RF_{A_i})$. If

$RA^+ = (\max_i(RT_i), \min_i(RI_i), \min_i(RF_i))$, $RA^- = (\min_i(RT_i), \max_i(RI_i), \max_i(RF_i))$, then

$$RA^- \le \mathrm{SVNNAAPG}\left(RA_1, RA_2, \cdots, RA_n\right) \le RA^+ \tag{13}$$

4. Methodology

Then, the SVNNAAPG technique is resolved for MADM with SVNSs. Let $RZ=\{RZ_1, RZ_2, \ldots, RZ_n\}$ be attributes. Let $RP=\{RP_1, RP_2, \ldots, RP_m\}$ be alternatives. $RQ=\left(RQ_{ij}\right)_{m\times n}=\left(RT_{ij}, RI_{ij}, RF_{ij}\right)_{m\times n}$ is the SVNN-matrix. The SVNNAAPG approach is resolved for MADM.

Step 1. Put forward the SVNN-matrix $RQ=\left(RQ_{ij}\right)_{m\times n}=\left(RT_{ij}, RI_{ij}, RF_{ij}\right)_{m\times n}$.

$$RQ=\left[RQ_{ij}\right]_{m\times n}=\begin{bmatrix} RQ_{11} & RQ_{12} & \ldots & RQ_{1n} \\ RQ_{21} & RQ_{22} & \ldots & RQ_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ RQ_{m1} & RQ_{m2} & \ldots & RQ_{mn} \end{bmatrix} \tag{14}$$

$$RQ=\left[RQ_{ij}\right]_{m\times n}=\left(RT_{ij}, RI_{ij}, RF_{ij}\right)_{m\times n}$$
$$=\begin{bmatrix} \left(RT_{11}, RI_{11}, RF_{11}\right) & \left(RT_{12}, RI_{1}, RF_{12}\right) & \ldots & \left(RT_{1n}, RI_{1n}, RF_{1n}\right) \\ \left(RT_{21}, RI_{21}, RF_{21}\right) & \left(RT_{22}, RI_{22}, RF_{22}\right) & \ldots & \left(RT_{2n}, RI_{2n}, RF_{2n}\right) \\ \vdots & \vdots & \vdots & \vdots \\ \left(RT_{m1}, RI_{m1}, RF_{m1}\right) & \left(RT_{m2}, RI_{m2}, RF_{m2}\right) & \ldots & \left(RT_{mn}, RI_{mn}, RF_{mn}\right) \end{bmatrix} \tag{15}$$

Step 2. Normalize the $RQ=\left(RQ_{ij}\right)_{m\times n}=\left(RT_{ij}, RI_{ij}, RF_{ij}\right)_{m\times n}$ to $NRQ=\left[NRQ_{ij}\right]_{m\times n}=\left(NRT_{ij}, NRI_{ij}, NRF_{ij}\right)_{m\times n}$.

$$NRQ_{ij}=\left(NRT_{ij}, NRI_{ij}, NRF_{ij}\right)$$
$$=\begin{cases}\left(RT_{ij}, RI_{ij}, RF_{ij}\right), & RZ_j \text{ is a benefit criterion} \\ \left(RF_{ij}, 1-RI_{ij}, RT_{ij}\right), & RZ_j \text{ is a cost criterion}\end{cases} \tag{16}$$

Step 3. In light with $NRQ=\left[NRQ_{ij}\right]_{m\times n}=\left(NRT_{ij}, NRI_{ij}, NRF_{ij}\right)_{m\times n}$, $NRQ=\left[NRQ_{ij}\right]_{m\times n}=\left(NRT_{ij}, NRI_{ij}, NRF_{ij}\right)_{m\times n}$ with SVNNAAPG to get the SVNNs $NRQ_i=\left(NRT_i, NRI_i, NRF_i\right)(i=1,2,\cdots,m)$:

$$NRQ_i = \text{SVNNAAPG}(NRQ_{i1}, NRQ_{i2}, \cdots, NRQ_{in})$$
$$= \overset{n}{\underset{j=1}{\otimes}} \left(NRQ_{ij} \right)^{\left(\pi + RT\left(NRQ_{ij}\right)\right) \Big/ \sum_{j=1}^{n}\left(\pi + RT\left(NRQ_{ij}\right)\right)}$$
$$= \left(\begin{array}{c} e^{-\left(\sum_{i=1}^{n} \frac{\left(\pi + RT\left(NRQ_{ij}\right)\right)}{\sum_{j=1}^{n}\left(\pi + RT\left(NRQ_{ij}\right)\right)} \left(-\ln\left(1-NRT_{ij}\right)\right)^{r\theta} \right)^{1/r\theta}}, \\ 1-e^{-\left(\sum_{j=1}^{n} \frac{\left(\pi + RT\left(NRQ_{ij}\right)\right)}{\sum_{j=1}^{n}\left(\pi + RT\left(NRQ_{ij}\right)\right)} \left(-\ln\left(NRI_{ij}\right)\right)^{r\theta} \right)^{1/r\theta}}, 1-e^{-\left(\sum_{j=1}^{n} \frac{\left(\pi + RT\left(NRQ_{ij}\right)\right)}{\sum_{j=1}^{n}\left(\pi + RT\left(NRQ_{ij}\right)\right)} \left(-\ln\left(NRF_{ij}\right)\right)^{r\theta} \right)^{1/r\theta}} \end{array} \right) \tag{17}$$

Step 4. Construct the $RSV\left(NRQ_i\right)$, $RAV\left(NRQ_i\right)\left(i = 1, 2, \cdots, m\right)$.

$$\begin{aligned} RSV\left(NRQ_i\right) &= \frac{\left(2 + NRT_i - NRI_i - NRF_i\right)}{3}, \\ RAV\left(NRQ_i\right) &= NRT_i - NRF_i \end{aligned} \tag{18}$$

Step 5. Rank the alternatives through $RSV\left(NRQ_i\right)$, $RAV\left(NRQ_i\right)$.

5. Applications

5.1. Case Study Description

The quality evaluation of university general education courses is a systematic and comprehensive assessment process for courses designed to develop students' overall competencies. These courses aim to cultivate skills such as critical thinking, communication, teamwork, and social responsibility. The quality of these courses not only directly impacts students' learning experiences and skill development but also plays a crucial role in the overall educational standards of the university. Hence, a scientific and reasonable course evaluation system is essential to ensure and enhance the quality of general education. First, the core of course quality evaluation lies in assessing teaching effectiveness. By evaluating the achievement of course objectives, students' learning outcomes, the design and use of teaching materials, and the scientific and cutting-edge nature of the content, one can determine whether the course truly meets the goal of fostering students' overall competencies. Additionally, student engagement and satisfaction are important indicators. Students' classroom performance, feedback, and learning experience surveys provide a direct reflection of how well the course is received and its actual impact on students. Second, the quality of the instructor is a key factor influencing the quality of the course. The teacher's professional competence, innovation in teaching methods, and ability to interact with students all significantly affect the course. Therefore, course quality evaluation should also include assessments of the instructor's teaching ability, attitude, and methods to ensure that teachers can use effective teaching strategies to engage students and promote their holistic development. Third, the design and arrangement of the course content are also central to the evaluation. General education courses should be broadly applicable and diverse, catering to students from different academic backgrounds and interests. Additionally, the course should integrate theory with practice, ensuring sufficient theoretical depth while enhancing students' practical abilities and

problem-solving skills through hands-on activities. Finally, establishing a course quality evaluation system should involve multi-perspective assessments, including peer reviews and external expert evaluations, to ensure objectivity and comprehensiveness. These multi-dimensional evaluations help identify areas for improvement and provide a foundation for course enhancement. In summary, the quality evaluation of university general education courses is a complex and multi-layered task. It not only assesses the courses themselves but also provides feedback on the broader educational system. Through scientific and systematic evaluation, courses can be continuously optimized, teaching quality improved, and the ultimate goal of cultivating highly qualified talents achieved. The quality evaluation of university general education courses is useful MADM. In this paper, numerical example for quality evaluation of university general education courses is resolved through employing SVNNAAPG approach. Five colleges are evaluated with four attributes: ①RZ_1 is Teaching Effectiveness-This indicator assesses whether the course has achieved its intended teaching objectives, including students' knowledge acquisition, improvement of overall competencies, and the development of critical thinking, communication, and problem-solving skills; ②RZ_2 is Instructor Quality-This evaluates the instructor's professional expertise, innovation in teaching methods, classroom management, and interaction with students, ensuring that the instructor can effectively guide students and stimulate their interest in learning; ③RZ_3 is Course Content Design-This indicator focuses on the rationality and relevance of the course content, including the integration of theory and practice, whether the content meets students' needs, and its adaptability to students from different academic backgrounds.; ④RZ_4 is Student Feedback and Engagement-By analyzing students' classroom performance, attendance, engagement, and feedback, this indicator evaluates student participation and satisfaction with the course, reflecting the course's actual teaching effectiveness and the overall learning experience.

5.2. Application of the Framework to Quality Evaluation

Then, the SVNNAAPG approach is put forward quality evaluation of university general education courses under SVNNs.

Step 1. Implement the SVNN-matrix $RQ = \left(RQ_{ij} \right)_{5\times4}$ as in Table 1 in light with statistical analysis.

Table 1. SVNN matrix

	RZ_1	RZ_2	RZ_3	RZ_4
RP_1	(0.36, 0.23, 0.35)	(0.31, 0.24, 0.36)	(0.34, 0.62, 0.36)	(0.28, 0.34, 0.39)
RP_2	(0.33, 0.16, 0.62)	(0.62, 0.23, 0.28)	(0.39, 0.35, 0.63)	(0.32, 0.36, 0.38)
RP_3	(0.61, 0.35, 0.63)	(0.64, 0.29, 0.61)	(0.31, 0.26, 0.63)	(0.67, 0.16, 0.34)
RP_4	(0.69, 0.24, 0.42)	(0.65, 0.36, 0.34)	(0.62, 0.61, 0.38)	(0.69, 0.34, 0.47)
RP_5	(0.64, 0.28, 0.67)	(0.68, 0.27, 0.62)	(0.62, 0.27, 0.67)	(0.37, 0.26, 0.42)

Step 2. Normalize $RQ = \left(RQ_{ij} \right)_{5\times4}$ to $NRQ = \left[NRQ_{ij} \right]_{5\times4}$ (See the Table 2).

Table 2. The NRQ matrix

	RZ_1	RZ_2	RZ_3	RZ_4
RP_1	(0.36, 0.23, 0.35)	(0.31, 0.24, 0.36)	(0.34, 0.62, 0.36)	(0.28, 0.34, 0.39)
RP_2	(0.33, 0.16, 0.62)	(0.62, 0.23, 0.28)	(0.39, 0.35, 0.63)	(0.32, 0.36, 0.38)
RP_3	(0.61, 0.35, 0.63)	(0.64, 0.29, 0.61)	(0.31, 0.26, 0.63)	(0.67, 0.16, 0.34)
RP_4	(0.69, 0.24, 0.42)	(0.65, 0.36, 0.34)	(0.62, 0.61, 0.38)	(0.69, 0.34, 0.47)
RP_5	(0.64, 0.28, 0.67)	(0.68, 0.27, 0.62)	(0.62, 0.27, 0.67)	(0.37, 0.26, 0.42)

Step 3. Put forward the $NRQ_i\ (i=1,2,3,4,5)$ with SVNNAAPG approach (Table 3).

Table 3. The NRQ_i by SVNNAAPG operator ($\theta=2,\pi=2$)

Alternatives	NRQ_i
RP_1	(0.4015,0.3324,0.2961)
RP_2	(0.7165,0.2213,0.3658)
RP_3	(0.5053,0.5327,0.2985)
RP_4	(0.4645,0.5031,0.3332)
RP_5	(0.5214,0.4325,0.3101)

Step 4. Obtain $RSV\left(NRQ_i\right)\left(i=1,2,3,4,5\right)$ (See Table 4).

Table 4. The $RSV\left(NRQ_i\right)$

Alternatives	$RSV\left(NRQ_i\right)$	Order
RP_1	0.6918	5
RP_2	0.7777	1
RP_3	0.7175	4
RP_4	0.7138	3
RP_5	0.7179	2

Step 5. From Table 4, the order is $RP_2 > RP_5 > RP_3 > RP_4 > RP_1$, and the best colleges is XP_2.

5.3. Influence analysis

To administrate the effects for final results according to parameters of SVNNAAPG approach, the results are administrated in Tables 5-6 and Figure 1.

Table 5. Parameter values for SVNNAAPG approach

θ	$RSV(RP_1)$	$RSV(RP_2)$	$RSV(RP_3)$	$RSV(RP_4)$	$RSV(RP_5)$
1	0.4890	0.6159	0.5157	0.5259	0.5383
2	0.6918	0.7777	0.7175	0.7138	0.7179
3	0.7476	0.8075	0.7689	0.7529	0.7715
4	0.7689	0.8155	0.7883	0.7648	0.7946
5	0.7792	0.8188	0.7983	0.7697	0.8072
6	0.7852	0.8204	0.8047	0.7724	0.8149
7	0.7891	0.8216	0.8091	0.7742	0.8200
8	0.7920	0.8226	0.8126	0.7757	0.8237
9	0.7941	0.8234	0.8154	0.7769	0.8266
10	0.7959	0.8244	0.8178	0.7778	0.8288

Table 6. Order for SVNNAAPG approach

$r\theta$	Order
1	$RP_2 > RP_5 > RP_4 > RP_3 > RP_1$
2	$RP_2 > RP_5 > RP_3 > RP_4 > RP_1$
3	$RP_2 > RP_5 > RP_3 > RP_4 > RP_1$
4	$RP_2 > RP_5 > RP_3 > RP_1 > RP_4$
5	$RP_2 > RP_5 > RP_3 > RP_1 > RP_4$
6	$RP_2 > RP_5 > RP_3 > RP_1 > RP_4$
7	$RP_2 > RP_5 > RP_3 > RP_1 > RP_4$

8	$RP_2 > RP_5 > RP_3 > RP_1 > RP_4$
9	$RP_2 > RP_5 > RP_3 > RP_1 > RP_4$
10	$RP_2 > RP_5 > RP_3 > RP_1 > RP_4$

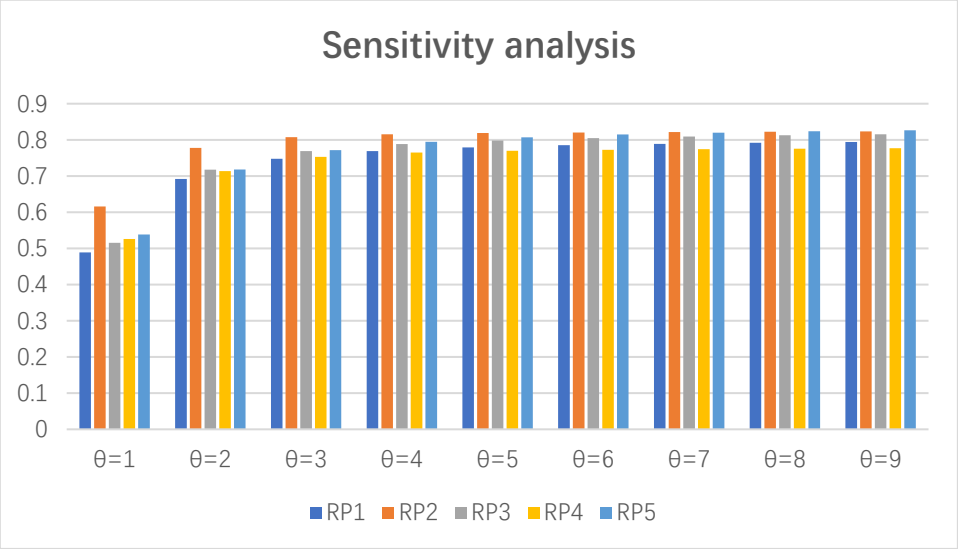


Figure 1. Different parameters for SVNNAAPG approach

It could be seen from Table 5-6 when different parameter values are resolved, the order is slightly different.

5.4. Comparative analysis

The SVNNAAPG approach is compared with SVNNWA approach and SVNNWG approach [32], SVNN-EDAS technique [35] and SVNN-CODAS technique[36] and. The results are resolved in Table 7.

Table 7. Results for different approaches

Approaches	Order
SVNNWA technique [32]	$RP_2 > RP_5 > RP_4 > RP_3 > RP_1$
SVNNWG technique [32]	$RP_2 > RP_5 > RP_3 > RP_4 > RP_1$
SVNN-EDAS technique [35]	$RP_2 > RP_5 > RP_3 > RP_4 > RP_1$
SVNN-CODAS technique [36]	$RP_2 > RP_5 > RP_3 > RP_4 > RP_1$
SVNNAAPG technique	$RP_2 > RP_5 > RP_3 > RP_4 > RP_1$

Obtained from Table 7, it is administrated that the order of these approaches is slightly different, however, the optimal business English college is RP_2 and the worst college is RP_1.

5.5. Discussion analysis

The SVNNAAPG approach presents distinct advantages and some drawbacks in multi-attribute decision-making, particularly in complex scenarios such as evaluating business English teaching quality. Some advantages for SVNNAAPG approach are outlined: (1) Effective handling of uncertainty: The SVNNAAPG method uses SVNSs to manage complex fuzzy and uncertain information. This three-dimensional representation captures the truth, falsity, and indeterminacy of information, allowing decision-makers to better understand and manage complexity. This capability enhances the accuracy and reliability of evaluations, helping to make more precise judgments, especially in subjective assessments. (2) High flexibility: The method incorporates Aczel-Alsina operations, providing the ability to flexibly adjust the weights of various factors. This flexibility allows decision-makers to tailor the evaluation model to specific needs and environmental conditions, adapting to different assessment scenarios. This is particularly important in educational quality assessments where conditions and priorities may change over time. (3) Comprehensiveness and inclusiveness: The introduction of the power geometric method allows SVNNAAPG to perform weighted averaging of multiple attributes, ensuring that all relevant factors are reasonably considered in the decision-making process. This comprehensiveness not only ensures fairness but also balances different factors, making the evaluation results more objective and credible.

6. Conclusion

6.1. Summary of Key Findings

The quality evaluation of university general education courses is a crucial aspect of MADM. In this study, the SVNNAAPG approach is applied, leveraging Aczel-Alsina operations and the PG approach within the framework of SVNSs. This method is specifically designed for MADM applications, aiming to enhance the accuracy and reliability of the evaluation process. A numerical example is provided to illustrate the application of the SVNNAAPG approach in evaluating business English teaching quality. The major contributions of this work are highlighted as follows: (1) Enhancement of Aczel-Alsina operations and PG approach under SVNSs: The study enhances the Aczel-Alsina operations and the PG approach within the context of SVNSs. These operations and methods are particularly effective in handling uncertainty and vagueness, which are inherent in decision-making processes. By applying these enhanced operations, the study aims to improve the precision and robustness of the evaluation. (2) Administration of the SVNNAAPG approach: The SVNNAAPG approach is systematically applied, integrating Aczel-Alsina operations and the PG approach. This integration ensures a comprehensive evaluation framework that can address the complexities associated with quality evaluation of university general education courses. (3) Application of SVNNAAPG for MADM with SVNSs: The SVNNAAPG approach is tailored for MADM scenarios involving SVNSs. SVNSs are adept at representing uncertain and imprecise information, making them suitable for evaluating qualitative aspects of teaching quality. This application demonstrates the method's capability to handle complex decision-making environments effectively. (4) Validation through numerical example: A numerical example is employed to validate the SVNNAAPG approach in the context of quality evaluation of university general education courses. This example showcases the practical application and potential benefits of the method, highlighting its effectiveness in real-world scenarios. The example provides a step-by-step implementation, demonstrating how the SVNNAAPG approach can be utilized to derive meaningful insights and support decision-making.

In conclusion, the application of the SVNNAAPG approach in quality evaluation of university general education courses offers significant improvements in the accuracy and reliability of the evaluation process. By enhancing Aczel-Alsina operations and the PG approach within SVNSs, the study provides a robust

framework for MADM. The numerical example further validates the method's practical applicability, making it a valuable tool for educators and decision-makers in the field of university general education courses. This approach not only addresses the inherent uncertainties in teaching quality evaluation but also offers a systematic method for making informed decisions.

6.1. Future Research Directions

In light of the above analysis, this study faces several challenges. First, the complexity of the SVNNAAPG method might make it difficult for educators and administrators to use, particularly those without a strong mathematical background. This could limit the widespread adoption of the method in the quality evaluation of university general education courses. Additionally, the research may rely on a limited dataset, which could affect its applicability across diverse educational contexts. Another challenge is that the model may not be dynamic enough to keep pace with rapid changes in the demands of education and student needs, potentially reducing its effectiveness over time. Looking ahead, there are several promising directions for further research. One important step is to simplify the model, making it more user-friendly and practical for a broader audience of educators and administrators. Developing intuitive software tools could assist non-technical users in applying the method more easily. Furthermore, it is crucial to validate the method internationally by testing it in various cultural and educational settings to ensure its wider applicability and effectiveness. Lastly, enhancing the model's adaptability to changes in educational policies, student expectations, and technological advancements would be beneficial. By regularly updating the model and incorporating machine learning, it could adjust to new trends and requirements in real-time. These efforts could significantly improve the quality evaluation of university general education courses and ensure its long-term relevance and utility.

Acknowledgment

The work was supported by the Research on the Dynamic Development Trajectory of College Counselors' Work Effectiveness and Its Influence Mechanism Based on Students' Evaluation, 2024 Higher Education Scientific Research Planning Project of the Chinese Society of Higher Education under Grant No. 24FD0416.

References

[1] L. Zhao, Research on the construction of university general education curriculum systems: From the perspective of teacher training in basic education, Talent Development, (2016) 199-201.

[2] F. Wang, J. Tang, H. Zhang, Reflections on the reform of practical teaching systems for electronic information courses in universities under the background of general education: A case study at henan university of technology, Henan Education (Higher Education), (2016) 123-124.

[3] H. Ma, Research on the design of general education courses in universities, Knowledge Library, (2018) 153-153.

[4] M. Yang, Exploration of mooc construction for university general education courses: A case study of "ten lectures on chinese and foreign cultural spirit" at yangtze university, Journal of Yangtze University (Social Science Edition), 42 (2019) 99-103.

[5] Q. Zu, C. Feng, Constructing the general education curriculum system in universities under the background of emerging engineering education, Journal of Qiqihar Normal College, (2019) 8-9.

[6] C. Yan, Y. Zhou, The influence of university general education on students' healthy lifestyles: A study based on the "five-classroom teaching" fitness course of shandong modern college, Cultural Innovation

Comparative Studies, 4 (2020) 25-27.

[7] X. Hao, Discussion on general education courses for property management majors in universities, Journal of Luohe Vocational and Technical College, 20 (2021) 106-108.

[8] J. Gao, On the general education goals and implementation path of ideological and political courses in universities, Jiangsu Higher Education, (2021) 75-79.

[9] N. Zeng, Research on the optimization and innovation of the college employment guidance curriculum system from the perspective of general education, Modern Vocational Education, (2022) 130-132.

[10] Y. Guo, L. Wang, Y. Gao, B. Wei, The construction logic, current situation, and practice strategies of general education courses in local universities under the background of general education, Western Quality Education, 10 (2024) 19-23.

[11] A.K. Das, N. Gupta, C. Granados, R. Das, S. Das, Neutrosophic approach to water quality assessment: A case study of gomati river, the largest river in tripura, india, Neutrosophic Systems with Applications, 22 (2024) 1-12.

[12] R. Imran, K. Ullah, Z. Ali, M. Akram, An approach to multi-attribute decision-making based on single-valued neutrosophic hesitant fuzzy aczel-alsina aggregation operator, Neutrosophic Systems with Applications, 22 (2024) 43-57.

[13] M. Karak, A. Mahata, S. Paul, S. Mukherjee, S. Biswas, B. Roy, An attempt on investment selection problem under neutrosophic environment based on weighted value method, Neutrosophic Optimization and Intelligent Systems, 4 (2024) 39-50.

[14] M. Luo, Z. Sun, L. Wu, Fuzzy inference quintuple implication method based on single valued neutrosophic t-representable t-norm, Neutrosophic Optimization and Intelligent Systems, 3 (2024) 8-22.

[15] S.I. Abdel Aal, Neutrosophic framework for assessment challenges in smart sustainable cities based on iot to better manage energy resources and decrease the urban environment's ecological impact, Neutrosophic Systems with Applications, 6 (2023) 9-16.

[16] M. Abouhawwash, M. Jameel, Evaluation factors of solar power plants to reduce cost under neutrosophic multi-criteria decision making model, Sustainable Machine Intelligence Journal, 2 (2023) 1-11.

[17] S. Bhuvaneshwari, C.A.C. Sweety, A. Singh, S. Broumi, M. Talea, P.K. Raut, A novel and an efficient codas technique to solve real-life magdm problems in fermatean neutrosophic environment, Neutrosophic Sets and Systems, 72 (2024) 41-59.

[18] M.R. Seikh, S. Dutta, Solution of matrix games with pay-offs of single-valued neutrosophic numbers and its application to market share problem, in: Mathematics and computer science volume 1, 2023, pp. 213-230.

[19] M.R. Seikh, S. Dutta, Solution of matrix games with payoffs of single-valued trapezoidal neutrosophic numbers, Soft Computing, 26 (2022) 921-936.

[20] M.R. Seikh, S. Dutta, A nonlinear programming model to solve matrix games with pay-offs of single-valued neutrosophic numbers, Neutrosophic Sets and Systems, 47 (2021) 366-383.

[21] H.M. Nehi, A. Keikha, Topsis and choquet integral hybrid technique for solving magdm problems with interval type-2 fuzzy numbers, Journal of Intelligent & Fuzzy Systems, 30 (2016) 1301-1310.

[22] Ö. Kabak, B. Ervural, Multiple attribute group decision making: A generic conceptual framework and a classification scheme, Knowledge-Based Systems, 123 (2017) 13-30.

[23] P.D. Liu, F. Teng, Multiple attribute group decision making methods based on some normal neutrosophic number heronian mean operators, Journal of Intelligent & Fuzzy Systems, 32 (2017) 2375-2391.

[24] X.Y. Zhang, J.Q. Wang, H.Y. Zhang, J.H. Hu, A heterogeneous linguistic magdm framework to classroom teaching quality evaluation, Eurasia Journal of Mathematics Science and Technology Education, 13 (2017) 4929-4956.

[25] H. Wang, F. Smarandache, Y.Q. Zhang, R. Sunderraman, Single valued neutrosophic sets, Multispace Multistruct, (2010) 410-413.

[26] S. Ashraf, S. Ahmad, M. Naeem, M. Riaz, M.A. Alam, Novel edas methodology based on single-valued neutrosophic aczel-alsina aggregation information and their application in complex decision-making, Complexity, 2022 (2022) 18.

[27] W. Ali, T. Shaheen, I. Ul Haq, H. Toor, F. Akram, H. Garg, M.Z. Uddin, M.M. Hassan, Aczel-alsina-based aggregation operators for intuitionistic hesitant fuzzy set environment and their application to multiple attribute decision-making process, Aims Mathematics, 8 (2023) 18021-18039.

[28] Z.S. Xu, R.R. Yager, Power-geometric operators and their use in group decision making, Ieee Transactions on Fuzzy Systems, 18 (2010) 94-105.

[29] S.-H. Xiong, Z.-S. Chen, J.-P. Chang, K.-S. Chin, On extended power average operators for decision-making: A case study in emergency response plan selection of civil aviation, Computers & Industrial Engineering, 130 (2019) 258-271.

[30] T. Senapati, G.Y. Chen, R. Mesiar, R.R. Yager, Intuitionistic fuzzy geometric aggregation operators in the framework of aczel-alsina triangular norms and their application to multiple attribute decision making, Expert Systems with Applications, 212 (2023) 15.

[31] T. Senapati, L. Martinez, G.Y. Chen, Selection of appropriate global partner for companies using q-rung orthopair fuzzy aczel-alsina average aggregation operators, International Journal of Fuzzy Systems, 25 (2023) 980-996.

[32] J.J. Peng, J.Q. Wang, J. Wang, H.Y. Zhang, X.H. Chen, Simplified neutrosophic sets and their applications in multi-criteria group decision-making problems, International Journal of Systems Science, 47 (2016) 2342-2358.

[33] S. Ashraf, S. Ahmad, M. Naeem, M. Riaz, M.A. Alam, Z. Stevic, Novel edas methodology based on single-valued neutrosophic aczel-alsina aggregation information and their application in complex decision-making, Complexity, 2022 (2022) 1-18.

[34] R. Yong, J. Ye, S.G. Du, A.Q. Zhu, Y.Y. Zhang, Aczel-alsina weighted aggregation operators of simplified neutrosophic numbers and its application in multiple attribute decision making, Cmes-Computer Modeling in Engineering & Sciences, 132 (2022) 569-584.

[35] D. Stanujkic, D. Karabasevic, G. Popovic, D. Pamucar, Z. Stevic, E.K. Zavadskas, F. Smarandache, A single-valued neutrosophic extension of the edas method, Axioms, 10 (2021) 13.

[36] E. Bolturk, A. Karasan, Prioritization of investment alternatives for a hospital by using neutrosophic codas method, Journal of Multiple-Valued Logic and Soft Computing, 33 (2019) 381-396.

Received: June 26, 2024. Accepted: Oct 13, 2024

Neutrosophic Sets and Systems (NSS) is an academic journal, published quarterly online and on paper, that has been created for publications of advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics etc. and their applications in any field.

All submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

It is an open access journal distributed under the Creative Commons Attribution License that permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

ISSN (print): 2331-6055, ISSN (online): 2331-608X
Impact Factor: 1.739

NSS has been accepted by SCOPUS. Starting with Vol. 19, 2018, all NSS articles are indexed in Scopus.

NSS is also indexed by Google Scholar, Google Plus, Google Books, EBSCO, Cengage Thompson Gale (USA), Cengage Learning, ProQuest, Amazon Kindle, DOAJ (Sweden), University Grants Commission (UGC) - India, International Society for Research Activity (ISRA), Scientific Index Services (SIS), Academic Research Index (ResearchBib), Index Copernicus (European Union),CNKI (Tongfang Knowledge Network Technology Co., Beijing, China), etc.

Google Dictionary has translated the neologisms "neutrosophy" (1) and "neutrosophic" (2), coined in 1995 for the first time, into about 100 languages.

FOLDOC Dictionary of Computing (1, 2), Webster Dictionary (1, 2), Wordnik (1), Dictionary.com, The Free Dictionary (1),Wiktionary (2), YourDictionary (1, 2), OneLook Dictionary (1, 2), Dictionary / Thesaurus (1), Online Medical Dictionary (1, 2), and Encyclopedia (1, 2) have included these scientific neologisms.

DOI numbers are assigned to all published articles.

Registered by the Library of Congress, Washington DC, United States, https://lccn.loc.gov/2013203857.

Recently, NSS was also approved for Emerging Sources Citation Index (ESCI) available on the Web of Science platform, starting with Vol. 15, 2017.

Editors-in-Chief:

Prof. Dr. Florentin Smarandache, Postdoc, Department of Mathematics, University of New Mexico, Gallup, NM 87301, USA, Email: smarand@unm.edu.
Dr. Mohamed Abdel-Basset, Faculty of Computers and Informatics, Zagazig University, Egypt, Email: mohamed.abdelbasset@fci.zu.edu.eg.
Dr. Said Broumi, Laboratory of Information Processing, Faculty of Science Ben M'Sik, University of Hassan II, Casablanca, Morocco, Email: s.broumi@flbenmsik.ma.

23316055

$39.95